Pramāṇa - J. Phys., Vol. 29, No. 1, July 1987, pp. 1–19. © Printed in India.

$N=1$ supergravity theories coupled to matter and duality transformations

G M STAEBLER and K T MAHANTHAPPA*

Physics Department, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA

*Department of Physics, Box 390, University of Colorado, Boulder, CO 80309, USA

MS received 5 March 1987

Abstract. The most general action for chiral and complex linear superfields coupled to the $N=1$ old minimal supergravity is given. Scalar potentials for pure complex linear and mixed cases are found. A condition for the breakdown of the duality transformation, which transforms a theory with complex linear superfields to one with chiral scalar superfields, is obtained. When this condition is satisfied, the potentials and couplings cannot be transformed, in general, into a Kähler form; examples are given. Some aspects of vanishing cosmological constant are considered in this context.

Keywords. Supergravity; duality transformation; Kähler potential.

PACS Nos 04·65; 11·30; 12·25

1. Introduction

Recently we have constructed a theory in which a matter chiral scalar superfield (CSS) is coupled to $N=1$ supergravity with a reducible set of three compensators—a chiral scalar superfield and two complex linear superfields (CLSs) whose components are non-propagating auxiliary fields (Mahanthappa and Staebler 1984, 1985). It is reducible because one of the CLSs, being a Lagrange multiplier, determines the other CLS. Such a reducible set of multiple compensators have been introduced before (Aulakh *et al* 1983). In this theory the supersymmetry is broken and the couplings and potentials turn out to be non-Kähler. This is in contradiction with the duality transformation (DT) (Lindstrom and Roček 1983; Deo and Gates Jr 1985; Gates Jr *et al* 1983) which transforms a theory with CSSs and CLSs into one with only CSSs and hence implies that couplings and potentials have to be Kähler (Bagger and Witten 1982; Cremmer *et al* 1979, 1983). In this paper we resolve this problem. We construct a general action for CSSs and CLSs coupled to the $N=1$ old minimal supergravity; the CSSs and CLSs represent physical $(0^+, 0^-, \frac{1}{2})$ matter superfields and/or Lagrange multiplier superfields made out of non-propagating components. It is to be noted that CSSs and CLSs are two of the three off-shell inequivalent descriptions of $(0^+, 0^-, \frac{1}{2})$ supermultiplet which can couple to Yang-Mills superfields; the third one is the Lagrange multiplier multiplet (Howe *et al* 1983), which we do not consider here. In this framework we derive a necessary and sufficient condition for DT to fail. Our previously constructed (Mahanthappa and Staebler 1985) theory satisfies this condition. We give other examples which satisfy this condition. The couplings and potentials obtained in these models after the elimination of auxiliary fields can be non-Kähler. We also consider the conditions needed for the cosmological constant to be zero.

It is apparent from our work that the canonically accepted view, that matter couplings to supergravity are Kähler in nature is not, in general, true. Although it is true that CSSs have a Kähler type coupling to the old minimal supergravity, CLSs, which cannot be transformed by DT to CSSs, will not necessarily have Kähler type couplings (Mahanthappa and Staebler 1985). It is also known that CSSs coupled to other off-shell representations of $N=1$ supergravity, that involve linear superfields, do not need to have Kähler kinetic terms for the scalar fields (Lang *et al* 1985). However, in all the previous examples with non-Kähler couplings supersymmetry was spontaneously broken on-shell, either by an auxiliary CLSs (Mahanthappa and Staebler 1985), or by the presence of a cosmological term (chiral density) in a supergravity which cannot support a deSitter geometry (Gates Jr *et al* 1983). In the present work we find that supersymmetry does *not* need to be broken on-shell in order to have a breakdown of DT. Since CLSs have kinetic (and other) terms which are not in Kähler form (Deo and Gates Jr 1985), before being transformed by DT to Kähler ones, it is expected that when DT fails the kinetic terms can be, in general, non-Kähler equivalent as well. We do not explicitly check the kinetic terms here, but an example where this happens can be found in Mahanthappa and Staebler (1985). An example in which the DT fails, but the Kähler geometry remains, is given in this paper.

The paper is organized as follows. In § 2 the most general coupling of both CLSs and CSSs to old minimal supergravity is constructed. Section 3 is devoted to finding the scalar potential for this general case and for the case where only CLSs are present. The scalar auxiliary field equations are also given with all derivatives and non-scalar fields set to zero. Section 4 introduces DTs, and derives the necessary and sufficient conditions for the breakdown (non-invertability) of DTs. Section 5 treats the mixed CSS and linear CLS case, and finds the condition for which DT breaks down in this case. Section 6 determines the necessary and sufficient conditions for a vanishing cosmological constant for the potentials of § 2. It is shown how the conditions for the breakdown of DT are compatible with a zero cosmological constant. Section 7 gives three examples of models which cannot be transformed by DT. The final section is the conclusion and discussion of the results. Some of the results of this paper have been previously reported in Mahanthappa and Staebler (1987).

2. General coupling of chiral and complex linear super-fields to supergravity

In this section we will derive the most general coupling of CLSs and CSSs to old minimal supergravity (Stelle and West 1978; Ferrara and Van Nieuwenhuizen 1978). The superfield formulation and conventions of Siegel and Gates Jr (1979) will be used throughout.

The $(0^+, 0^-, \frac{1}{2})$ supermultiplet can be represented off-shell by either a chiral superfield S, or a linear one Σ. They satisfy the covariant constraints:

$$\bar{\nabla}_{\dot{\alpha}} S = 0, \tag{1}$$

$$(\bar{\nabla}_{\dot{\alpha}} \bar{\nabla}^{\dot{\alpha}} + R)\Sigma = 0. \tag{2}$$

All $N=1$ supergravities (in $3+1$ dimensions) have the conformal superfield prepotential

$$e^{2u}, u = uMi\partial_M = u^m i\partial_m + u^\alpha i\partial_\alpha + u_{\dot\alpha} i\bar\partial^{\dot\alpha}. \tag{3}$$

The prepotential has the graviton, gravitino and a real vector auxiliary field as components. The old minimal supergravity has in addition a chiral density compensator ϕ as an auxiliary superfield (with only an auxiliary complex scalar field):

$$\bar\nabla_{\dot\alpha}\phi = 0. \tag{4}$$

It has a density type transformation rule under superspace coordinate transformations with parameter $\Lambda = \Lambda^M i\partial_M$:

$$\delta\phi = [i\Lambda, \phi] + \tfrac{1}{3}(\partial_\alpha\Lambda^\alpha - \partial_m\Lambda^m)\phi. \tag{5}$$

The superfield Lagrangian for old minimal supergravity is just the inverse of the superdeterminant of the super-vielbien (E^{-1}):

$$E^{-1}(\phi, u) = \phi\tilde{\bar\phi}(1\cdot e^{-2\bar u})^{1/3}\hat{E}^{1/3}, \tag{6}$$

where $\hat{E}$ and $(1\cdot e^{-2\bar u})$ are defined by Siegel and Gates Jr (1979) and they only depend on u. (Note: $\tilde{\bar\phi} = e^{-2u}\bar\phi$.)

We will also make use of the so-called superscale compensator Ψ, which for the old minimal theory is given by

$$\Psi(\phi, u) = \phi^{-1}\tilde{\bar\phi}^{1/2}(1\cdot e^{-2\bar u})\hat{E}^{-1/6}. \tag{7}$$

The non-minimal theory (Siegel and Gates 1979; Breitenlohner 1977) does not have a chiral density compensator, but rather has a linear density compensator γ as an auxiliary superfield. The linear compensator satisfies

$$\bar\partial^2\gamma = 0, \tag{8a}$$

$$\delta\gamma = [i\Lambda, \gamma] + (\partial^{\dot\alpha}\Lambda_{\dot\alpha})\gamma + \frac{n+1}{3n+1}(\partial_\alpha\Lambda^\alpha - \partial_m\Lambda^m)\gamma. \tag{8b}$$

The superfield Lagrangian density is given by

$$E^{-1}(\gamma, u) = (\gamma\tilde{\bar\gamma})^{\frac{1}{2}(3n+1)}(1\cdot e^{-2\bar u})^{\frac{1}{2}(n+1)}\hat{E}^n. \tag{9}$$

Now let us consider the coupling of CSS and CLS matter fields to old minimal supergravity (ϕ, u).

Let $\{S_a\}$ and $\{\Sigma_x\}$ be some set of CSSs and CLSs superfields respectively. We will show that the most general coupling of these fields to old minimal supergravity is

$$\begin{aligned} A_{\min} = &\int d^4x d^2\theta d^2\bar\theta[-3E^{-1}(\phi, u)K(\Sigma_x, \tilde{\bar\Sigma}_{\bar x}, S_a, \tilde{\bar S}_{\bar a})] \\ &-[\int d^4x d^2\theta\lambda\phi^3 + \text{h.c.}], \end{aligned} \tag{10}$$

where K is an arbitrary self-conjugate $(K = \tilde{\bar K})$ function, and λ is a constant parameter introduced just for book-keeping ($\lambda = 0$ removes the cosmological ϕ^3 term). We emphasize that CSSs and CLSs in (10) represent physical $(0^+, 0^-, \frac{1}{2})$ matter fields *and/or* Lagrange multiplier superfields made out of nonpropagating components like in Mahanthappa and Staebler 1985.

To see that (10) is the most general action let us consider what types of couplings are possible. If we had only chiral fields ($\Sigma_x = 0$), then (10) reduces to the action of Cremmer *et al* (1979). This is the most general action for chiral matter. Including linear CLSs in (10) is thus the most obvious extension, but we have to show that it is the most general one. This requires some subtle properties of CLSs.

Due to the linearity constraint (2) on Σ_x we can write Σ_x in terms of the superscale compensator Ψ, given by (7), and a density type superfield γ_x:

$$\Sigma_x = \Psi^{-2}\phi^{[-3(n+1)/(3n+1)]}\gamma_x, \quad \bar{\partial}^2\gamma_x = 0. \tag{11}$$

The independent degrees of freedom in Σ_x are all in γ_x, which transforms as a density compensator (8b). The old minimal supergravity fields $\Psi(\phi, u)$ and ϕ cannot both be absorbed into γ_x by a field redefinition, since γ_x has too few components. In particular

$$\bar{\partial}^2\Psi^2 = R(x, \theta), \quad \text{but} \quad \bar{\partial}^2\gamma_x = 0. \tag{12}$$

Of course $\phi^{[-3(n+1)/(3n+1)]}$ alone could be absorbed into the definition of γ_x changing its transformation property (8b).

Since the independent part of Σ_x is in γ_x, we need to consider whether or not all of the γ_x couplings can be written in terms of Σ_x. Since γ_x acts like a linear compensator, we can couple it directly to the prepotential as in (9). The quantity

$$E_{xy}^{-1} \equiv (\gamma_x\tilde{\gamma}_y)^{\frac{1}{4}(3n+1)}(1\cdot e^{-2\bar{u}})^{\frac{1}{4}(n+1)}\hat{E}^n, \tag{13}$$

is a Lagrangian density which involves γ_x and not Σ_x. However, it is easily verified using (11) and (7) that (13) is equivalent to

$$E_{xy}^{-1} = (\Sigma_x\tilde{\Sigma}_y)^{\frac{1}{4}(3n+1)}E^{-1}(\phi, u), \tag{14}$$

where $E^{-1}(\phi, u)$ is (6). This is just a special form of

$$E^{-1}(\phi, u)K(\Sigma_x, \tilde{\Sigma}_{\bar{x}}, S_a, \tilde{S}_{\bar{a}}).$$

Another Lagrangian density is

$$\phi^{[6n/(3n+1)]}\gamma_x + \text{h.c.}, \tag{15}$$

but this is equivalent to

$$E^{-1}(\phi, u)\Sigma_x + \text{h.c.}, \tag{16}$$

as can be seen from (11) and (6). Again this is in the form $E^{-1}K$.

Finally γ_x can be used to make chiral densities just as in the non-minimal supergravity (Siegel and Gates Jr 1979; Galperin *et al* 1983). For example

$$\phi_x^3 \equiv \frac{\bar{\partial}^2 E^{-1}(\gamma_x, u)}{\bar{\partial}^2\Psi^2(\phi, u)}, \tag{17}$$

is a chiral superfield with a density transformation rule like ϕ^3 (see (5)). In (17) $E^{-1}(\gamma_x, u)$ and $\Psi^2(\phi, u)$ are given by (9) and (7) respectively. These chiral densities are singular in general, but they can, in principle, be included in a superpotential type term $g(S_x)$ in the action:

$$\int d^4x d^2\theta \lambda \phi^3 g(S_x), \quad S_x \equiv \phi_x \phi^{-1}. \tag{18}$$

The arbitrary function g can be absorbed into ϕ^3 by a Weyl transformation (field redefinition), thus transforming (18) to the form in (10). This Weyl rescaling must be compensated by a redefinition of K:

$$K(\Sigma_x, \tilde{\Sigma}_{\bar{x}}, S_a, \tilde{S}_{\bar{a}}) \to g^{1/3} \tilde{g}^{1/3} K(g^{1/3[(3n-1)/(3n+1)]} \tilde{g}^{-1/3} \Sigma_x, \tilde{g}^{1/3[(3n-1)/(3n+1)]} \times g^{-1/3} \tilde{\Sigma}_{\bar{x}}, S_a, \tilde{S}_{\bar{a}}), \tag{19}$$

in order to leave (10) invariant. The transformation (19) is similar to the Kähler transformation (Lindstrom and Roček 1983) of the pure chiral matter action.

The transformation (19) is only possible when g is non-singular. The redefinition (19) appears to require that K be a function of not just Σ_x and S_a but of densities such as (17) as well. We can fix this by using a Lagrange multiplier construction. For example, we could introduce the density (17) through the Lagrange multiplier action

$$\int d^4x d^2\theta d^2\bar{\theta} [E^{-1}(\phi, u) S_0 ((\Sigma_x \tilde{\Sigma}_{\bar{x}})^{\frac{1}{2}(3n+1)} - S_x^3)], \tag{20}$$

where S_0 and S_x are CSSs. The Lagrange multiplier S_0 produces (17), where $S_x = \phi_x \phi^{-1}$. We can add (20) to (10), which just redefines K. With this Lagrange multiplier formulation, adding a superpotential term like (18) does not amount to adding any new type of coupling involving densities like (17), since S_x is just an ordinary chiral superfield. It is always possible to use Lagrange multipliers to construct the known chiral densities from linear compensators.

From the above discussion, it is clear that all of the couplings involving CLSs and CSSs can be put into the form of the action (10), for the old minimal supergravity.

The action (10) has an additional invariance property. It is invariant under the transformation

$$K \to K + \Sigma^x f_x(S_a) + \tilde{\Sigma}^{\bar{x}} \tilde{f}_{\bar{x}}(\tilde{S}_{\bar{a}}), \tag{21}$$

where $\{f_x\}$ is a set of arbitrary chiral functions. Invariance of (10) follows from the linearity constraint (2) on Σ_x.

3. Scalar potentials and auxiliary field equations

Now let us turn to the task of determining the scalar potential for the general action (10).

Since we are only interested in the scalar potential, we can set all non-scalar components of the superfields to zero. This also sets all derivatives to zero. The scalar potential obtained in this way is completely general, but the auxiliary field equations will only be valid when the non-scalar component fields are absent.

With all but scalar fields vanishing we have:

$$
\begin{aligned}
&(1\cdot e^{-2\bar{u}})=1, \quad \hat{E}=1,\\
&\psi^2=\phi^{-2}\bar{\phi},\\
&S^a=C^a+\theta^2F^a,\\
&E^{-1}=K(C,\bar{C})^{-1}(1+\theta^2B)(1+\bar{\theta}^2\bar{B}),\\
&\phi=K^{-1/2}(C,\bar{C})(1+\theta^2B),\\
&\Sigma^x=C^x+\theta^2G^x-\bar{\theta}^2C^x\bar{B}-\theta^2\bar{\theta}^2\bar{B}G^x.
\end{aligned} \tag{22}
$$

In writing (22) we have chosen the special gauge $\phi|_{\theta=\bar{\theta}=0}=K^{-1/2}(C,\bar{C})$. This choice is needed in order to remove the $K(C,\bar{C})$ factor that would otherwise appear in front of the Einstein-Hilbert part of the action for the graviton. This gauge is equivalent to performing Weyl rescalings of the tetrad (Cremmer *et al* 1983). The CLS components have been redefined to put them in the simplest form possible.

The scalar potential is defined by

$$
V=-[\int d^2\theta d^2\bar{\theta}[-3E^{-1}K]-\int d^2\theta\lambda\phi^3-\int d^2\bar{\theta}\lambda\bar{\phi}^3]. \tag{23}
$$

In terms of components it is

$$
\begin{aligned}
V= 3K^{-1}[&B\bar{B}\,P+B(K_{\bar{a}}\bar{F}^{\bar{a}}+\lambda K^{-1/2}-K_{\bar{x}\bar{y}}\bar{C}^{\bar{y}}\bar{G}^{\bar{x}}-K_{\bar{a}\bar{x}}\bar{F}^{\bar{a}}\bar{C}^{\bar{x}})\\
&+\bar{B}(K_aF^a+\lambda K^{-1/2}-K_{xy}G^xC^y-K_{ax}F^aC^x)\\
&+K_{a\bar{b}}F^a\bar{F}^{\bar{b}}+K_{x\bar{y}}G^x\bar{G}^{\bar{y}}+K_{a\bar{x}}F^a\bar{G}^{\bar{x}}+K_{\bar{a}x}\bar{F}^{\bar{a}}G^x],
\end{aligned} \tag{24a}
$$

where

$$
P\equiv K-K_xC^x-K_{\bar{x}}\bar{C}^{\bar{x}}+K_{x\bar{y}}C^x\bar{C}^{\bar{y}}, \tag{24b}
$$

and

$$
K_x\equiv\partial K(C,\bar{C})/\partial C_x, \quad K_a=\partial K/\partial C_a, \quad \text{etc.} \tag{24c}
$$

A sum over repeated indices is taken.

The fields B, G^x and F^a are all auxiliary, so they may be eliminated by their Euler-Lagrange equations.

The auxiliary field equations are:

$$
BP+K_aF^a+\lambda K^{-1/2}-K_{xy}G^xC^y-K_{ax}F^aC^x=0, \tag{25}
$$

$$
K_{\bar{a}b}F^b+K_{\bar{a}x}G^x+BK_{\bar{a}}-BK_{\bar{a}\bar{x}}\bar{C}^{\bar{x}}=0, \tag{26}
$$

$$
K_{\bar{x}y}G^y+K_{\bar{x}a}F^a-BK_{\bar{x}\bar{y}}\bar{C}^{\bar{y}}=0. \tag{27}
$$

These equations can be solved simultaneously even for the general case, but first let us consider two special cases.

The first special case is the pure CSS model ($\Sigma_x=0$). For this case the auxiliary field equations reduce to

$$
BK+K_aF^a+\lambda K^{-1/2}=0, \tag{28}
$$

$$K_{\bar{a}b}F^b+BK_{\bar{a}}=0. \tag{29}$$

If we change functions from K to G by

$$K=e^{G/3}, \tag{30}$$

and solve (28) and (29), we obtain the result of Cremmer *et al* 1979 (when $\lambda^2=1$).

$$V=-\lambda^2\bar{e}^{G}(3+G_aG^{-1a\bar{b}}G_{\bar{b}}). \tag{31}$$

The other special case is the purely CLS model. Setting $S_a=0$ gives the auxiliary field equations

$$BP+\lambda K^{-1/2}-K_{xy}G^xC^y=0, \tag{32}$$

$$K_{\bar{x}y}G^y-BK_{\bar{x}\bar{y}}\bar{C}^{\bar{y}}=0. \tag{33}$$

We can eliminate B and G_y using (32) and (33) provided

$$\det(M_{x\bar{y}})\neq 0,\quad M_{x\bar{y}}\equiv K_{x\bar{y}}-P^{-1}C^yK_{xy}K_{\bar{y}\bar{x}}\bar{C}^{\bar{x}}, \tag{34a}$$

$$P\neq 0. \tag{34b}$$

The potential for this case is

$$V=-3P^{-2}K^{-2}\lambda^2[P+C^xK_{xy}M^{-1y\bar{x}}K_{\bar{x}\bar{y}}\bar{C}^y],\quad P\neq 0. \tag{35}$$

If P is identically zero K must be of the form

$$K=C_xf^x(\bar{C}_{\bar{x}})+\bar{C}_{\bar{x}}\bar{f}^{\bar{x}}(C_x), \tag{36}$$

for some arbitrary holomorphic function $f^x[(\partial/\partial C_x)f^y=0]$. If $K_{x\bar{y}}$ is invertible we can solve (32) and (33). The potential in this case is

$$V=3\lambda^2K^{-2}[C_xK^{xy}K_{x\bar{y}}^{-1}K^{\bar{x}\bar{y}}\bar{C}_{\bar{y}}]^{-1},\quad P=0. \tag{37}$$

Now let us eliminate the auxiliary fields in the general case. If $P\neq 0$ we can solve for B in (25) and substitute the result into (26) and (27). The resulting equations have the form of a matrix equation.

$$M_{\bar{i}j}F^j=\lambda K^{-1/2}P^{-1}\bar{J}_{\bar{i}}, \tag{38a}$$

where

$$F^j\equiv\begin{pmatrix}F^a\\G^x\end{pmatrix},\quad \bar{J}_{\bar{i}}\equiv\begin{pmatrix}K_{\bar{a}}-K_{\bar{a}\bar{x}}\bar{C}^{\bar{x}}\\-K_{\bar{x}\bar{y}}\bar{C}^{\bar{y}}\end{pmatrix}, \tag{38b}$$

and

$$C_j\equiv\begin{pmatrix}C^a\\C^x\end{pmatrix},\quad K_i\equiv\partial K/\partial C_i, \tag{38c}$$

$$M_{\bar{i}j}=K_{\bar{i}j}-P^{-1}\bar{J}_{\bar{i}}J_j. \tag{38d}$$

Provided $\det(M_{i\bar{j}})\neq 0$, (38) can be inverted, and the potential becomes $(P\neq 0)$

$$V=-3\lambda^2P^{-2}K^{-2}(P+J^iM_{i\bar{j}}^{-1}J)^{\bar{j}}. \tag{39}$$

Note that although the form of this potential (and (35)) is superficially similar to the pure CSS case (31), a simple change of functions like (30) cannot reduce $M_{i\bar{j}}$ to $G_{i\bar{j}}$, as it happened in obtaining (31). This makes it difficult for finding an easily invertible form for $M_{i\bar{j}}$.

4. Duality transformation between linear and chiral superfields: pure linear case

The scalar potentials (35) and (39) found in the last section can, in some cases, be transformed into the form of the pure CSS potential (31). This is accomplished by using so-called duality transformations (DT) (Lindstrom and Roček 1983) between CLSs and CSSs. In this section we will derive necessary and sufficient conditions for DTs to fail. For these special cases, the coupling of supergravity to CLSs will yield *new* on-shell couplings to the scalar $(0^+, 0^-, \frac{1}{2})$ multiplet, which cannot be obtained from the purely CSS action.

First we will specialize to the purely CLS case $(S_a=0)$. The mixed CSS plus CLS case is treated in § 5.

The DT between CLSs and CSSs is accomplished by introducing a set of CSSs $\{\sigma_x\}$ and relaxing the linearity constraint on Σ_x, so that they are now general complex superfields.

The action (10) is replaced by

$$A_D=\int d^4x d^2\theta d^2\bar{\theta}[-3E^{-1}(\phi, u)(K(\Sigma_x, \bar{\Sigma}_{\bar{x}})-\sigma_x\Sigma^x-\bar{\sigma}_{\bar{x}}\bar{\Sigma}^{\bar{x}})]$$
$$-[\int d^4x d^2\theta\lambda\phi^3+\text{h.c.}]. \tag{40}$$

Variation of (40) with respect to σ_x reproduces the linearity constraint (2) on Σ_x, the use of which in (40) yields (10) (with $S_a=0$). The form of A_D is *uniquely* determined by the requirement that it reduces to the original action with Σ_x constrained to be a CLS, when σ_x and $\bar{\sigma}_{\bar{x}}$ are eliminated. If we vary the unconstrained Σ_x in A_D we get the superfield equation

$$K_x(\Sigma_x, \bar{\Sigma}_{\bar{x}})=\sigma_x. \tag{41}$$

This equation, combined with the linearity constraint from σ_x variation, imposes the field equations on Σ_x that one would get from the original action (10). It also gives new equations which determine the components of σ_x in terms of those of Σ_x. These are the DT equations. If we can invert these equations, we obtain an action for the CSSs only:

$$\hat{A}=\int d^4x d^2\theta d^2\bar{\theta}[-3E^{-1}(\phi, u)\hat{K}(\sigma_x, \bar{\sigma}_{\bar{x}})]-[\int d^4x d^2\theta\lambda\phi^3+\text{h.c.}], \tag{42}$$

where $\hat{K}$ is the Legendre transform of K. This dual action will then yield identical equations of motion for the physical on-shell fields as the original action (10).

However, the duality equations cannot always be inverted as we shall see. For our purposes it is sufficient just to consider DT for the scalar components only. The disadvantage of this approach is that we do not then know the kinetic terms for the on-shell scalars, as these depend in an important way on the equations of motion for the auxiliary vector component of the linear superfield. It is known however that the pure chiral theory has Kähler type kinetic terms (Lindstrom and Roček 1983). The linear multiplets on the other hand have kinetic terms which appear non-Kähler (Deo and Gates Jr 1985). If DT, which would put the kinetic terms in Kähler form, fail, then it becomes possible that not all of the kinetic terms could be put in a Kähler form. Thus, it is only the cases where linear and chiral couplings, are inequivalent, that can have couplings which are non-Kähler. An explicit example where this happens is contained in Mahanthappa and Staebler (1985).

The duality transformations for the scalar components of σ_x:

$$\sigma_x = D_x + \theta^2 E_x, \tag{43}$$

can be deduced from (41). They are

$$K_x(C_x, \bar{C}_{\bar{x}}) = D_x, \tag{44}$$

$$K_{xy} G^y - K_{x\bar{y}} \bar{C}^{\bar{y}} B = E_x. \tag{45}$$

The auxiliary field equation (33) can be combined in matrix form with (45) to read

$$K_{rs} G^s = E_r, \quad r = (x, \bar{x}), \quad s = (y, \bar{y}), \tag{46a}$$

where

$$G_s \equiv \begin{pmatrix} G_x \\ -C_{\bar{x}} B \end{pmatrix}, \quad E_s \equiv \begin{pmatrix} E_x \\ 0 \end{pmatrix}, \quad C_r \equiv \begin{pmatrix} C_x \\ \bar{C}_{\bar{x}} \end{pmatrix}, \tag{46b}$$

$$K_{rs} = \partial^2 K / \partial C_r \partial C_s. \tag{46c}$$

We can invert (46a) provided

$$\det(K_{rs}) \neq 0. \tag{47}$$

The Legendre transform function $\hat{K}(D_x, \bar{D}_{\bar{x}})$ is

$$\hat{K} = K - C^x D_x - \bar{C}^{\bar{x}} \bar{D}_{\bar{x}} = K - C^x K_x - \bar{C}^{\bar{x}} K_{\bar{x}}. \tag{48}$$

With the definitions

$$\hat{K}_x = \partial \hat{K} / \partial D_x, \quad D_r = \begin{pmatrix} D_x \\ \bar{D}_{\bar{x}} \end{pmatrix}, \tag{49}$$

it is easy to prove the following by partial differentiation:

$$C_r = \hat{K}_r, \tag{50}$$

$$-\mathrm{d}C_r/\mathrm{d}C_s = \hat{K}^r_t \, K^t_s = -\delta^r_s. \tag{51}$$

The last equation (51) implies that $\hat{K}_{rs} = -K_{rs}^{-1}$. Therefore, if the invertibility condition (47) is satisfied $-\hat{K}_{rs}$ is the *unique* inverse of K_{rs}. Now we can use (51) to invert (46a)

$$G_r = -\hat{K}_{rs} E^s. \tag{52}$$

In terms of the original fields (52) reads

$$G_x = -\hat{K}_{xy}E^y, \tag{53a}$$

$$-\bar{C}_{\bar{x}}B = -\hat{K}_{\bar{x}y}E^y. \tag{53b}$$

Substituting (50) for $\bar{C}_{\bar{x}}$, we may write (53b) in the form of the auxiliary field equation for E_x in the chiral superfield σ_x (compare with (29)):

$$\hat{K}_{\bar{x}y}E^y + \hat{K}_{\bar{x}}B = 0. \tag{54}$$

The above result gives us one necessary condition (47) for the duality transformation to go through.

We still have to convert the other auxiliary field equation (32). Multiplying (35) by C_x and contracting, we can replace the $C_xG_yK^{xy}$ term in (32) by one involving E_x:

$$B(P - K_{x\bar{y}}C^x\bar{C}^{\bar{y}}) + \lambda K^{-1/2} - C^xE_x = 0. \tag{55}$$

Now, if we use the definitions of P (24b) and $\hat{K}$ (48), and substitute $-\hat{K}_x$ for C_x, (55) reads

$$B\hat{K} + \lambda K^{-1/2} + \hat{K}_xE^x = 0. \tag{56}$$

This is in the correct form for a chiral theory (compare with (28)), except for the $\lambda K^{-1/2}$ term. The proper form for this term ($\lambda\hat{K}^{-1/2}$) is easily obtained by changing the choice of gauge for the chiral compensator ϕ used in (24).

In summary, the conversion of both auxiliary field equations from linear to chiral form using DTs (44) and (45) required the single condition (47). This invertibility condition is only necessary, but not sufficient, to ensure the successful completion of DT. The inclusion of the non-scalar components may give rise to additional conditions, which are necessary for the DT to go through. However,

$$\det(K_{rs}) = 0, \quad r = (x, \bar{x}), \quad s = (y, \bar{y}) \tag{57}$$

is a necessary and sufficient condition for DT to fail, when we have only linear multiplets. This condition is equally good for the globally supersymmetric nonlinear σ-model (Mahanthappa and Staebler 1986).

Discussion: (A) If (57) holds globally then at least one of the scalar component fields must be non-propagating, as an entire column of K_{rs} must vanish in some particular basis, which implies a vanishing kinetic term for one of the basis scalar fields. This field would thus act like a Lagrange multiplier field. Note that *at least one* of the superfields, Σ's and S's, in (10) must represent physical $(0, \frac{1}{2})$ fields in order to have a non-trivial matter-coupled supergravity theory. Thus, even though the total kinetic matrix for scalar fields would be semi-positive definite, the kinetic submatrix for physical fields would still be positive definite. There are three ways in which (57) can be satisfied globally. The first is when K is independent of some holomorphic linear combination of the CLSs: $\Sigma'_x = \Lambda^y_x\Sigma_y$, for some constant matrix Λ^y_x. If K is independent of say Σ'_N, then (57) is satisfied trivially, as we should not have taken derivatives with respect to Σ'_N. The second case is when (57) is satisfied identically due to the functional form of K. This can happen, for example, if K is independent of some new coordinate basis element

$\Sigma'_x = \Sigma'_x(\Sigma_y, \bar{\Sigma}_{\bar{y}})$, which does not preserve the linearity constraint (i.e. $(\bar{\nabla}^2 + R)\Sigma'_x \neq 0$). For example if $f \equiv \frac{1}{2}(\Sigma + \bar{\Sigma})$, and $g \equiv \frac{i}{2}(\Sigma - \bar{\Sigma})$, and $K = K(f)$, then (57) is satisfied non-trivially. We cannot perform a duality transformation between just f and a CSS since the dual Lagrangian $K - \frac{1}{2}(\sigma + \bar{\sigma}) f$ does not reproduce the *original* constraint on f when $(\sigma + \bar{\sigma})$ is eliminated. The unique DT between CLSs and CSSs fails due to (57) in this case. Note that a duality transformation could go through if f were a real linear superfield rather than the real part of a CLS. Another example, for which (57) is satisfied identically, is when the CLS is a Lagrange multiplier, as discussed in § 7. The third case is when $\det(K_{rs})$ is forced to vanish by the equations of motion, an example of which is given in Aulakh *et al* (1983) and § 7.

(B) We do not need to require (57) to hold globally in order to have DT fail. Even if only the classical vacuum expectation value of K_{rs} has a vanishing determinant, the linear theory will have a vacuum state which is inequivalent to the chiral theory. This is more interesting than case (A) because it does not require any scalar fields to be non-propagating. An example of this case is given in § 7. Such a case can also arise for the *global* SUSY nonlinear σ-model where all superfields represent physical $(0^+, 0^-, \frac{1}{2})$ fields (Mahanthappa and Staebler 1986). Notice that if the determinant of K_{rs} only vanishes for the vacuum of the theory, that supersymmetry will only be broken in the vacuum, if at all, as well. Thus, supersymmetry need not be broken in the process of eliminating non-propagating fields, as happened in Mahanthappa and Staebler (1985), in order to obtain theories inequivalent to the CSS theories.

5. Duality transformations continued: mixed case

We have seen that under certain conditions DT can fail in the case where only linear matter superfields are present. Now we will investigate the case where both chiral and linear matter superfields are present, and see if new conditions occur.

As we did in the purely linear case, we introduce chiral superfields $\{\sigma_x\}$, and relax the linearity constraint on $\{\Sigma_x\}$. We write an action identical to (40), where now K is a function of both linear Σ_x and chiral S_a superfields. The duality equation obtained in this way for the components of σ_x are

$$K_x = D_x, \tag{58}$$

$$K_{xy} G^y + K_{xa} F^a - K_{x\bar{y}} \bar{C}^{\bar{y}} B = E_x. \tag{59}$$

We have only performed DT for the linear superfields, but (59) involves the chiral auxiliary field F^a as well. First let us find the dual to the auxiliary field equation for B (equation (25)). Contracting (59) with C^x and using the definition of $\hat{K}$ (48) and P (24b) we obtain

$$B\hat{K} + K_a F^a + \lambda K^{-1/2} - C^x E_x = 0. \tag{60}$$

Now,

$$\hat{K}(D_x, \bar{D}_{\bar{x}}, C_a, \bar{C}_{\bar{a}}) = K(C_x, \bar{C}_{\bar{y}}, C_a, \bar{C}_{\bar{a}}) - D^x C_x - \bar{D}^{\bar{x}} \bar{C}_{\bar{x}}, \tag{61}$$

where C_x is taken as an implicit function of the new variables $(D_x, \bar{D}_{\bar{x}}, C_a, \bar{C}_{\bar{a}})$.

Differentiating (61) yields

$$\hat{K}_a = K_a, \tag{62a}$$

and

$$\hat{K}_x = -C_x. \tag{62b}$$

So (60) can be put into the form (28), after changing the gauge choice for ϕ to convert $\lambda K^{-1/2}$, as before.

Now we want to convert (26) and (27). We may combine (27) and (59), with the identity $F^a = F^a$, to get the equations for the change of variables from $(\Sigma_x, S_a) \to (\sigma_x, S_a)$ in a matrix form:

$$\begin{pmatrix} E_x \\ 0 \\ F_a \end{pmatrix} = \begin{pmatrix} K_{xy} & K_{x\bar{y}} & K_{xa} \\ K_{\bar{x}y} & K_{\bar{x}\bar{y}} & K_{\bar{x}a} \\ 0 & 0 & \delta_{ab} \end{pmatrix} \begin{pmatrix} G^y \\ -\bar{C}^{\bar{y}}B \\ F^b \end{pmatrix}. \tag{63}$$

This can be inverted provided the Jacobian determinant doesn't vanish, which reduces to the same condition as for the pure CLS case (47). The inverse transformation reads

$$\begin{pmatrix} G_y \\ -\bar{C}_{\bar{x}}B \\ F_a \end{pmatrix} = -\begin{pmatrix} \hat{K}_{xy} & \hat{K}_{x\bar{y}} & \hat{K}_{xb} \\ \hat{K}_{\bar{x}y} & \hat{K}_{\bar{x}\bar{y}} & \hat{K}_{\bar{x}a} \\ 0 & 0 & \delta_{ab} \end{pmatrix} \begin{pmatrix} E_y \\ 0 \\ F^b \end{pmatrix}. \tag{64}$$

As before $-\hat{K}_{rs}$ is the unique inverse of K_{rs}. We also need the relations

$$\mathrm{d}D_x/\mathrm{d}S^a = -K_{xy}\hat{K}^y_a - K_{x\bar{y}}\hat{K}^{\bar{y}}_a + K_{xa} = 0, \tag{65a}$$

$$\mathrm{d}D_x/\mathrm{d}\bar{S}^{\bar{a}} = -K_{xy}\hat{K}^y_{\bar{a}} - K_{x\bar{y}}\hat{K}^{\bar{y}}_{\bar{a}} + K_{x\bar{a}} = 0, \tag{65b}$$

in order to invert (63). These follows from differentiating the equation $D_x = K_x$.

Substituting (63) into (26) and (27), we find they can be put into the form we would get from the Kähler potential $\hat{K}(\sigma^x, \bar{\sigma}^{\bar{x}}, S^a, \bar{S}^{\bar{a}})$, by using (65a, b), (51) and the equation

$$\hat{K}_{\bar{a}b} = K_{\bar{a}b} - K_{\bar{a}x}\hat{K}^x_b - K_{\bar{a}\bar{x}}\hat{K}^{\bar{x}}_b, \tag{66}$$

which may be obtained by differentiating (61). Hence all of the auxiliary field equations can be converted to the pure CSS form provided the determinant condition (47) holds. No new conditions were needed, and so the DT will fail if (57) holds for either the pure CLS case or the mixed CLS and CSS case.

6. Conditions for a vanishing cosmological constant

Now that it has been shown that there are cases where the potentials (35) and (39) are inequivalent to the potentials obtained from purely chiral matter couplings, when (57) is satisfied, we will determine the general conditions for these potentials to have a vanishing cosmological constant. Of course if the duality transformations go through nothing new is obtained, so we will use the duality failure requirement, in conjunction with a vanishing cosmological constant, to explore the new potentials.

First let us consider the couplings in the case of purely CLS matter the potential (for $P \neq 0$) is given in (35). It can be written as

$$V = 3\lambda K^{-3/2} B, \quad \lambda \neq 0. \tag{67}$$

In order to have a vanishing cosmological constant, the classical vacuum expectation value (VEV) of the potential must vanish: $\langle V \rangle = 0$. This can only occur if $\langle K^{-3/2} \rangle = 0$, or $\langle B \rangle = 0$. The former condition is unrealistic as it requires $\langle K \rangle \to \infty$. We will always assume a finite VEV for K in what follows. The latter condition $\langle B \rangle = 0$, can be substituted into the equations obtained by taking VEV of (32) and (33) to get

$$\lambda \langle K^{-3/2} \rangle - \langle K_{xy} G^x C^y \rangle = 0, \tag{68}$$

$$\langle K_{\bar{x}y} G^y \rangle = 0. \tag{69}$$

Since (68) requires at least one non-vanishing VEV for G_x, (69) is equivalent to

$$\langle \det(K_{\bar{x}y}) \rangle = 0. \tag{70}$$

The other necessary condition for $\langle V \rangle = 0$ is

$$\langle K_{xy} G^x C^y \rangle \neq 0 \quad \text{if} \quad \lambda \neq 0. \tag{71}$$

The equations (69) and (71) imply that $\langle K_{rs} G^s \rangle \neq 0$ for all solutions $\langle G_s \rangle$. Thus $\langle K_{rs} \rangle$ cannot have any zero eigenvalues, and the condition (47) for DT to go through is satisfied. Since the invertability of K_{rs} is sufficient to carry out DT, at least for the vacuum field equations, we can convert (68) and (69) into the purely chiral case (when $\lambda \neq 0$), and obtain the conditions necessary for $\langle V \rangle = 0$ there:

$$\det \langle \hat{K}_{\bar{x}y} \rangle = 0, \quad \lambda \neq 0, \tag{72a}$$

$$\langle E^x \hat{K}_x \rangle \neq 0. \tag{72b}$$

It has been shown by Barbieri *et al* (1985), that (72a) and (72b) are both necessary and sufficient for $\langle V \rangle = 0$. We can verify this by substituting (71) and (69) into the auxiliary field equation (33), which would yield $\langle B \rangle = 0$.

If DT fails because (57) holds in the vacuum, then we will have new results. It was found above that $\lambda \neq 0$ required the determinant of $\langle K_{rs} \rangle$ not to vanish if $\langle V \rangle = 0$. If $\lambda = 0$ then (68) and (69) together are solved by

$$\langle K_{rs} G^s \rangle = 0. \tag{73}$$

Notice that if $\lambda = 0$ the potential is always flat ($V \equiv 0$), so the cosmological constant vanishes identically even if $B \neq 0$, thus (73) is not the most general solution since we have also taken $\langle B \rangle = 0$. If $\det \langle K_{rs} \rangle = 0$, then $\langle K_{rs} \rangle$ has at least one zero eigenvalue, so that (73) is satisfied. Note that $\langle G_r \rangle$ is undetermined in some direction, so that the scale of supersymmetry breaking is not fixed at the tree level.

Now let us examine the mixed chiral and linear superfield case. Again the potential can be written in the form (67) if $\lambda \neq 0$, so $\langle V \rangle = 0$ implies $\langle B \rangle = 0$, if $\langle K^{-3/2} \rangle \neq 0$.

Setting $B=0$ in the auxiliary field equations (25), (26) and (27), and using the definitions (38b, c), we find

$$\langle J_i F^i + \lambda K^{-1/2} \rangle = 0, \tag{74}$$

$$\langle K_{i\bar{j}} \bar{F}^{\bar{j}} \rangle = 0. \tag{75}$$

Now $\lambda \neq 0$ and (74) requires some $\langle F^i \rangle \neq 0$. So (75) is equivalent to

$$\langle \det(K_{i\bar{j}}) \rangle = 0, \quad i = (a, x). \tag{76}$$

This must hold regardless of whether or not DT can be made. The other necessary condition is

$$\langle J_i F^i \rangle \neq 0. \tag{77}$$

It is easy to verify that (76) and (77) are also sufficient conditions as they force $\langle B \rangle = 0$ and $\langle V \rangle = 0$.

Failure of DT in vacua occurs if

$$\langle K_{\bar{x}y} G^y + K_{\bar{x}\bar{y}} \bar{G}^{\bar{y}} \rangle = 0, \tag{78a}$$

$$\langle K_{xy} G^y + K_{x\bar{y}} \bar{G}^{\bar{y}} \rangle = 0, \tag{78b}$$

for some non-zero $\langle G_x \rangle$. We can rewrite (74) as

$$\langle K_a F^a - C^x K_{xa} F^a - K_{xy} C^x G^y + \lambda K^{-1/2} \rangle = 0, \tag{79}$$

and (75) as

$$\langle K_{\bar{a}a} F^a + K_{\bar{a}y} G^y \rangle = 0, \tag{80a}$$

$$\langle K_{\bar{x}a} F^a + K_{\bar{x}y} G^y \rangle = 0. \tag{80b}$$

The set of equations (78), (79) and (80) can be solved simultaneously with some $\langle G_x \rangle \neq 0$, only if at least one of the CSS auxiliary fields also has $\langle F_a \rangle \neq 0$; in this case the vacuum of the model will be inequivalent to the dual pure CSS model. Thus we do not require $\lambda = 0$ in order to have both failed DT and a zero cosmological constant for the mixed CLS + CSS case.

7. Examples of failed duality transformations

In this section three examples in which DT to chiral superfields do not go through, are given. The first example has $\det(K_{rs}) = 0$ by virtue of the auxiliary field equations, even away from the vacuum. The second has the determinant vanish only at the minimum of the scalar potential. The third example has only one CLS and it is a Lagrange multiplier.

The first example is taken from Mahanthappa and Staebler (1985). We will simplify the version given thereby leaving out the chiral superfields. The model has two linear superfields Σ_1 and Σ_2 with K given by

$$K=\Sigma_1\left(\Sigma_2+\frac{a^n}{n-1}\Sigma_2^{-(n-1)}\right)+f(\Sigma_2,\bar{\Sigma}_2)+\text{h.c.} \tag{81}$$

In (81) f is an arbitrary function which was not introduced in Mahanthappa and Staebler (1985), and (a, n) are real constant parameters. Note that Σ_1 is a Lagrange multiplier, and that both Σ_1 and Σ_2 are non-propagating auxiliary fields. The matrix elements K_{rs} are

$$K_{11}=K_{1\bar{1}}=K_{1\bar{2}}=K_{\bar{1}2}=0, \tag{82a}$$

$$K_{12}=1-a^nC_2^{-n}, \tag{82b}$$

$$K_{22}=f_{22}, \quad K_{2\bar{2}}=f_{2\bar{2}}. \tag{82c}$$

The inclusion of the fermionic terms in the auxiliary field equations will not change our results (see Mahanthappa and Staebler 1985), so we ignore them, and use (32) and (33) to get the equations for B and G_1:

$$\bar{B}\,K_{12}\,C_2=0, \tag{83a}$$

$$BP+\lambda K^{-1/2}-2K_{12}G^1C^2-K_{22}G^2C^2=0. \tag{83b}$$

Away from the vacuum $B\neq 0$ as can be seen by (83b), so (83a) is solved by either $C_2=0$ or $K_{12}=0$ globally. We cannot take $C_2=0$ if $a\neq 0$, and $n-1>0$, since then $K\to\infty$; so the only solution to (83a) is $K_{12}=0$. From (82a), it is clear that if $K_{12}=0$ then $K_{1s}=0$, so det $(K_{rs})=0$ everywhere on-shell. Thus, the on-shell theory, with auxiliary fields eliminated, must be inequivalent to a chiral superfield action, when $1-n>0$, and $a\neq 0$. It turns out, that for this particular model, supersymmetry is broken in a novel way: by the vacuum expectation value of the non-propagating auxiliary superfield Σ_2.

The second example has only one propagating linear superfield Σ. The conditions for the minimization of the potential, for a single superfield, can be derived directly from the form (24) for the potential (with $S_a=0$). Using (67), it follows that, at the minimum C_0, we must have

$$B\Big|_{C_0}=0, \quad \frac{\mathrm{d}B}{\mathrm{d}C}\Big|_{C_0}=0, \tag{84a}$$

and

$$\frac{\mathrm{d}^2V}{\mathrm{d}C^2}\Big|_{C_0}=3\lambda K^{-3/2}\frac{\mathrm{d}^2B}{\mathrm{d}C^2}\Big|_{C_0} \quad \text{etc.} \tag{84b}$$

Using these equations, and differentiating (24), we find the extremum condition $(\mathrm{d}V/\mathrm{d}C)|_{C_0}=0$ implies

$$K_{11\bar{1}}G\bar{G}=0, \tag{85}$$

and the minimum condition reads

$$(G\bar{G})^2(K_{11\bar{1}\bar{1}}{}^2-K_{111\bar{1}}K_{\bar{1}\bar{1}\bar{1}1})<0, \quad K_{111\bar{1}}>0. \tag{86}$$

We need to look for a supersymmetry breaking solution with $\langle G\rangle \neq 0$ in order to have zero cosmological constant.

The following particular model will do

$$K=\alpha_1+\alpha_2\Sigma\bar{\Sigma}+\frac{\beta}{3!}(\Sigma^3\bar{\Sigma}+\bar{\Sigma}^3\Sigma)-\frac{\gamma}{2}(\Sigma^2\bar{\Sigma}+\Sigma\bar{\Sigma}^2), \tag{87}$$

where $(\alpha_1, \alpha_2, \beta, \gamma)$ are real parameters. This function yields a minimum at $C_0=\gamma/\beta$. The DT fails if

$$K_{1\bar{1}}|_{C_0}=0 \Rightarrow \alpha_2=\gamma^2/\beta. \tag{88}$$

The other matrix elements, K_{11} and $K_{\bar{1}\bar{1}}$ vanish at C_0, without fine tuning of the parameters. In this example the cosmological constant is

$$V|_{C_0}=3\lambda^2 K^{-2}|_{C_0}\alpha_1^{-1}. \tag{89}$$

If we had set $\lambda=0$, then the potential would have vanished, but $C_0=\gamma/\beta$ would still be a solution to the auxiliary field equations, for which the DT fails (using (88)). In either case the VEV of G is not determined at the tree level, which is a general characteristic of models where DT breaks down, for the purely linear superfield case.

The third example has both a CLS and CSSs, but the CLS is a Lagrange multiplier, which imposes constraints on the manifold of the nonlinear sigma model of CSSs.

Consider the general coupling of a single Lagrange multiplier CLS to a set of CSSs:

$$K=f(S_a, \bar{S}_{\bar{a}})+\Sigma g(S_a, \bar{S}_{\bar{a}})+\bar{\Sigma}\bar{g}(S_a, \bar{S}_{\bar{a}}), \tag{90}$$

where f is a real function but g is complex in general. The constraint equation imposed by the CLS Σ is

$$\bar{\nabla}_{\dot{\alpha}} g(S_a, \bar{S}_{\bar{a}})=0. \tag{91}$$

If g is a holomorphic function of the CSSs; $g=g(S_a)$, then (91) is satisfied trivially as the Lagrange multiplier term in (90) would vanish. If g is antiholomorphic $g=g(\bar{S}_{\bar{a}})$, then the constraint (91) is equilvalent to $g(\bar{S}_{\bar{a}})=\text{constant}$. In this case the constraint could have been applied by using a CSS Lagrange multiplier ($\bar{S}_0$), since

$$\int d^4x d^2\theta d^2\bar{\theta}[E^{-1}\Sigma g(\bar{S}_{\bar{a}})]=\int d^4x d^2\bar{\theta}[\bar{\phi}^3\bar{S}_0 g(\bar{S}_{\bar{a}})], \tag{92}$$

where $\bar{S}_0 \equiv -\frac{1}{4}\nabla^2\Sigma$. We therefore only have a truly CLS Lagrange multiplier when the constraint functional is non-holomorphic $g=g(S_a, \bar{S}_{\bar{a}})$. The condition (57) for the breakdown of DT is satisfied since

$$\partial^2K/\partial\Sigma\partial\Sigma=\partial^2K/\partial\Sigma\partial\bar{\Sigma}=\partial^2K/\partial\bar{\Sigma}\partial\bar{\Sigma}=0. \tag{93}$$

The functional $f(S_a, \bar{S}_{\bar{a}})$ is the Kähler potential for the manifold described by the CSS coordinates S_a. The non-holomorphic constraint (91) not only prevents DT from going through, it also reduces the sigma model manifold to a smaller one which is, in general, non-Kähler. This is due to the fact that a non-holomorphic constraint does not preserve the complex structure of a Kähler manifold, so that the constrained manifold is non-Kähler, except in special cases where a non-holomorphic change of variables exists which restores the hermiticity of the metric. Non-holomorphic constraints can also be imposed by CSS Lagrange multipliers, but they are not usually equivalent to those constraints imposed by CLS Lagrange multipliers.

As a specific example let us take

$$f = S_1\tilde{S}_1 + S_2\tilde{S}_2, \quad g = (S_1 + \tilde{S}_2)^2, \tag{94}$$

for the functions in (90). The equations of motion for the CSSs S_1, S_2 are

$$(\bar{\nabla}^2 + R)(\tilde{S}_1 + 2\Sigma(S_1 + \tilde{S}_2)) = 0, \tag{95a}$$

$$(\bar{\nabla}^2 + R)(\tilde{S}_2 + 2\tilde{\Sigma}(\tilde{S}_1 + S_2)) = 0, \tag{95b}$$

and the constraint equation is

$$\bar{\nabla}_{\dot{\alpha}}(S_1 + \tilde{S}_2)^2 = 0. \tag{96}$$

The constraint (96) is solved by $\tilde{S}_2 = \text{constant}$, which reduces (95a, b) to

$$(\bar{\nabla}^2 + R)\tilde{S}_1 = 0, \tag{97a}$$

$$(\bar{\nabla}^2 + R)(A + 2\tilde{\Sigma}(\tilde{S}_1 + A)), \quad A = \text{constant}. \tag{97b}$$

The first of these is just the free field equation for a CSS, and the second determines the Lagrange multiplier Σ in terms of S_1. This model describes a single propagating CSS.

If we try to make a duality transformation by introducing a CSS (σ) in the usual way we obtain the dual equation

$$\sigma = g = (S_1 + \tilde{S}_2)^2. \tag{98}$$

This equation does not involve Σ, so it is impossible to replace Σ by the dual variable σ, and the DT fails. One might think that (98) could be solved for say S_1 as a function of $(\sigma, \tilde{S}_2)$, which would eliminate Σ from the action leaving only a pure CSS term $K = f(\sigma, \tilde{S}_2)$. In this way the duality transformation would result in a pure chiral theory anyway. Although it is true that (98) can be solved formally for $S_1 = \sigma^{1/2} - \tilde{S}_2$, the resulting theory has two propagating CSSs, S_2 and σ, whereas the original theory only has one. The reason a formal solution of (98) fails to reproduce the original theory is that the chirality of σ implies the constraint (91), which eliminates some of the degrees of freedom, and makes the formal solution invalid. In this particular example the constraint (96) is solved by $\tilde{S}_2 = \text{constant}$, which is equivalent to a constraint which results from a CSS Lagrange multiplier, as mentioned above. This model is therefore equivalent to a pure CSS model, but the DT has failed to produce the appropriate CSS model. In general the constraint (91) will determine the same number of degrees of freedom as two CSSs, but a CSS Lagrange multiplier would yield the constraint

$$(\bar{\nabla}^2 + R)g = 0, \tag{99}$$

which is less restrictive than (91), and only determines the degrees of freedom of one CSS. For the special case (94), the two constraints (91) and (99) have the same solution, which is why the model is equivalent to a pure CSS model.

From this example it is apparent that the condition (57) ensures that the DT will not produce an action which is equivalent to the original one involving CLSs. However, the failure of duality, by itself, is not sufficient to ensure that the resulting theory has no equivalent pure CSS theory. Since the final equations (97a, b) are not only supersymmetry preserving, but are Kähler (trivially) in nature, it is also true that failure of DT does not necessarily imply either broken supersymmetry or non-Kähler geometry. Thus, we can only consider (98) as one of the conditions needed in order to be able to construct supersymmetric theories with non-Kähler geometry.

8. Discussion and conclusion

In this paper the breakdown of DT between chiral and linear matter superfields has been demonstrated, in the presence of the old-minimal supergravity. Two cases have been distinguished: the purely linear and the mixed case of chiral and linear.

The DT is not invertible if the determinant of the matrix K_{rs} (46c) vanishes, either globally, or in the vacuum. This result can be understood in simple geometric terms. The DT converts K_{rs} into its unique inverse $-\hat{K}_{rs}$ which contains the Hermitian Kähler metric $\hat{K}_{x\bar{y}}$. This is accomplished by a non-holomorphic change of the complex coordinates $C_x = \Sigma_x|_{\theta=\bar{\theta}=0}$. If K_{rs} is not invertible, its determinant vanishes, and no change of coordinates exist to convert it to $\hat{K}_{rs}$. This, combined with the fact that CLSs have couplings and kinetic terms (Deo and Gates 1985) which are not in Kähler form before a DT is made, leads one to expect that models with CLSs not admitting DT will usually have non-Kähler couplings and kinetic terms on-shell. The third example in § 7 shows that this is not always the case however. This example also shows that the failure of DT does not necessarily imply broken supersymmetry, although we have not constructed an example where DT fails due to (57) *globally*, and yet supersymmetry is preserved for a *non-Kähler* model. The most interesting possibility is that both supersymmetry and DT break down only in vacua for physical CLSs, resulting in a non-Kähler equivalent theory, as demonstrated by the second example in § 7.

We have seen, that in order to have a realistic model, with a zero cosmological constant, the cosmological term in the action (10) must be absent ($\lambda = 0$), for pure CLS theories, which cannot be transformed to chiral ones by DT. This results in a flat potential ($V \equiv 0$). When the potential is flat, supersymmetry can be broken only if there exist non-supersymmetric solutions to the auxiliary field equations, since there are no "extrema" of V.

The mixed case is more flexible, as it can have a zero cosmological constant even when $\lambda \neq 0$, and the DT breaks down. We conclude from this that the scalar multiplet couplings, which have been used in almost all $N = 1$ supergravity theories, are a special case, and that there is a wide class of inequivalent couplings, which are equally phenomenologically viable.

In the third example of § 7 we saw that even when all the physical supermultiplets are represented by CSSs, the manifold of the nonlinear sigma model can become non-Kähler, if CLSs are used as Lagrange multipliers to impose (non-holomorphic) constraints. This indicates that the presence of auxiliary fields in the off-shell representations of N-extended supersymmetries could result in the manifold of scalar fields being non-Kähler on-shell. It is clear that the assumption that the scalar fields, in extended supergravities and superstrings, are coordinates of a Kähler manifold needs to be reexamined.

The importance of the off-shell structure of supersymmetric theories is obvious from these results. There has been a lot of activity to find on-shell representations in the case of extended supergravities. It cannot be assumed that finding these on-shell representations is the end of the story, especially when external matter is present. We have seen, that even in the simple case of $N=1$ supersymmetry in four dimensions, that the off-shell structure can dramatically affect the on-shell theory. It would be very surprising if the off-shell structure of extended supersymmetric theories did not produce equally dramatic effects.

Acknowledgements

This research was supported in part by the U.S. Department of Energy Contracts No. DE-AS05-80ER10713 (GMS) and DE-AC02-86ER40253 (KTM).

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