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Finite Larmor radius and Hall effects on thermal instability of a compressible plasma

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Abstract. The effects of compressibility, finite Larmor radius (FLR) and Hall currents are considered on the thermal instability of a plasma in the presence of a uniform horizontal magnetic field. For stationary convection, the compressibility has a stabilizing effect whereas FLR and Hall currents have stabilizing as well as destabilizing effects. For $(C_p\beta/g) < 1$, the system is stable. The magnetic field, FLR and Hall currents introduce oscillatory modes in the system for $(C_p\beta/g) > 1$.

Keywords. Finite Larmor radius; Hall currents; compressibility; thermal instability; plasma.

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1. Introduction

A detailed account of the thermal instability of a fluid layer heated from below under varying assumptions of hydromagnetics was given by Chandrasekhar (1961). The effects of finiteness of the ion Larmor radius, which exhibits itself in the form of a magnetic viscosity in the fluid equations, have been studied by Rosenbluth *et al* (1962), Roberts and Taylor (1962), Vandakurov (1964) and Jukes (1964). The effect of FLR on the thermal instability of a plasma has been considered by Sharma and Prakash (1975) in the presence of a horizontal magnetic field. Gupta (1967) studied the thermal instability of fluids in the presence of Hall currents. The Boussinesq approximation has been used in all the above studies.

When the fluids are compressible, the equations governing the system become quite complicated. To simplify the set of equations governing the flow of compressible fluids, Spiegel and Veronis (1960) made the following assumptions: (i) the depth of fluid layer is much smaller than the scale height as defined by them and (ii) the fluctuations in temperature, pressure and density, introduced due to motion, do not exceed their static variations.

Under the above assumptions, the flow equations are the same as those for incompressible fluids except that the static temperature gradient is replaced by its excess over the adiabatic. The thermal instability in compressible fluids in the presence of rotation and magnetic field has been considered by Sharma (1977). Sharma (1976) also studied the thermal instability of a compressible Hall plasma. Sharma and Sharma (1978) considered the thermal instability of a partially ionized plasma in the presence of compressibility and collisional effects while the thermal instability of a compressible plasma with FLR has been studied by Sharma *et al* (1983). Finite Larmor radius (FLR) effects are likely to be important in "weakly" unstable systems such as high beta stellarator, mirror machines, slowly rotating plasmas, large aspect tori etc. The Hall effects are likely to be important in many astrophysical situations as well as in flows of laboratory plasma. Sherman and Sutton (1962) considered the effect of Hall current on the efficiency of a magneto-fluid dynamic generator while Sato (1961) and Tani (1962) studied the incompressible viscous flow of an ionized gas with tensor conductivity in channels with consideration of Hall effect. Sonnerup (1961) and Uberoi and Devanathan (1963) investigated the effects of Hall current on the propagation of small amplitude waves taking compressibility into account. The object of the present paper is to study FLR and Hall effects on thermal instability of a compressible plasma. The effects of compressibility, FLR and Hall currents are more realistic in the physics of atmosphere and astrophysics especially in the case of ionosphere and outer layers of the sun's atmosphere.

2. Formulation of the problem and perturbation equations

Consider an infinite horizontal layer of compressible, viscous, heat-conducting and finite electrically conducting fluid of thickness d in which a uniform temperature gradient $\beta(=|dT/dz|)$ is maintained. Consider the cartesian coordinates (x, y, z) with origin on the lower boundary $z=0$ and the z-axis perpendicular to it along the vertical. The fluid is acted on by a horizontal magnetic fluid $\mathbf{H}(H, 0, 0)$ and gravity force $\mathbf{g}(0, 0, -g)$.

Following Spiegel and Veronis (1960) and Sharma *et al* (1983), the linearized hydromagnetic perturbation equations appropriate to the problem are

$$\partial\mathbf{q}/\partial t = -(1/\rho_m)\nabla\delta\overset{\leftrightarrow}{P} + \nu\nabla^2\mathbf{q} + \frac{1}{4\pi}(\nabla\times\mathbf{h})\times\mathbf{H} - \mathbf{g}\alpha\theta, \tag{1}$$

$$\nabla\cdot\mathbf{q}=0, \tag{2}$$

$$\partial\mathbf{h}/\partial t = \nabla\times(\mathbf{q}\times\mathbf{H}) + \eta\nabla^2\mathbf{h} - (1/4\pi Ne)\nabla\times[(\nabla\times\mathbf{h})\times\mathbf{H}], \tag{3}$$

$$\nabla\cdot\mathbf{h}=0, \tag{4}$$

$$\partial\theta/\partial t = \left(\beta - \frac{g}{C_p}\right)w + \kappa\nabla^2\theta, \tag{5}$$

where $\mathbf{q}(u, v, w)$, $\mathbf{h}(h_x, h_y, h_z)$, δp, $\delta\rho$ and θ denote respectively the perturbations in velocity, magnetic field $\mathbf{H}$, pressure p, density ρ and temperature T. $\delta\overset{\leftrightarrow}{P}$, ρ_m, μ, $\nu(=\mu/\rho_m)$, κ', $\kappa(=\kappa'/\rho_m C_p)$, η, g/C_p, g, α, N and e stand for stress tensor perturbation, constant space average of ρ, viscosity, kinematic viscosity, thermal conductivity, thermal diffusivity, resistivity, adiabatic gradient, acceleration due to gravity, coefficient of thermal expansion, electron number density and charge of an electron respectively.

For the horizontal magnetic field $\mathbf{H}(H, 0, 0)$, the stress tensor $\delta\overset{\leftrightarrow}{P}$, taking into account the finite ion gyration (Vandakurov 1964), has the components

$$\delta P_{xx} = \delta p, \quad \delta P_{xy} = \delta P_{yx} = -2\rho\nu_0\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right),$$

$$\delta P_{xz}=\delta P_{zx}=2\rho\nu_0\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),\quad \delta P_{yy}=\delta p-\rho\nu_0\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right),$$

$$\delta P_{yz}=\delta P_{zy}=\rho\nu_0\left(\frac{\partial v}{\partial y}-\frac{\partial w}{\partial z}\right),\quad \delta P_{zz}=\delta p+\rho\nu_0\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right). \tag{6}$$

Here $\rho\nu_0=NT/4\,\omega_H$ where N, T and ω_H denote respectively the number density, the ion temperature and the ion gyration frequency.

3. Dispersion relation

Analysing the disturbances into normal modes, we assume that perturbation quantities are of the form

$$[w,\theta,h_z,\zeta,\xi]=[W(z),\Theta(z),K(z),Z(z),X(z)]\exp(\imath k_x x+\imath k_y y+nt), \tag{7}$$

where k_x, k_y are the wave numbers along the x and y directions respectively, $k=(k_x^2+k_y^2)^{1/2}$ is the resultant wave number and n is the frequency of oscillation. ζ and ξ stand for the z-components of vorticity and current density respectively.

Let $a=kd$, $\sigma=nd^2/\nu$, $p_1=\nu/\kappa$, $p_2=\nu/\eta$, $D=\mathrm{d}/\mathrm{d}z$ and x, y, z stand for the coordinates in the new unit of length d. Equations (1)–(6), using expression (7), give

$$(D^2-a^2)(D^2-a^2-\sigma)W-\left(\frac{g\alpha d^2}{\nu}\right)a^2\Theta+\frac{\imath k_x Hd^2}{4\pi\rho_m\nu}(D^2-a^2)K$$
$$+\left(\frac{\imath k_x\nu_0 d^2}{\nu}\right)\left(D^2-a^2+3\frac{k_x^2}{k^2}a^2\right)Z=0, \tag{8}$$

$$(D^2-a^2-\sigma)Z=\left(\frac{\imath k_x\nu_0}{\nu}\right)\left(D^2-a^2+3\frac{k_x^2}{k^2}a^2\right)W-\frac{\imath k_x Hd^2}{4\pi\rho_m\nu}X, \tag{9}$$

$$(D^2-a^2-p_2\sigma)X=-\left(\frac{\imath k_x Hd^2}{\eta}\right)Z-\left(\frac{\imath k_x H}{4\pi Ne\eta}\right)(D^2-a^2)K, \tag{10}$$

$$(D^2-a^2-p_2\sigma)K=-\left(\frac{\imath k_x Hd^2}{\eta}\right)W+\left(\frac{\imath k_x Hd^2}{4\pi Ne\eta}\right)X, \tag{11}$$

$$(D^2-a^2-p_1\sigma)\Theta=-\frac{d^2}{\kappa}\left(\beta-\frac{g}{C_p}\right)W. \tag{12}$$

Eliminating Θ, K, X and Z between equations (8)–(12), we obtain

$$[(D^2-a^2-p_2\sigma)^2(D^2-a^2-\sigma)+Qk_x^2d^2(D^2-a^2-p_2\sigma)$$
$$-Mk_x^2d^2(D^2-a^2)(D^2-a^2-\sigma)]$$
$$\times\left[Ra^2\left(\frac{G-1}{G}\right)+(D^2-a^2)(D^2-a^2-\sigma)(D^2-a^2-p_1\sigma)\right]W$$

$$
\begin{aligned}
&+Qk_x^2d^2(D^2-a^2)(D^2-a^2-p_1\sigma)\Big[\{(D^2-a^2-\sigma)(D^2-a^2-p_2\sigma)\\
&+Qk_x^2d^2\}-M^{1/2}N^{1/2}k_x^2d^2(D^2-a^2-p_2\sigma)\\
&-M^{1/2}N^{1/2}k_x^2d^2\left(D^2-a^2+3\frac{k_x^2}{k^2}a^2\right)\Big]W\\
&-Nk_x^2d^2(D^2-a^2-p_1\sigma)\left(D^2-a^2+3\frac{k_x^2}{k^2}a^2\right)^2[(D^2-a^2-p_2\sigma)^2\\
&-Mk_x^2d^2(D^2-a^2)]W=0,
\end{aligned}
\tag{13}
$$

where $Q=H^2d^2/4\pi\rho_m\nu\eta$ is the Chandrasekhar number, $R=g\alpha\beta d^4/\nu\kappa$ is the Rayleigh number, $M=(H/4\pi Ne\eta)^2$ is the non-dimensional number accounting for Hall currents, $N=(\nu_0/\nu)^2$ is a non-dimensional number accounting for FLR effect and $G=C_p\beta/g$. Consider the case in which both the boundaries are free and the medium adjoining the fluid is non-conducting. The appropriate boundary conditions for this case are (Chandrasekhar 1961)

$$
\left.\begin{array}{l}
W=D^2W=0,\ \Theta=0,\ DZ=0,\ X=0 \text{ at } z=0 \text{ and } 1\\
\text{and } \mathbf{h} \text{ is continuous.}
\end{array}\right\}.
\tag{14}
$$

The case of two free boundaries, though little artificial, is the most appropriate for stellar atmospheres (Spiegel 1965). Using the boundary conditions (14), one can show that all the even derivatives of W must vanish for $z=0$ and 1 and hence the proper solution of (13) characterizing the lowest mode is

$$
W=W_0\sin\pi z,
\tag{15}
$$

where W_0 is a constant. Substituting (15) in (13) and letting $a^2=\pi^2x$, $R_1=R/\pi^4$, $Q_1=Q/\pi^2$, $k_x=k\cos\theta$ and $\imath\sigma_1=\sigma/\pi^2$, we obtain the dispersion relation

$$
\begin{aligned}
R_1=&\left(\frac{G}{G-1}\right)\Bigg(\left(\frac{1+x}{x}\right)(1+x+\imath\sigma_1)(1+x+\imath p_1\sigma_1)\\
&+[Q_1\cos^2\theta(1+x)(1+x+\imath p_1\sigma_1)\\
&\times\{(1+x+\imath\sigma_1)(1+x+\imath p_2\sigma_1)+Q_1x\cos^2\theta\}\\
&+N\cos^2\theta(1+x+\imath p_1\sigma_1)(1+x-3x\cos^2\theta)^2\\
&\times\{(1+x+\imath p_2\sigma_1)^2+Mx(1+x)\cos^2\theta\}\\
&+M^{1/2}N^{1/2}Q_1x(1+x)(1+x+\imath p_1\sigma_1)\cos^4\theta\\
&\times\{(1+x+\imath p_2\sigma_1)+(1+x-3x\cos^2\theta)\}][(1+x-\imath p_2\sigma_1)^2(1+x+\imath\sigma_1)\\
&+Q_1x\cos^2\theta(1+x+\imath p_2\sigma_1)+Mx\cos^2\theta(1+x)(1+x+\imath\sigma_1)]^{-1}\Bigg).
\end{aligned}
\tag{16}
$$

Equation (16) is the required dispersion relation studying the effects of FLR and Hall currents on thermal instability of a compressible plasma. In the absence of Hall currents ($M\to 0$), (16) reduces to the dispersion relation (Sharma *et al* 1983).

4. The stationary convection

When the instability sets in as stationary convection, the marginal state will be characterized by $\sigma=0$. Putting $\sigma=0$, the dispersion relation (16) reduces to

$$R_1=\left(\frac{G}{G-1}\right)\left(\frac{(1+x)^3}{x}+[Q_1\cos^2\theta(1+x)\{(1+x)^2+Q_1x\cos^2\theta\}\right.$$
$$+M^{1/2}N^{1/2}Q_1x\cos^4\theta(1+x)\{(1+x)+(1+x-3x\cos^2\theta)\}$$
$$+N\cos^2\theta(1+x)(1+x-3x\cos^2\theta)^2(1+x+Mx\cos^2\theta)]$$
$$\left.\times[(1+x)^2+Q_1x\cos^2\theta+Mx(1+x)\cos^2\theta]^{-1}\right), \tag{17}$$

which expresses the modified Rayleigh number R_1 as a function of the dimensionless wave number x and the parameters Q_1, M, N and G.

Let $\bar{R}_c$ and R_c denote respectively the critical Rayleigh numbers in the presence and in the absence of compressibility. For fixed values of Q_1, M and N, let the non-dimensional number G accounting for the compressibility effects be also kept fixed, then we find that

$$\bar{R}_c=[G/(G-1)]R_c. \tag{18}$$

The effect of compressibility is thus to postpone the onset of thermal instability. Hence compressibility has a stabilizing effect. $G>1$ is relevant here. The cases $G<1$ and $G=1$ correspond to negative and infinite values of critical Rayleigh numbers in the presence of compressibility which are not relevant in the present study. From (17), it follows that

$$\frac{\mathrm{d}R_1}{\mathrm{d}N}=\left(\frac{G}{G-1}\right)\left[(1+x)(1+x-3x\cos^2\theta)^2(1+x+Mx\cos^2\theta)\cos^2\theta\right.$$
$$\left.+\frac{1}{2}\left(\frac{M}{N}\right)^{1/2}Q_1x(1+x)\cos^4\theta\{(1+x)+(1+x-3x\cos^2\theta)\}\right]$$
$$\times[(1+x)^2+Q_1x\cos^2\theta+Mx\cos^2\theta(1+x)]^{-1}, \tag{19}$$

which is positive if $2(1+x)>3x\cos^2\theta$ i.e. the wave number range satisfying

$$\cos\theta<\left\{\frac{2(1+x)}{3x}\right\}^{1/2}.$$

This shows that FLR has a stabilizing effect for the above wave-number range. In the absence of Hall currents ($M\to 0$), FLR always has a stabilizing effect. But in the presence of FLR and Hall effects on thermal instability, the FLR effect may be both stabilizing as well as destabilizing but completely stabilizes the above wave-number range.

Equation (17) also yields

$$\frac{\mathrm{d}R_1}{\mathrm{d}M}=\Bigg[Q_1x(1+x)\cos^4\theta\{(1+x)^2+Q_1x\cos^2\theta\}$$

$$\times\left\{\frac{1}{2}\left(\frac{N}{M}\right)^{1/2}(1+x-3x\cos^2\theta)+(1+x)\frac{(N^{1/2}-2M^{1/2})}{2M}\right\}$$

$$+N^{1/2}Q_1x^2(1+x)\cos^6\theta\Bigg\{N^{1/2}(1+x-3x\cos^2\theta)^2$$

$$-\frac{M^{1/2}}{2}(1+x)(2\,\overline{1+x}-3x\cos^2\theta)\Bigg\}\Bigg]$$

$$\times[(1+x)^2+Q_1x\cos^2\theta+Mx\cos^2\theta(1+x)^2]^{-2}, \tag{20}$$

which is positive if $N>4M$ and $\cos\theta>\{[2(1+x)]/3x\}^{1/2}$ i.e. if

$$\frac{\nu_0}{\nu}>2\frac{cH}{4\pi Ne\eta}\quad\text{and}\quad\cos\theta>\left\{\frac{2(1+x)}{3x}\right\}^{1/2}.$$

In the absence of FLR, (20) yields that $\mathrm{d}R_1/\mathrm{d}M$ is always negative thus indicating the destabilizing effect of Hall currents. In the presence of FLR and Hall effects, the Hall currents may have both destabilizing as well as stabilizing effects and there is competition between the destabilizing role of Hall currents and stabilizing role of FLR but completely stabilizes the above wave-number range if $\nu_0/\nu>2[cH/(4\pi Ne\eta)]$.

5. Stability of the system and oscillatory modes

Multiplying (8) by W^*, the complex conjugate of W, integrating over the range of z, and making use of equations (9) to (12), we obtain

$$I_1+\sigma I_2+\frac{C_p\alpha\kappa a^2}{\nu(1-G)}(I_3+p_2\sigma^*I_4)+\frac{\eta}{4\pi\rho_m\nu}(I_5+p_2\sigma^*I_6)$$

$$+\frac{\eta d^2}{4\pi\rho_m\nu}(I_7+p_2\sigma I_8)+d^2(I_9+\sigma^*I_{10})=0, \tag{21}$$

where

$$I_1=\int_0^1(|D^2W|^2+2a^2|DW|^2+a^4|W|^2)\,\mathrm{d}z, \tag{22a}$$

$$I_2=\int_0^1(|DW|^2+a^2|W|^2)\,\mathrm{d}z, \tag{22b}$$

$$I_3=\int_0^1(|D\Theta|^2+a^2|\Theta|^2)\mathrm{d}z, \tag{22c}$$

$$I_4=\int_0^1(|\Theta|^2)\mathrm{d}z, \tag{22d}$$

$$I_5=\int_0^1 (|D^2K|^2+2a^2|DK|^2+a^4|K|^2)\mathrm{d}z, \tag{22e}$$

$$I_6=\int_0^1 (|DK|^2+a^2|K|^2)\,\mathrm{d}z, \tag{22f}$$

$$I_7=\int_0^1 (|DX|^2+a^2|X|^2)\,\mathrm{d}z, \tag{22g}$$

$$I_8=\int_0^1 |X|^2\mathrm{d}z, \tag{22h}$$

$$I_9=\int_0^1 (|DZ|^2+a^2|Z|^2)\,\mathrm{d}z, \tag{22i}$$

$$I_{10}=\int_0^1 |Z|^2\mathrm{d}z, \tag{22j}$$

which are all positive definite. The real and imaginary parts of (21) give

$$\sigma_r\left[I_2+\frac{C_p\alpha\kappa a^2}{\nu(1-G)}p_1I_4+\frac{\eta}{4\pi\rho_m\nu}p_2(I_6+d^2I_8)+d^2I_{10}\right]$$
$$=-\left[I_1+\frac{C_p\alpha\kappa a^2}{\nu(1-G)}I_3+\frac{\eta}{4\pi\rho_m\nu}(I_5+d^2I_7)+d^2I_9\right], \tag{23}$$

and

$$\sigma_i\left[I_2-\frac{C_p\alpha\kappa a^2}{\nu(1-G)}p_1I_4-\frac{\eta}{4\pi\rho_m\nu}p_2(I_6-d^2I_8)-d^2I_{10}\right]=0. \tag{24}$$

It follows from (23) that σ_r is negative if $G<1$. The system is therefore stable for $G<1$. It is evident from (24) that σ_i may be zero or non-zero. Thus the modes may be oscillatory or non-oscillatory. The oscillatory modes are introduced due to the presence of magnetic field (and hence the presence of Hall currents and FLR effects). In the absence of a magnetic field, the oscillatory modes are not allowed for $G>1$ but in the presence of magnetic field, Hall currents and FLR effects, oscillatory modes come into play.

6. Conclusions

The effects of compressibility, FLR and Hall currents are considered on the thermal instability of a plasma in the presence of a uniform horizontal magnetic field. When the instability sets in as stationary convection, the compressibility is found to have a stabilizing effect. The FLR effect may be both stabilizing as well as destabilizing in the presence of Hall currents but completely stabilizes certain wave-number range. The Hall currents also have the dual effect and there is competition between the stabilizing role of FLR and the destabilizing role of Hall currents but completely stabilizes a wave-number band if the non-dimensional number accounting for FLR exceeds four times the non-dimensional number accounting for Hall currents. For $(C_p\beta/g)<1$, with C_p, g

and β denoting the specific heat at constant pressure, the acceleration due to gravity and the uniform temperature gradient, respectively, the system is shown to be stable. The magnetic field, finite Larmor radius and the Hall currents introduce oscillatory modes in thermal instability of compressible plasma, which were completely missing for $(C_p\beta/g) > 1$ in their absence.

It has been shown by Sato (1961) and Tani (1962) that inclusion of Hall currents gives rise to a cross-flow i.e. a flow at right angles to the primary flow in a channel in the presence of a transverse magnetic field. Tani (1962) found that Hall effect produces a cross-flow of double-swirl pattern in incompressible flow through a straight channel with arbitrary cross-section. This breakdown of the primary flow and formation of a secondary flow may be attributed to the inherent instability of the primary flow in the presence of Hall current. Our stability analysis lends support to this finding. We found that the presence of Hall current induces a vertical component of vorticity and this may well be the reason for the destabilizing effect. The stabilizing role of FLR has been depicted by Rosenbluth *et al* (1962), Roberts and Taylor (1962) and Vandakurov (1964). In the simultaneous presence of FLR and Hall effects on thermal instability of a compressible plasma, the FLR and Hall currents have stabilizing as well as destabilizing effects as there is competition between the stabilizing role of FLR and the destabilizing role of Hall currents.

References

Chandrasekhar S 1961 *Hydrodynamic and hydromagnetic stability* (Oxford: Clarendon Press)
Gupta A S 1967 *Rev. Roum. Math. Pures. Appl.* **12** 665
Jukes J D 1964 *Phys. Fluids* **7** 52
Roberts K V and Taylor J B 1962 *Phys. Rev. Lett.* **8** 197
Rosenbluth M N, Krall N and Rostoker N 1962 *Nucl. Fusion Suppl.* **1** 143
Sato H 1961 *J. Phys. Soc. Jpn* **16** 1427
Sharma R C 1976 *Z. Naturforsch.* **A31** 1011
Sharma R C 1977 *J. Math. Anal. Appl.* **60** 227
Sharma R C and Prakash K 1975 *Z. Naturforsch.* **A30** 461
Sharma R C and Sharma K C 1978 *Pramana – J. Phys.* **10** 267
Sharma R C, Nyland E and Thakur K P 1983 *Physica* **C122** 341
Sherman A and Sutton G W 1962 *Magnetohydrodynamics.* (Evanston, Illinois: Northwestern Univ. Press)
Sonnerup B 1961 *J. Aerosp. Sci.* **28** 612
Spiegel E A 1965 *Astrophys. J.* **141** 1068
Spiegel E A and Veronis G 1960 *Astrophys. J.* **131** 442
Tani I 1962 *J. Aerosp. Sci.* **29** 297
Uberoi C and Devanathan C 1963 *Proc. Summer Seminar in Magnetohydrodynamics* (Bangalore: Indian Institute of Science) p. 296.
Vandakurov Ju V 1964 *Prik. Mat. Mech.* **1** **28** 69