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Microscopic theory of soliton propagation in a mixture of two boson fluid films

A C BISWAS
Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

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Abstract. A microscopic theory of soliton propagation in a mixture of two boson fluids at $T = 0°K$ has been provided.

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Introduction

In a recent paper Warke and the author gave a microscopic theory of soliton propagation in a film of liquid ^4He. In this paper we give a microscopic theory of soliton propagation in a film of a mixture of two boson fluids. Such a mixture can be visualized as a mixture of superfluid ^3He and liquid ^4He. We can also think of a mixture of liquid ^6He and liquid ^4He.

The Hamiltonian is

$$H = \sum \frac{P_i^2}{2M} + \sum \frac{P_i^2}{2m} + \sum u(r_i - r_j) + \sum V(r_i - r_j) + \sum_{i,j} W(r_i - r_j) + \sum u_1(r_i) + \sum u_2(r_j), \tag{1}$$

where $u(r_i - r_j)$ is the interaction between the particles of system (1) and $V(r_i - r_j)$ is the interaction between particles of system (2), $W(r_i - r_j)$ is the interaction between particles of system (1) and system (2), $u_1(r_i)$ and $u_2(r_j)$ are the Van der Waal's interaction for the system (1) and system (2) particles respectively.

The action is

$$L = \int \left\langle \psi \middle| i\hbar \frac{\partial}{\partial t} - H \middle| \psi \right\rangle \mathrm{d}t. \tag{2}$$

We use the Hartree-Fock decomposition of the wave function and write

$$\psi = \prod_{i=1}^{N_1} \phi_1(r_i, t) \prod_{j=1}^{N_2} \phi_2(r'_j, t). \tag{3}$$

We put

$$\phi_1(r_j, t) = [\rho_{01} \text{①} (z_j - \eta)]^{1/2} \exp[i\theta(\eta_j, t)], \tag{4}$$

$$\phi_2(r'_j, t) = [\rho_{02} \text{①} (z_j - \eta)]^{1/2} \exp[i\theta(\eta_j, t)], \tag{5}$$

where ① (x) is the Heaviside theta function (since $z_j \leqslant \eta$ the surface height).

The action is composed of

$$I_1 = \int dt\, dx\, dz [\rho_{01} ①(z_i-\eta)]^{1/2} \exp(-i\theta_i) [\rho_{02} ①(z_j'-\eta)]^{1/2} \exp(-i\theta_j)$$

$$i\hbar \frac{\partial}{\partial t} [\rho_{01} ①(z_i-\eta)]^{1/2} [\rho_{02} ①(z_j'-\eta)]^{1/2} \exp(i\theta_i) \exp(-i\theta_j)$$

$$= \int dt\, dx\, dz\, \rho_{01} \tfrac{1}{2} i\hbar\, \delta(z_i-\eta)(-\partial/\partial t)$$

$$+ \int dt\, dx\, dz\, \rho_{02} i\hbar \tfrac{1}{2} \delta(z_j-\eta)(-\partial/\partial t)$$

$$- \int dt\, dx\, dz\, \hbar\, \rho_{01} ①(z_i-\eta)(\partial\theta_i/\partial t) - \int dt\, dx\, dz\, \rho_{02} ①$$

$$(z_j-\eta)(\partial\theta_j/\partial t)$$

$$= \int dt\, dx\, \rho_{01} i\hbar(-\partial\eta/\partial t) + \int dt\, dx\, \rho_{02} \tfrac{1}{2} i\hbar(-\partial\eta/\partial t) - \int dt\, dx\, \hbar\, \rho_{01}$$

$$\times \int_0^\eta \frac{\partial\theta_1}{\partial t} dz - \int dt\, dx\, \hbar\, \rho_{02} \int_0^\eta \frac{\partial\theta_2}{\partial t} dz. \tag{6}$$

Using the fact that the total number of particles in the system is conserved we derive,

$$\int \frac{\partial\eta}{\partial t} dx = 0,$$

$$I_2 = \int dt\, dx \frac{\hbar^2}{2m} \rho_{01} \left[\frac{1}{4}\frac{1}{h_0}\left(-\frac{\partial\eta}{\partial x}\right)^2 + \frac{1}{2}\left(-\frac{\partial^2\eta}{\partial x^2}\right) + \frac{1}{2} i \left(-\frac{\partial\eta}{\partial x}\right) \frac{\partial\theta_1}{\partial x}\bigg|_{z=\eta} \right.$$

$$\left. - \int_0^\eta \left(\frac{\partial\theta_1}{\partial x}\right)^2 dz + i \int_0^\eta \frac{\partial^2\theta_1}{\partial x^2} dz \right] + \int dt\, dx \frac{\hbar^2}{2M} \rho_{02} \left[\frac{1}{4}\left(-\frac{\partial\eta}{\partial x}\right)^2 \right.$$

$$\left. + \frac{1}{2}\left(-\frac{\partial^2\eta}{\partial x^2}\right) + \frac{1}{2} i \left(-\frac{\partial\eta}{\partial x}\right) \frac{\partial\theta_2}{\partial x}\bigg|_{z=\eta} - \int_0^\eta \left(\frac{\partial\theta_2}{\partial x}\right)^2 dz + i \int_0^\eta \frac{\partial^2\theta_2}{\partial x^2} dz \right]$$

$$- \frac{\hbar^2}{2m} \int_\eta dt\, dx\, \rho_{01} \left[\frac{1}{4}\frac{1}{h_0} + i \frac{\partial\theta_1}{\partial z}\bigg|_{z=\eta} - \int_0^\eta \left(\frac{\partial\theta_1}{\partial z}\right)^2 dz \right.$$

$$\left. + i \int_0^\eta \frac{\partial^2\theta_1}{\partial z^2} dz \right] - \frac{\hbar^2}{2M} \int dt\, dx\, \rho_{02} \left[\frac{1}{4}\frac{1}{h_0} + i \frac{\partial\theta_2}{\partial z}\bigg|_{z=\eta} - \int_0^\eta \left(\frac{\partial\theta_2}{\partial z}\right)^2 dz \right.$$

$$\left. + i \int_0^\eta \frac{\partial^2\theta_2}{\partial z^2} dz \right] - \int dt\, dx'\, dy\, dy'\, dx \int_0^\eta \int_0^{\eta'} u(|r-r'|)\, dz\, dz'$$

$$+ \int dt\, dx\, dy \int_0^\eta W_1(r)\, dz - \int dt\, dx\, dy \int_0^\eta W_2(r)\, dz. \tag{7}$$

We scale all lengths with respect to the length l which is the extent in the x-direction ($\eta \ll l$). The action L then comes out to be

$$L=-\int \mathrm{d}t\,\mathrm{d}\bar{x}\,\hbar\,\rho_{01}\,l\int_0^{\bar{\eta}}\frac{\partial\theta_1}{\partial t}\,\mathrm{d}\bar{z}-\int \mathrm{d}t\,\mathrm{d}\bar{x}\,\hbar\,\rho_{02}\,l\int_0^{\bar{\eta}}\frac{\partial\theta_2}{\partial t}\,\mathrm{d}\bar{z}$$

$$-\frac{\hbar^2}{2m}\int \mathrm{d}t\,\mathrm{d}\bar{x}\,\rho_{01}\left[\frac{1}{4}\frac{1}{h_0}\left(-\frac{\partial\bar{\eta}}{\partial\bar{x}}\right)^2+\frac{1}{2}\left(-l\frac{\partial^2\bar{\eta}}{\partial\bar{x}^2}\right)+\frac{1}{2}i\left(-\frac{\partial\bar{\eta}}{\partial\bar{x}}\frac{1}{l}\frac{\partial\theta_1}{\partial\bar{x}}\bigg|_{\bar{z}=\bar{\eta}}\right)\right.$$

$$\left.-\int_0^{\bar{\eta}}\left(\frac{\partial\theta_1}{\partial\bar{x}}\right)^2\frac{1}{l}\,\mathrm{d}\bar{z}+i\int_0^{\bar{\eta}}\frac{\partial^2\theta_1}{\partial\bar{x}^2}\frac{1}{l}\,\mathrm{d}\bar{z}\right]-\frac{\hbar^2}{2M}\int \mathrm{d}t\,\mathrm{d}\bar{x}\,\rho_{01}\left[\frac{1}{4}\frac{1}{h_0}\left(-\frac{\partial\bar{\eta}}{\partial\bar{x}}\right)^2\right.$$

$$+\frac{1}{2}\left(-l\frac{\partial^2\bar{\eta}}{\partial\bar{x}^2}\right)+\frac{1}{2}i\left(-\frac{\partial\bar{\eta}}{\partial\bar{x}}l\frac{\partial\theta_2}{\partial\bar{x}}\bigg|_{\bar{z}=\bar{\eta}}\right)-\int_0^{\bar{\eta}}\left(\frac{\partial\theta_2}{\partial\bar{x}}\right)^2\frac{1}{l}\,\mathrm{d}\bar{z}$$

$$\left.+i\int_0^{\bar{\eta}}\frac{\partial^2\theta_2}{\partial\bar{x}^2}\frac{1}{l}\,\mathrm{d}\bar{z}\right]-\frac{\hbar^2}{2m}\int \mathrm{d}t\,\mathrm{d}x\,\rho_{01}\left[\frac{1}{4}\frac{1}{h_0}+i\frac{1}{l}\frac{\partial\theta_1}{\partial\bar{z}}-\frac{1}{l}\int_0^{\bar{\eta}}\left(\frac{\partial\theta_1}{\partial\bar{z}}\right)^2\mathrm{d}\bar{z}\right.$$

$$\left.+i\int_0^{\bar{\eta}}\frac{1}{l}\frac{\partial^2\theta_2}{\partial\bar{z}^2}\,\mathrm{d}\bar{z}\right]-\int \mathrm{d}t\,\mathrm{d}\bar{x}'\,\mathrm{d}\bar{y}\,\mathrm{d}\bar{y}'\,\mathrm{d}\bar{x}\int_0^{\bar{\eta}}\int_0^{\bar{\eta}}u(|r-r'|)\,\mathrm{d}\bar{z}\,\mathrm{d}\bar{z}'$$

$$-\int \mathrm{d}t\,\mathrm{d}\bar{x}'\,\mathrm{d}\bar{y}\,\mathrm{d}\bar{y}'\,\mathrm{d}\bar{x}\int_0^{\bar{\eta}}\int_0^{\bar{\eta}}V(|r-R|)\,\mathrm{d}\bar{z}\,\mathrm{d}\bar{z}'$$

$$-\int \mathrm{d}t\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{y}\int_0^{\bar{\eta}}W_1(r)\,\mathrm{d}\bar{z}-\int \mathrm{d}t\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{y}\int_0^{\bar{\eta}}W_2(r)\,\mathrm{d}\bar{z}. \qquad (8)$$

In the above h_0 is the minimum thickness of the film for which our treatment is valid.

By varying w.r.t. $\bar{\eta}$, θ_1, θ_2 we get the following equations

$$-\frac{\partial\bar{\eta}}{\partial\bar{x}}\frac{\partial\theta_1}{\partial\bar{x}}+\frac{\partial^2\theta_1}{\partial\bar{x}^2}+\frac{\partial^2\theta_1}{\partial\bar{z}^2}-\frac{\partial\theta_1}{\partial\bar{z}}=0, \qquad (9)$$

$$-\frac{\partial\bar{\eta}}{\partial\bar{x}}\frac{\partial\theta_2}{\partial\bar{x}}+\frac{\partial^2\theta_2}{\partial\bar{x}^2}+\frac{\partial^2\theta_2}{\partial\bar{z}^2}-\frac{\partial\theta_2}{\partial\bar{z}}=0, \qquad (10)$$

$$-\hbar C_1\frac{\partial\theta_1}{\partial t}-\hbar C_2\frac{\partial\theta_2}{\partial t}-\left(\frac{\hbar^2C_1}{2m}+\frac{\hbar^2C_2}{2M}\right)\frac{1}{4lh_0}\frac{\partial^2\bar{\eta}}{\partial\bar{x}^2}-\frac{\hbar^2C_1}{2m}\left(\frac{\partial\theta_1}{\partial\bar{z}}\right)^2\bigg|_{\bar{z}=\bar{\eta}}$$

$$-\frac{\hbar^2C_2}{2M}\left(\frac{\partial\theta_2}{\partial\bar{x}}\right)^2\bigg|_{\bar{z}=\bar{\eta}}-C_1^2\int u(|r-r'|)\bigg|_{\substack{\bar{z}=\bar{\eta}\\ \bar{z}'=\bar{\eta}'}}\mathrm{d}x\,\mathrm{d}x'\,\mathrm{d}y\,\mathrm{d}y'-C_1C_2$$

$$\int W(|r-r'|)\bigg|_{\substack{\bar{z}=\bar{\eta}\\ \bar{z}'=\bar{\eta}'}}\mathrm{d}x\,\mathrm{d}x'\,\mathrm{d}y\,\mathrm{d}y'-C_2^2\int V(|r-r'|)\bigg|_{\substack{\bar{z}=\bar{\eta}\\ \bar{z}'=\bar{\eta}'}}\mathrm{d}x\,\mathrm{d}x'\,\mathrm{d}y\,\mathrm{d}y'$$

$$-C_1\int W_1(r)\bigg|_{\bar{z}=\bar{\eta}}\mathrm{d}x\,\mathrm{d}y-C_2\int W_2(r)\bigg|_{\bar{z}=\bar{\eta}}\mathrm{d}x\,\mathrm{d}y=0. \qquad (11)$$

Let us consider the equation

$$-\frac{\partial \bar{\eta}}{\partial \bar{x}}\frac{\partial \theta_1}{\partial \bar{x}}+\frac{\partial^2 \theta_1}{\partial \bar{x}^2}+\frac{\partial^2 \theta_1}{\partial \bar{z}^2}-\frac{\partial \theta_1}{\partial \bar{z}}=0. \tag{12}$$

We put

$$\theta_1(\bar{x}, \bar{z}, t)=\sum_{n=0}^{\infty} \theta_1^{(n)}(x, t)\bar{z}^n,$$

$$\theta_{1\bar{z}}=0 \quad \text{at} \quad \bar{z}=0 \quad \text{gives} \quad \theta_1^{(1)}=0. \tag{13}$$

We have

$$-\eta_x \sum_{n=0}^{\infty} \theta_{1x}^{(n)}(x, t)z^n+\sum_{n=0}^{\infty} \theta_{1xx}^{(n)}z^n+\sum_{n=0}^{\infty} \theta_1^{(n)}n(n-1)z^{n-1}$$

$$-\sum_{n=0}^{\infty} \theta_1^{(n)}nz^{n-1}=0. \tag{14}$$

From (14) we have

$$(n+2)(n+1)\,\theta_1^{(n+2)}-(n+1)\,\theta_1^{(n+1)}+\theta_{1xx}^{(n)}-\bar{\eta}_x\theta_{1x}^{(n)}=0 \tag{15}$$

$$2\cdot 1\;\theta_1^{(2)}=-\theta_{1\bar{x}\bar{x}}^{(0)}+\bar{\eta}_{\bar{x}}\theta_{1\bar{x}}^{(0)}.$$

Similarly

$$2\cdot 1\;\theta_2^{(2)}=-\theta_{2\bar{x}\bar{x}}^{(0)}+\bar{\eta}_{\bar{x}}\theta_{2\bar{x}}^{(0)}. \tag{16}$$

Assuming both θ_1 and θ_2 slowly varying functions of $\bar{z}$ we have

$$\theta_{1\bar{x}\bar{x}}^{0}=\bar{\eta}_{\bar{x}}\theta_{1\bar{x}}^{(0)}, \tag{17}$$

$$\theta_{2\bar{x}\bar{x}}^{0}=\bar{\eta}_{\bar{x}}\,\theta_{2\bar{x}}^{(0)}. \tag{18}$$

Integrating (17) and (18) we have

$$\theta_{1\bar{x}}^{(0)}=C_1\,e^{\bar{\eta}}, \tag{19}$$

$$\theta_{2\bar{x}}^{(0)}=C_2\,e^{\bar{\eta}}. \tag{20}$$

C_1 and C_2 are constants to be determined. Let V be the velocity of the fluid at an infinitesimal thickness. Then

$$C_1=\frac{mVl}{\hbar}, \qquad C_2=\frac{MVl}{\hbar}. \tag{21}$$

We have then

$$\theta_{1\bar{x}}^{(0)}=\frac{mVl}{\hbar}e^{\bar{\eta}}, \qquad \theta_{2\bar{x}}^{(0)}=\frac{MVl}{\hbar}e^{\bar{\eta}}. \tag{22}$$

Expanding w.r.t. $\bar{\eta}$ and retaining upto quadratic terms

$$\theta_{1\bar{x}}^{(0)}=\frac{mVl}{\hbar}\left(1+\bar{\eta}+\frac{\bar{\eta}^2}{2}\right), \tag{23}$$

$$\theta_{2\bar{x}}^{(0)} = \frac{MVl}{\hbar}\left(1 + \bar{\eta} + \frac{\bar{\eta}^2}{2}\right). \tag{24}$$

Equation (11) can be written as (using expansion of θ_1 and θ_2 and retaining the first terms only).

$$\begin{aligned} &-\hbar C_1 \frac{\partial \theta_1^{(0)}}{\partial t} - \hbar C_2 \frac{\partial \theta_2^{(0)}}{\partial t} - \left(\frac{\hbar^2 C_1}{2m} + \frac{\hbar^2 C_2}{2M}\right) \frac{1}{4lh_0} \frac{\partial^2 \bar{\eta}}{\partial \bar{x}^2} \\ &\quad - C_1^2(\alpha_1 \bar{\eta}^2 - 2\alpha_1 \bar{\eta} d + d^2) - C_1 C_2(\beta_1 \bar{\eta}^2 - 2\beta_1 \bar{\eta} d + d^2) \\ &\quad - C_2^2(\gamma_1 \bar{\eta}^2 - 2\gamma_1 \bar{\eta} d + d^2) - C_1(\delta_0 + \delta_1 \bar{\eta}^2) - C_2(\delta_2 + \delta_3 \bar{\eta}^2) = 0, \end{aligned} \tag{25}$$

where $\alpha_1, \beta_1, \gamma_1, \delta_0, \delta_1, \delta_2, \delta_3$ are interaction constants. d is the equilibrium thickness of the film. C_1, C_2 are concentrations of the two types of boson fluids in the film.

Differentiating (25) w.r.t. $\bar{x}$ we have

$$\begin{aligned} &-\hbar C_1 \frac{\partial \theta_{1\bar{x}}^{(0)}}{\partial t} - \hbar C_2 \frac{\partial \theta_{2\bar{x}}^{(0)}}{\partial t} - \left(\frac{\hbar^2 C_1}{2m} + \frac{\hbar^2 C_2}{2M}\right) \frac{1}{4lh_0} \frac{\partial^3 \bar{\eta}}{\partial \bar{x}^3} \\ &\quad - C_1^2\left(2\alpha_1 \bar{\eta} \frac{\partial \bar{\eta}}{\partial \bar{x}} - 2\alpha_1 d \frac{\partial \bar{\eta}}{\partial \bar{x}}\right) - C_1 C_2\left(2\beta_1 \bar{\eta} \frac{\partial \bar{\eta}}{\partial \bar{x}} - 2\beta_1 d \frac{\partial \bar{\eta}}{\partial \bar{x}}\right) \\ &- C_2^2\left(2\gamma_1 \bar{\eta} \frac{\partial \bar{\eta}}{\partial \bar{x}} - 2\gamma_1 d \frac{\partial \bar{\eta}}{\partial \bar{x}}\right) - C_1\left(2\delta_1 \bar{\eta} \frac{\partial \bar{\eta}}{\partial \bar{x}}\right) - C_2\left(2\delta_3 \bar{\eta} \frac{\partial \bar{\eta}}{\partial \bar{x}}\right) = 0, \end{aligned} \tag{26}$$

$$\begin{aligned} &-\hbar C_1 \frac{\partial}{\partial t} \frac{mVl}{\hbar}\left(1 + \bar{\eta} + \frac{\bar{\eta}^2}{2}\right) - \hbar C_2 \frac{\partial}{\partial t} \frac{MVl}{\hbar}\left(1 + \bar{\eta} + \frac{\bar{\eta}^2}{2}\right) \\ &\quad - \left(\frac{\hbar^2 C_1}{2m} + \frac{\hbar^2 C_2}{2M}\right) \frac{1}{4lh_0} \frac{\partial^3 \bar{\eta}}{\partial \bar{x}^3} - C_1^2\left(\alpha_1 2\bar{\eta} \frac{\partial \bar{\eta}}{\partial \bar{x}} - 2\alpha_1 d \frac{\partial \bar{\eta}}{\partial \bar{x}}\right) \\ &\quad - C_1 C_2\left(2\beta_1 \bar{\eta} \frac{\partial \bar{\eta}}{\partial \bar{x}} - 2\beta_1 d \frac{\partial \bar{\eta}}{\partial \bar{x}}\right) - C_2\left(2\gamma_1 \bar{\eta} \frac{\partial \bar{\eta}}{\partial \bar{x}} - 2\gamma_1 d \frac{\partial \bar{\eta}}{\partial \bar{x}}\right) \\ &\quad - C_1\left(2\delta_1 \bar{\eta} \frac{\partial \bar{\eta}}{\partial \bar{x}}\right) - C_2\left(2\delta_3 \bar{\eta} \frac{\partial \bar{\eta}}{\partial \bar{x}}\right) = 0, \end{aligned} \tag{27}$$

$$\begin{aligned} &-\hbar C_1 \frac{mVl}{\hbar}\left(\frac{\partial \bar{\eta}}{\partial t} + \bar{\eta} \frac{\partial \bar{\eta}}{\partial t}\right) - \hbar C_2 \frac{MVl}{\hbar}\left(\frac{\partial \bar{\eta}}{\partial t} + \bar{\eta} \frac{\partial \bar{\eta}}{\partial t}\right) \\ &\quad - \left(\frac{\hbar^2 C_1}{2m} + \frac{\hbar^2 C_2}{2M}\right) \frac{1}{4lh_0} \frac{\partial^3 \bar{\eta}}{\partial \bar{x}^3} - C_1^2\left(2\alpha_1 \bar{\eta} \frac{\partial \bar{\eta}}{\partial x} - 2\alpha_1 d \frac{\partial \bar{\eta}}{\partial x}\right) \\ &\quad - C_1 C_2\left(2\beta_1 \bar{\eta} \frac{\partial \bar{\eta}}{\partial x} - 2\beta_1 d \frac{\partial \bar{\eta}}{\partial x}\right) - C_2^2\left(2\gamma_1 \bar{\eta} \frac{\partial \bar{\eta}}{\partial \bar{x}} - 2\gamma_1 d \frac{\partial \bar{\eta}}{\partial x}\right) \\ &\quad - C_1\left(2\delta_1 \bar{\eta} \frac{\partial \bar{\eta}}{\partial x}\right) - C_2\left(2\delta_3 \bar{\eta} \frac{\partial \bar{\eta}}{\partial x}\right) = 0, \end{aligned} \tag{28}$$

$$-Vl(mC_1+MC_2)\left(\frac{\partial\bar{\eta}}{\partial t}+\bar{\eta}\frac{\partial\bar{\eta}}{\partial t}\right)-\left(\frac{\hbar^2 C_1}{2m}+\frac{\hbar^2 C_2}{2M}\right)\frac{1}{4lh_0}\frac{\partial^3\bar{\eta}}{\partial\bar{x}^3}$$

$$-(2\alpha_1 C_1^2+2\beta_1 C_1 C_2+2\gamma_1 C_2^2+2\delta_1 C_1+2\delta_3 C_2)\bar{\eta}\frac{\partial\bar{\eta}}{\partial\bar{x}}$$

$$+(2\alpha_1 C_1^2+2\beta_1 C_1 C_2+2\gamma_1 C_2^2)d\frac{\partial\bar{\eta}}{\partial\bar{x}}=0. \tag{29}$$

Put

$$Vl(mC_1+MC_2)=A, \tag{30}$$

$$(2\alpha_1 C_1^2+2\beta_1 C_1 C_2+2\gamma_1 C_2^2)d=B, \tag{31}$$

$$2\alpha_1 C_1^2+2\beta_1 C_1 C_2+2\gamma_1 C_2^2+2\delta_1 C_1+2\delta_3 C_2=C, \tag{32}$$

$$\left(\frac{\hbar^2 C_1}{2m}+\frac{\hbar^2 C_2}{2M}\right)\frac{1}{4lh_0}=D. \tag{33}$$

From (29) we have

$$A\left(\frac{\partial\bar{\eta}}{\partial t}+\bar{\eta}\frac{\partial\bar{\eta}}{\partial t}\right)+D\frac{\partial^3\bar{\eta}}{\partial\bar{x}^3}+C\,\bar{\eta}\frac{\partial\bar{\eta}}{\partial\bar{x}}-B\frac{\partial\bar{\eta}}{\partial\bar{x}}=0. \tag{34}$$

It is clear from equation (34) that the two fluids act as a single mixture producing a common third sound. Let us put

$$\bar{\eta}=f(\bar{x}-ct)=f(\xi). \tag{35}$$

From (34) we have

$$Df'''-(Ac+B)f'+(C-Ac)ff'=0. \tag{36}$$

Integrating (36) we have

$$f=\frac{3(Ac+B)}{C-Ac}\,\text{sech}^2\frac{1}{6}\frac{[3(Ac+B)]^{1/2}}{D}(\bar{x}-ct). \tag{37}$$

This defines a soliton moving with velocity c, amplitude $3(Ac+B)/(C-Ac)$ and width $6\{D/[3(Ac+B)]\}^{1/2}$. Note that the amplitude is a function of velocity c and the soliton is a crest if $c>C/A$ and a trough if $c<C/A$. A soliton with $c=C/A$ is unstable.

References

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