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Relativistic effects on meson spectra

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Abstract. The meson masses are considered to retain total relativistic effects for spinors instead of Fermi-Breit approximation. This necessitates a diagonalization approach instead of solutions of differential equations. Correction over Fermi-Breit form appears to be significant. For heavy quark systems agreement with experiment is found. The method will be quite useful if the quark-antiquark dynamics becomes sufficiently known.

Keywords. Relativistic meson spectra; Fermi-Breit approximation; decay widths.

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1. Introduction

Spectroscopy of hadrons in the quark model has been dealt with by many authors in the recent past (Eichten *et al* 1978; Schnitzer 1978; Richardson 1979; Buchmuller *et al* 1980; Martin 1980; Stanley and Robson 1980; Barik and Jena 1981; Gupta and Radford 1982; Gupta *et al* 1982; Mitra 1981; Mitra and Santhanam 1981; Mittal and Mitra 1984; Gupta 1977). The hyperfine and fine structure splittings of heavy quarkonia are usually considered by Fermi-Breit Hamiltonian as a perturbation over spin-averaged levels, where terms are retained only upto the leading orders. On the other hand we aim here to derive the same retaining the total relativistic effects for different phenomenological potentials.

In § 2, we describe the framework of the approach to evaluate the effective Hamiltonian in field theory. Here the full contribution from the Dirac spinors is retained, instead of the lowest order approximation. Unlike solving the Schrödinger's differential equation for spin-averaged state and doing perturbation calculation for spin structure, we adopt here the diagonalization approach for obtaining the relevant spectra. This permits a fully relativistic treatment of the spinor components. Section 3 is devoted to the discussion of some approximation schemes. In § 4, we apply the scheme to illustrate with different potentials. With Cornell type of potential (Eichten *et al* 1978) and with a vector exchange even for a heavy quark system, a significant correction to Fermi-Breit form is observed. As usual, we also include a scalar confining potential (Bardeen *et al* 1975). We further observe a variation of the potential parameters according to the mass of the fermion constituents and the size of the mesons (Gupta and Radford 1982), as expected. parallel to the renormalization scheme.

Our objective here has been to develop a fully relativistic approximation scheme and apply it for a few simple cases. Although we have succeeded in some of the applications, it appears that the problem should be really tackled when we can be in a position to *derive* the potential. The method indicated here will enable us to obtain full relativistic corrections in such an eventuality. It is quite possible that the calculations of lattice QCD may help us in this direction.*

2. General theory

As mentioned earlier, we consider here the matrix elements of the Hamiltonian operator for an enumerable quark-antiquark basis and then diagonalize this operator as approximated by a finite number of elements of this basis. We retain the four-component Dirac spinors instead of making Fermi-Breit approximation.

2.1 *Effective Hamiltonian*

We first consider the potential in the context of quantum chromodynamics (QCD) as an illustration, and later, introduce phenomenological potentials. Thus, in the second order in quantum chromodynamics (QCD) the S-matrix element corresponding to one-gluon exchange will simply be related to the *effective* potential $v(x)$ by the equation,

$$\begin{aligned} S_2 &= \frac{(-i)^2}{2!}\int :\bar{Q}(x)\gamma^{\mu}\frac{\lambda_a}{2}Q(x)\bar{\tilde{Q}}(y)\gamma^{\nu}\frac{\lambda_0}{2}\tilde{Q}(y):\mathrm{d}^4x\,\mathrm{d}^4y \\ &\quad\times\frac{ig^2}{(2\pi)^4}D_{\mu\nu}(q)\exp\left[-iq\cdot(x-y)\right]\mathrm{d}^4q \\ &\equiv(-i)\int v(x)\,\mathrm{d}^4x. \end{aligned} \tag{1}$$

In the above, we choose the Coulomb gauge (Berestetskii *et al* 1971) such that $D_{00}(q)=1/q^2$, $D_{0i}=D_{i0}=0$ and $D_{ij}(q)=(-\delta_{ij}+q_iq_j/|\mathbf{q}|^2)/q^2$. This yields

$$\begin{aligned} v(x) &= -\frac{1}{(2\pi)^4}\int :\bar{Q}(x)\gamma^{\mu}\frac{\lambda_a}{2}Q(x)\bar{\tilde{Q}}(y)\gamma^{\nu}\frac{\lambda_a}{2}\tilde{Q}(y): \\ &\quad\times g^2D_{\mu\nu}(q)\exp(-iq(x-y))\,\mathrm{d}^4y\,\mathrm{d}^4q. \end{aligned} \tag{2}$$

In general, $v(x)$ will be a function of space and time. $Q(x)$ and $\tilde{Q}(y)$ in (2) represent the *four-component* quark annihilation and anti-quark creation operators (Misra 1978). For such quark field operators *for hadrons at rest* a specific *time dependence* had been suggested (Misra 1978). It was considered, *as an approximation scheme*, that $Q(x)=Q(\mathbf{x})\exp(-ik_1^0t)$ where k_1^0 is the energy value of the quark Q in the hadron *at rest*, and is a constant. We shall presently see that *with this assumption* $v(x)$ in (2) will become independent** of $x^0=t$. We thus have in equation (2), when we consider hadrons at rest, $\bar{\tilde{Q}}(y)=\bar{\tilde{Q}}(\mathbf{y})\exp(-ik_2^0y^0)$ and $\tilde{Q}(y)=\tilde{Q}(\mathbf{y})\exp(ik_2^0y^0)$, with the same value k_2^0 since both $\tilde{Q}$ and $\bar{\tilde{Q}}$ corresponding to the same constituent antiquark of the

* One example of this type is by one of the authors (Naik 1986) with a mean field approximation.

** The assumption considered implies that the *constituent* quark field operators (Misra 1978) yield quarks with variable momentum but fixed energy. The theoretical situation however is unclear (Caswell and Lepage 1978).

hadron under consideration[§]. Hence, when $v(x)$ is applied to hadrons at rest, as will be case for hadron spectroscopy[§§], y^0 and q^0 integrations in (2) can be trivially performed, and we obtain that

$$v(\mathbf{x}) = -\frac{1}{(2\pi)^3}\int :\bar{Q}(\mathbf{x})\gamma^\mu\frac{\lambda_a}{2}Q(\mathbf{x})\bar{\bar{Q}}(\mathbf{y})\gamma^\nu\frac{\lambda_a}{2}\tilde{Q}(\mathbf{y}): \mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{y}\cdot g^2\cdot D_{\mu\nu}(q)\exp(i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})). \tag{3}$$

The time independence of the potential "density" in (3) as advertised may be noted. This fact is only true when we consider the operator $v(x)$ applied to the hadrons in their rest frame (Misra 1978) as for hadron spectroscopy, and, is true to the extent quarks in hadrons at rest occupy *fixed* energy levels.

We can rewrite (3) as

$$v(\mathbf{x}) = \frac{1}{(2\pi)^3}\iint\Big[:\bar{Q}(\mathbf{x})\gamma^\mu\frac{\lambda_a}{2}Q(\mathbf{x})\bar{\bar{Q}}(\mathbf{y})\gamma_\mu\frac{\lambda_a}{2}\tilde{Q}(\mathbf{y}): + :\bar{Q}(\mathbf{x})(\gamma\cdot\mathbf{q})\frac{\lambda_a}{2}Q(\mathbf{x})\bar{\bar{Q}}(\mathbf{y})(\gamma\cdot\mathbf{q})\frac{\lambda_a}{2}\tilde{Q}(\mathbf{y}):/|\mathbf{q}|^2\Big] \times\tilde{V}(\mathbf{q})\exp(i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y}))\,\mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{y}, \tag{4}$$

where we have substituted

$$\tilde{V}(\mathbf{q}) = g^2(|\mathbf{q}|)/|\mathbf{q}|^2 = 4\pi\alpha_s(Q)/Q^2. \tag{5}$$

In (5), we have in mind, the renormalization group improved perturbation theory (Marciano and Pagels 1978) and, we may have, e.g. a *phenomenological* extrapolation of $\alpha_s(Q)$ to the infrared regime given as

$$\alpha_s(Q) = \frac{12\pi}{(33-2n_f)}\Big/\log\left(1+\frac{Q^2}{\Lambda^2}\right), \tag{6}$$

as considered by Richardson and others. The potential in coordinate space is given as

$$V(\mathbf{r}) = (2\pi)^{-3}\int\tilde{V}(\mathbf{q})\exp(i\mathbf{q}\cdot\mathbf{r})\mathrm{d}\mathbf{q}. \tag{7}$$

Whenever we take a *phenomenological* central potential (Rosner 1981) the identification will be made by (7), where while doing this it is to be recognised that the colour degree of freedom is still present in (4).

We now proceed to evaluate the *operator* part of (4), which, with the four-component structure of the operators will give rise to orbital, hyperfine, spin-orbit and tensor operators in the nonrelativistic limit (Berestetskii *et al* 1971; Rosner 1981), and in the general case will include momentum (velocity) dependence in the potential arising as

[§] The fact that having energy dependence as suggested may be a reasonable approximation finds further support (Biswal and Misra 1982; Misra 1983).

[§§] The idea may be good only for hadrons at rest. Otherwise one may take relativistic Bethe Salpeter equations (e.g. Mittal and Mitra 1984).

relativistic corrections. For this purpose we write (Rosner 1981)

$$Q(\mathbf{x}) = (2\pi)^{-3/2} \int u(\mathbf{k}) Q_I(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{x})\, d\mathbf{k}, \tag{8a}$$

and

$$\tilde{Q}(\mathbf{y}) = (2\pi)^{-3/2} \int v(\mathbf{k}) \tilde{Q}_I(\mathbf{k}) \exp(-i\mathbf{k}\cdot\mathbf{y})\, d\mathbf{k}, \tag{8b}$$

where

$$u(\mathbf{k}) = \begin{bmatrix} f(\mathbf{k}) \\ g\sigma\cdot\mathbf{k} \end{bmatrix}, \tag{9a}$$

and

$$v(\mathbf{k}) = \begin{bmatrix} g(\mathbf{k})\sigma\cdot\mathbf{k} \\ f(\mathbf{k}) \end{bmatrix}. \tag{9b}$$

We here note that $f(\mathbf{k}) = [(p^0 + m_Q)/2p^0]^{1/2}$ and $g(\mathbf{k}) = [2p^0(p^0+m_Q)]^{-1/2}$ with $p^0 = \sqrt{\mathbf{k}^2 + m_Q^2}$. Further, the two *component* annihilation and creation for the quark antiquark and in (8) are given as $(Q_I(\mathbf{k}) = \sum_r Q_{Ir}(\mathbf{k}) u_{Ir}$ and $\tilde{Q}_I(\mathbf{k}) = \sum_r \tilde{Q}_{Ir}(\mathbf{k})\, v_{Ir}$.

Now we have

$$\begin{aligned} P &\equiv \int v(\mathbf{x})\, d\mathbf{x} \\ &= (2\pi)^{-3} \int\Big[Q_I^\dagger(\mathbf{k}'_1)\bar{u}(\mathbf{k}'_1)\gamma^\mu \frac{\lambda_a}{2} u(\mathbf{k}_1) Q_I(\mathbf{k}_1) \\ &\quad \times :\tilde{Q}_I(\mathbf{k}_2)^\dagger \bar{v}(\mathbf{k}_2)\gamma_\mu \frac{\lambda_a}{2} v(\mathbf{k}'_2)\tilde{Q}_I(\mathbf{k}'_2): \\ &\quad + Q_I(\mathbf{k}'_1)^\dagger \bar{u}(\mathbf{k}'_1)(\gamma\cdot\mathbf{q})\frac{\lambda_a}{2} u(\mathbf{k}_1) Q_I(\mathbf{k}_1) \\ &\quad \times :\tilde{Q}_I(\mathbf{k}_2)^\dagger \bar{v}(\mathbf{k}_2)(\gamma\cdot\mathbf{q})\frac{\lambda_a}{2} v(\mathbf{k}'_2)\tilde{Q}_I(\mathbf{k}'_2):/|\mathbf{q}|^2 \Big] \\ &\quad \times \tilde{V}(\mathbf{q})\, d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{q}. \end{aligned} \tag{10}$$

In equation (10), $\mathbf{k}'_1 = \mathbf{k}_1 + \mathbf{q}$ and $\mathbf{k}'_2 = \mathbf{k}_2 - \mathbf{q}$.

In order to evaluate (10) we now note that

$$\begin{aligned} &\bar{u}(\mathbf{k}'_1)\gamma^\mu u(\mathbf{k}_1) \otimes \bar{v}(\mathbf{k}_2)\gamma_\mu v(\mathbf{k}'_2) \\ &+ \bar{u}(\mathbf{k}'_1)(\gamma\cdot\mathbf{q}) u(\mathbf{k}_1) \otimes \bar{v}(\mathbf{k}_2)(\gamma\,.\,\mathbf{q}) v(\mathbf{k}'_2)/|\mathbf{q}^2|^2 \\ &= U_1 + U_2 + U_3 + U_4, \end{aligned} \tag{11}$$

where

$$\begin{aligned} U_1 &= (f'_1 f_1 + g'_1 g_1 (\mathbf{k}'_1\cdot\mathbf{k}_1))(f'_2 f_2 + g'_2 g_2 (\mathbf{k}'_2\cdot\mathbf{k}_2)) \\ &\quad + \left\{ \frac{(\mathbf{k}_1\cdot\mathbf{q})(\mathbf{k}_2\cdot\mathbf{q})}{|\mathbf{q}|^2} - \mathbf{k}_1\cdot\mathbf{k}_2 \right\} \\ &\quad \times \{ f_2 f_1 g'_1 g'_2 + f'_1 f'_2 g_1 g_2 + f_1 f'_2 g'_1 g_2 + f'_1 f_2 g_1 g'_2 \}, \end{aligned} \tag{12}$$

$$U_2 = i(f'_2 f_2 + g'_2 g_2 (\mathbf{k}_2 \cdot \mathbf{k}'_2)) g'_1 g_1 \sigma \cdot (\mathbf{q} \times \mathbf{k}_1)$$
$$- i(f_2 g'_2 + f'_2 g_2) \sigma_1 \cdot \left\{ f_1 g'_1 (\mathbf{q} \times \mathbf{k}_2) \right.$$
$$\left. - (f'_1 g_1 - f_1 g'_1) \left(\mathbf{k}_1 \times \left(\mathbf{k}_2 - \frac{\mathbf{k}_2 \cdot \mathbf{q}}{|\mathbf{q}|^2} \mathbf{q} \right) \right) \right\}$$
$$+ i(f'_1 f_1 + g'_1 g_1 \mathbf{k}'_1 \cdot \mathbf{k}_1) g'_2 g_2 \sigma_2 \cdot (\mathbf{q} \times \mathbf{k}_2)$$
$$- i(f_1 g'_1 + f'_1 g_1) \sigma_2 \cdot \left\{ f_2 g'_2 (\mathbf{q} \times \mathbf{k}_1) \right.$$
$$\left. + (f_2 g'_2 - f'_2 g_2) \left(\mathbf{k}_1 - \frac{\mathbf{k}_1 \cdot \mathbf{q}}{|\mathbf{q}|^2} \mathbf{q} \right) \times \mathbf{k}_2 \right\}. \tag{13}$$

$$U_3 = (\sigma_1 . \sigma_2) [f'_1 f_2 g_1 g'_2 \, \mathbf{k}_1 \cdot \mathbf{k}'_2 + f_1 f'_2 g'_1 g_2 \, \mathbf{k}'_1 \cdot \mathbf{k}_2$$
$$- f_1 f_2 g'_1 g'_2 \, \mathbf{k}'_1 \cdot \mathbf{k}'_2 - f'_1 f'_2 g_1 g_2 \, \mathbf{k}_1 . \mathbf{k}_2], \tag{14}$$

$$U_4 = [f_1 f_2 g'_1 g'_2 (\sigma_1 \cdot \mathbf{k}'_2)(\sigma_2 \cdot \mathbf{k}'_1) + f'_1 f'_2 g_1 g_2 (\sigma_1 . \mathbf{k}_2)(\sigma_2 \cdot \mathbf{k}_1)$$
$$- f'_1 f_2 g_1 g'_2 (\sigma_1 \cdot \mathbf{k}'_2)(\sigma_2 \cdot \mathbf{k}_1) - f_1 f'_2 g'_1 g_2 (\sigma_1 \cdot \mathbf{k}_2)(\sigma_2 \cdot \mathbf{k}'_1)$$
$$- g_1 g_2 g'_1 g'_2 (\sigma_1 \cdot (\mathbf{q} \times \mathbf{k}_1))(\sigma_2 . (\mathbf{q} \times \mathbf{k}_2))$$
$$- (f'_1 g_1 - f_1 g'_1)(f_2 g'_2 - f'_2 g_2) \sigma_1 . (\mathbf{q} \times \mathbf{k}_1) \sigma_2 \cdot (\mathbf{q} \times \mathbf{k}_2)]. \tag{15}$$

In the above, we have substituted $\sigma_1 = \sigma \otimes I$ and $\sigma_2 = I \otimes \sigma$. For an identification of the above terms, and otherwise, we may verify the nonrelativistic limit as

$$U_1 \simeq 1 - \left(\frac{1}{8m_1^2} + \frac{1}{8m_2^2} \right) \mathbf{q}^2 + \frac{1}{m_1 m_2} \left(- \mathbf{k}_1 \cdot \mathbf{k}_2 + \frac{(\mathbf{k}_1 \cdot \mathbf{q})(\mathbf{k}_2 \cdot \mathbf{q})}{|\mathbf{q}|^2} \right), \tag{16}$$

$$U_2 \simeq \frac{i}{4m_1^2} \sigma_1 \cdot (\mathbf{q} \times \mathbf{k}_1) - \frac{i}{2m_1 m_2} \sigma_1 \cdot (\mathbf{q} \times \mathbf{k}_2)$$
$$+ \frac{i}{4m_2^2} \sigma_2 \cdot (\mathbf{q} \times \mathbf{k}_2) - \frac{i}{2m_1 m_2} \sigma_2 \cdot (\mathbf{q} \times \mathbf{k}_1), \tag{17}$$

$$U_3 \simeq (\sigma_1 \cdot \sigma_2) \mathbf{q}^2 / (4m_1 m_2), \tag{18}$$

and

$$U_4 \simeq - \frac{1}{4 m_1 m_2} (\sigma_1 \cdot \mathbf{q})(\sigma_2 \cdot \mathbf{q}). \tag{19}$$

The terms in this limit are, as expected, identical (Berestetskii *et al* 1971) except for an apparent change in sign for the terms involving σ_2. This arises due to the difference in the convention regarding the writing of the state. The masses of Q and $\tilde{Q}$ may be different. We note that in the above, U_2 yields the spin orbit term, (2/3) U_3 yields the hyperfine term and (1/3) $U_3 + U_4$ yields the tensor term (Rosner 1981). The completely nonrelativistic version corresponds to $U_1 = 1$ and $U_2 = U_3 = U_4 = 0$. In the present analysis however, we shall use the fully relativistic expressions (12) to (15).

To obtain the total Hamiltonian operator, we now note that the free field

Hamiltonian density is given as

$$\mathscr{H}_0(x) = \sum :\psi_Q^\dagger(x)\,(-i\boldsymbol{\alpha}\cdot\nabla + \beta m_Q)\psi_Q(x): \tag{20}$$

$$\begin{aligned} T &= \int \mathscr{H}_0(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= \int [Q_{I'}(\mathbf{k}_1)^\dagger Q_I(\mathbf{k})p_1^0 - :\tilde{Q}_I(\mathbf{k}_1)^\dagger \tilde{Q}_I(\mathbf{k}_1): p_2^0]\,\mathrm{d}\mathbf{k}_1, \end{aligned} \tag{21}$$

where, obviously, $p_1^0 = (m_1^2 + \mathbf{k}^2)^{1/2}$ and $p_2^0 = (m_2^2 + \mathbf{k}^2)^{1/2}$.

Thus the effective Hamiltonian operator for the quark Q of mass m_1 and antiquark $\tilde{Q}$ of mass m_2 is given by, with (10) and (21),

$$H = T + P. \tag{22}$$

We generate the nonrelativistic limit of the Hamiltonian by substituting e.g. $p_1^0 = m_1 + \mathbf{k}_1^2/(2m_1)$ and using equations (16) to (19), which will yield Schrödinger equation with appropriate spin orbit, hyperfine and tensor interactions. We shall here however retain the relativistic expressions everywhere. We *shall not* however, use differential equations (which will be obviously impossible to solve), but utilize instead the alternative scheme of approximations through choosing a basis and diagonalizing the Hamiltonian operator.

2.2 *The basis for mesons*

We shall now consider the *basis* for the quark-antiquark states. For this purpose, let the radial quantum number be n, and the orbital quantum numbers l and m, we write the $S = 0$ states as (Misra 1978)

$$\begin{aligned} |n, J, J_z, 0\rangle = \frac{1}{\sqrt{6}} \int & \tilde{u}_{nJ}(K_1)\,\mathscr{Y}_{JJ_z}(\hat{k}_1)\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2 \\ & \times \delta(\mathbf{k}_1 + \mathbf{k}_2)\,Q_I^i(\mathbf{k}_1)^\dagger \tilde{Q}_I^i(\mathbf{k}_2)|\mathrm{vac}\rangle. \end{aligned} \tag{23}$$

As earlier, (23) defines a "meson" state of zero momentum. Also $(l, m) = (J, J_z)$, i is the colour index which is summed. $K_1 = |\mathbf{k}_1|$, and $\hat{k}_1 = \mathbf{k}_1/K_1$. Also, the relationship with the basis in coordinate space is given by

$$\tilde{u}_{nl}(K) = (2/\pi)^{1/2} \int u_{nl}(r) j_l(Kr) r^2\,\mathrm{d}r \tag{24}$$

and

$$\mathscr{Y}_{lm}(\Omega) = (-i)^l Y_{lm}(\Omega).$$

Obviously, we have put

$$\tilde{u}_{nl}(K)\,\mathscr{Y}_{lm}(\hat{k}) = (2\pi)^{-3/2} \int u_{nl}(r) Y_{lm}(\hat{r}) \exp(-i\mathbf{k}\cdot\mathbf{r})\,\mathrm{d}\mathbf{r}. \tag{25}$$

We shall substitute $\tilde{u}_{nl}(K)\mathscr{Y}_{lm}(\hat{k}) \equiv \tilde{u}_{nlm}(\mathbf{k})$ and $u_{nl}(r)Y_{lm}(\hat{r}) \equiv u_{nlm}(\mathbf{r})$. These states satisfy the usual ortho-normalization condition. We note that here $\tilde{u}_{nlm}(\mathbf{k})$ or $u_{nlm}(\mathbf{r})$ form a *complete* set of orthonormal basis, such that (23) includes all possible spin singlet states which can be constructed with the given angular momenta. Construction of such states with a field theoretic *notation* in (23) is here merely a matter of convenience, which was used earlier*.

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* Incoherent phenomena with the same type of states as above have also been considered (Misra 1980; Misra and Panda 1980).

We next construct $S = 1$ states. This will involve addition of angular momenta. We however choose a Cartesian basis for spin and write

$$|n, l, JJ_z; \mathbf{0}\rangle = \frac{1}{\sqrt{6}} \int \tilde{u}_{nl}(K_1) g_m(l, J, J_z, \hat{k}_1) \delta(\mathbf{k}_1 + \mathbf{k}_2)\, d\mathbf{k}_1\, d\mathbf{k}_2$$
$$\times Q_I^i((\mathbf{k}_1)^\dagger \sigma_m \tilde{Q}_I^i(\mathbf{k}_2)|\text{vac}\rangle. \tag{26}$$

Here, there is summation over the dummy index m, and, as mentioned, σ_m are the Cartesian (usual) components of the Pauli spin vector. Further, $g_m(l, J; J_z; \hat{k}_1)$ is determined from the equation

$$g_m(l, J, J_z; \hat{k}_1)\sigma_m = \langle lm' 1 m'_1 | JJ_z \rangle \mathscr{Y}_{lm'}(\hat{k}_1)\sigma^P_{m'_1}. \tag{27}$$

In (27), $\sigma^P_{m'_1}$ are the Pauli spin matrices in polar co-ordinates given as

$$\sigma^P_1 = -\frac{1}{\sqrt{2}}(\sigma_1 + i\sigma_2); \quad \sigma^P_0 = \sigma_3; \quad \sigma^P_{-1} = \frac{1}{\sqrt{2}}(\sigma_1 - i\sigma_2). \tag{28}$$

Equation (26) yields all possible "meson" states having $S = 1$ and with zero momentum. Clearly $l = J + 1$, J or $J - 1$ and in contrast to (23), (26) implies that it is a state with $S = 1$. The quantities $g_m(l, J, J_z, \hat{k}_1)$ are explicitly given in Appendix A for $l = J - 1$, J and $J + 1$ for ready reference. Equations (23) and (26) yield a complete basis of states, and, we shall use this basis to obtain the matrix elements of the Hamiltonian operator in (22) which will be diagonalized. We next proceed to consider the matrix elements of H.

2.3 *Matrix elements of the Hamiltonian for vector exchange*

We note that the states as in (26) involve the normalization

$$\langle \mathbf{p} | \mathbf{p}' \rangle = \delta(\mathbf{p} - \mathbf{p}'), \tag{29}$$

and thus when $\mathbf{p} = \mathbf{p}' = 0$, there will be a spurious infinity. In order to get rid of the above and obtain the matrix elements of H or of T and P, we shall use the translational invariance. In fact, we then have, e.g.,

$$\langle n'J'J'_z | T | nJJ_z \rangle = (2\pi)^3 \langle n'J'J'_z; \mathbf{0} | \mathscr{H}_0(0) | nJJ_z; \mathbf{0} \rangle. \tag{30}$$

Similarly,

$$\langle n'J'J'_z | P | nJJ_z \rangle = (2\pi)^3 \langle n'J'J'_z; \mathbf{0} | v(0) | nJJ_z; \mathbf{0} \rangle. \tag{31}$$

These define the conventional matrix elements of $H = T + P$. Obviously from Wigner Eckart theorem the above matrix elements, and those with $|nlJJ_z; \mathbf{0}\rangle$ for $S = 1$ states, will automatically maintain the rotational invariance.

We may notice one aspect of (31) which will soon arise in applications. The confining potentials, strictly speaking, *do not* have Fourier transforms, and this gets reflected in there being *spurious* infrared divergences when we consider the integral in momentum space, as for example will be the case with $1/|\mathbf{q}|^4$ coming from the "Fourier transform" of the linear potential. This aspect will be tackled later. We shall conveniently adopt the harmonic oscillator basis. Thus we take (Davydov 1968)

$$\tilde{u}_{nl}(K) \equiv f_{nl}(K)(RK)^l \exp(-\tfrac{1}{2}R^2K^2), \tag{32}$$

where

$$f_{nl}(K) = (-1)^n N_{nl} R^{3/2} F(-n, l+\tfrac{3}{2}, R^2K^2). \tag{33}$$

In (33),

$$N_{nl} = [2\Gamma(n+l+\tfrac{3}{2})/\Gamma(n+1)]^{1/2}/\Gamma(l+\tfrac{3}{2}) \tag{34}$$

and the hypergeometric function is given as

$$F(-n, l+\tfrac{3}{2}, R^2K^2) = \sum_{k=0}^{n} A(n, l, k)(R^2K^2)^k, \tag{35}$$

where

$$A(n, l, k) = \frac{(-1)^k}{\Gamma(k+1)} \cdot \frac{\Gamma(n+1)}{\Gamma(n-k+1)} \cdot \frac{\Gamma(l+3/2)}{\Gamma(l+k+3/2)}. \tag{36}$$

(i) *Matrix elements for $S = 0$ states*

We shall now utilize equations (30) and (31). The states are given by (23). The matrix elements of T are trivially obtained. In fact, we have, for a given J,

$$\langle n'|T^{(J)}|n\rangle = \int f_{n'J}(K) f_{nJ}(K)(R^2K^2)^J (p_1^0 + p_2^0)\exp(-R^2K^2)\, K^2\, dK, \tag{37}$$

where, $p_1^0 = (m_1^2 + K^2)^{1/2}$ and $p_2^0 = (m_2^2 + K^2)^{1/2}$. The matrix elements for the potential P in (10) will need the evaluation of (11) corresponding to the state (23). In fact, we obtain that

$$\langle n'JJ_Z|P_V|nJJ_Z\rangle \equiv \langle n'|P_V^J|n\rangle \tag{38}$$

$$= (2\pi)^{-3} \int \tilde{u}_{n'J}(K_1')\mathcal{Y}_{JJ_z}(\hat{k}_1')^* F_{PS}^V(\mathbf{k}_1', \mathbf{k}_1) \times \tilde{u}_{nJ}(K_1)\mathcal{Y}_{JJ_z}(\hat{k}_1)\tilde{V}_c^V(\mathbf{q})\, d\mathbf{k}_1'\, d\mathbf{k}_1. \tag{39}$$

In (39) $\tilde{V}_c(\mathbf{q}) = (-4/3)\tilde{V}(\mathbf{q})$, and we have substituted

$$F_{PS}^V(\mathbf{k}_1', \mathbf{k}_1) = \tfrac{1}{2}\mathrm{Tr}[\bar{u}(\mathbf{k}_1')\gamma^\mu u(\mathbf{k}_1)v(-\mathbf{k}_1)\gamma_\mu v(-\mathbf{k}_1')] + \frac{1}{2|\mathbf{q}|^2}\mathrm{Tr}[\bar{u}(\mathbf{k}_1')(\gamma\cdot\mathbf{q})u(\mathbf{k}_1)\bar{v}(-\mathbf{k}_1)(\gamma\cdot\mathbf{q})v(-\mathbf{k}_1')]. \tag{40}$$

In (39) and (40) as before, $\mathbf{k}_1' = \mathbf{k}_1 + \mathbf{q}$. In the context of the potential operator in (10) and (11) we have in fact

$$F_{PS}^V = \tfrac{1}{2}\mathrm{Tr}[U_1 + U_2 + U_3 + U_4]. \tag{40a}$$

Here, with A and B as 2×2 matrices of U_i, $i = 1, 2, 3, 4$ in (12) to (15), and in the context of our definition of the states in (23), we have here used the notation that $\mathrm{Tr}[A\otimes B] = \mathrm{Tr}[AB]$. We have also substituted $\mathbf{k}_2 = -\mathbf{k}_1, \mathbf{k}_2' = -\mathbf{k}_1'$ in the matrices U_i for evaluation of F_{PS}. These can be explicitly verified while evaluating (31). We also substitute $\mathbf{k}_1 = \mathbf{K} - \tfrac{1}{2}\mathbf{q}$, $\mathbf{k}_1' = \mathbf{K} + \tfrac{1}{2}\mathbf{q}$ such that $\mathbf{k}_1'\cdot\mathbf{k}_1 = K^2 - \tfrac{1}{4}Q^2$, where

$Q = |\mathbf{q}|$. Also we take $\mathbf{k}.\mathbf{q} = KQ\cos\theta$. We then put, from rotational invariance, $F_{PS}(\mathbf{k}_1', \mathbf{k}_1) = F_{PS}(K, Q, \cos\theta)$. In the nonrelativistic limits with equations (16) to (19) we explicitly have

$$F^{V}_{PS} \simeq \left(1 + \frac{K^2\sin^2\theta}{m_1 m_2}\right) + \frac{Q^2}{8}\left(\frac{4}{m_1 m_2} - \frac{1}{m_1^2} - \frac{1}{m_2^2}\right). \tag{40b}$$

The relativistic expression is given in Appendix B. We also obtain, *summing* (39) *for different* J_z *values and dividing by* $(2J+1)$, that

$$\begin{aligned}\langle n'|P^J_V|n\rangle &= (2\pi)^{-3}\int f_{n'J}(K_1')f_{nJ}(K_1)(R^2K_1'K_1)^J\cdot\frac{1}{4\pi}\cdot P_J(\cos\theta_1)\\ &\quad\times F^{V}_{PS}(K, Q, \cos\theta)\,\tilde{V}_c(Q)\exp(-R^2K^2 - \tfrac{1}{4}R^2Q^2)\mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{K}\\ &= \frac{1}{4\pi^2}\int f_{n'J}(K_1')f_{nJ}(K_1)(R^2K_1'K_1)^J P_J(\cos\theta_1)\\ &\quad\times F^{V}_{PS}(K, Q, \cos\theta)\,\tilde{V}^{V}_c(Q)\exp(-R^2K^2 - \tfrac{1}{4}R^2Q^2)\\ &\quad\times K^2\mathrm{d}KQ^2\mathrm{d}Q\,\mathrm{d}(\cos\theta).\end{aligned} \tag{41}$$

We note that $|\mathbf{k}_1| = \left(K^2 + \dfrac{Q^2}{4} - KQ\cos\theta\right)^{1/2}$ and $|\mathbf{k}_1'| = \left(K^2 + \dfrac{Q^2}{4} + KQ\cos\theta\right)^{1/2}$.

As earlier stated for a confining potential like ar, $\tilde{V}(Q)$ varies as $1/Q^4$, and thus (41) will have an apparent infrared divergence. This however is spurious, since in coordinate space the corresponding matrix elements converge. This is associated with the fact that ar has *no* Fourier transform. We may replace r by $r\exp(-\lambda r)$, in which case the Fourier transform is $-8\pi(Q^2 - 3\lambda^2)/(Q^2 + \lambda^2)^3$, which has *no* infrared divergence. We can next take the limit $\lambda \to 0$.

We also now discuss an alternative procedure for dealing with the situation. Let us substitute

$$\begin{aligned}F_V(K, Q, \cos\theta) &= f_{n'J}(K_1')f_{nJ}(K_1)(R^2K_1'K_1)^J P_J(\cos\theta_1)\\ &\quad\times F^{V}_{PS}(K, Q, \cos\theta).\end{aligned} \tag{42}$$

When $Q = 0$, we obtain,

$$F_V(K, 0, \cos\theta) = f_{n'J}(K)f_{nJ}(K)(R^2K^2)^J F^{V}_{PS}(K, 0, \cos\theta). \tag{43}$$

We note that terms $(\mathbf{q}\cdot\mathbf{k}_1')(\mathbf{q}\cdot\mathbf{k}_1)/|\mathbf{q}|^2$ introduces $\cos\theta$ dependence for the other terms in (43). Clearly

$$F_V(K, Q, \cos\theta) - F_V(K, 0, \cos\theta) \tag{44}$$

will have a zero in Q, which, we shall see, will in general be adequate to cancel the divergence introduced in $\tilde{V}_c(Q)$ for small Q. We shall thus write

$$\langle n'|P^J_V|n\rangle = \langle n'|P^J_{V0}|n\rangle + \langle n'|\Delta P^J_V|n\rangle, \tag{45}$$

where

$$\langle n'|\Delta P_V^J|n\rangle = \frac{1}{4\pi^2}\int (F_V(K,Q,\cos\theta) - F_V(K,0,\cos\theta))$$

$$\times \exp(-R^2K^2 - \tfrac{1}{4}R^2Q^2)\tilde{V}_c^V(Q)K^2\,\mathrm{d}K\,Q^2\,\mathrm{d}Q\,\mathrm{d}(\cos\theta) \quad (46)$$

and

$$\langle n'|P_{V0}^J|n\rangle = \langle n'|I_{V0}^J|n\rangle V_0^V, \quad (47)$$

where

$$\langle n'|I_{V0}^J|n\rangle = \tfrac{1}{2}\int F_V(K,0,\cos\theta)\exp(-R^2K^2)K^2\,\mathrm{d}K\,\mathrm{d}(\cos\theta), \quad (48)$$

and

$$V_0^V(R) \equiv P_0 \equiv \frac{1}{2\pi^2}\int \tilde{V}_c^V(Q)\exp(-\tfrac{1}{4}R^2Q^2)Q^2\,\mathrm{d}Q \quad (49)$$

$$= \frac{4}{\pi^{1/2}R^3}\int V_c^V(r)\exp(-r^2/R^2)r^2\,\mathrm{d}r. \quad (50)$$

We note that separability of the integrals in the matrix elements of P^J enabled us to evaluate the *formal* integral (49) in coordinate space by equation (50), where the Fourier transform (7) has been utilized. For the QCD potential, usually well defined in the momentum space, a further variation of the technique will be needed, which will be described later at the appropriate place. For the phenomenological potentials defined in coordinate space, (50) will be adequate in dealing with the above spurious divergence. We may also note that P_0 is *independent* of the states under consideration, and, *depends only* on the basis through the variable R^2.

(ii) *Matrix elements for* $S = 1$ *states*

Here, for the matrix elements of H, or of T and P, we have to choose states as given by (26) instead of (23). We first note that corresponding to (21), the l dependence will be absent in T, such that, explicitly from an equation similar to (30), we get

$$\langle n'l'|T^l|nl\rangle = \delta_{ll'}\int f_{n'l}(K)f_{nl}(K)(R^2K^2)^l(p_1^0 + p_2^0)\exp(-R^2K^2)K^2\,\mathrm{d}K. \quad (51)$$

Equation (51) is parallel to (37) written down earlier.

We next consider the matrix elements of P in equation (10), which as before, will need the evaluation of $\langle n', l', J', J_z'; \mathbf{0}|v(0)|n, l, J, J_z; \mathbf{0}\rangle$ parallel to (31). Looking at the states as in (26), we note that for this purpose we shall need the expression

$$F_{m'm}^V(\mathbf{k}_1', \mathbf{k}_1) = \tfrac{1}{2}\mathrm{Tr}[\bar{u}(\mathbf{k}_1')\gamma^\mu u(\mathbf{k}_1)\sigma_m\bar{v}(-\mathbf{k}_1)\gamma_\mu v(-\mathbf{k}_1')\sigma_{m'}]$$
$$+ \frac{1}{2|\mathbf{q}|^2}\mathrm{Tr}[\bar{u}(\mathbf{k}_1')(\gamma\cdot\mathbf{q})u(\mathbf{k}_1)\sigma_m\bar{v}(-\mathbf{k}_1)(\gamma\cdot\mathbf{q})$$
$$v(-\mathbf{k}_1')\sigma_{m'}] \quad (52)$$

which is the parallel of (40). In fact, we really have, in the context of potential operator (10) and (11), the relationship

$$F_{m'm}^V(\mathbf{k}_1', \mathbf{k}_1) = \tfrac{1}{2}\mathrm{Tr}[(U_1 + U_2 + U_3 + U_4)\sigma_m \otimes \sigma_{m'}] \quad (52a)$$

where, as in (40), $\mathbf{k}_2 = -\mathbf{k}_1$ and $\mathbf{k}_2' = -\mathbf{k}_1'$ in equations (12) to (15). Further, as earlier we have used the notation that

$$\mathrm{Tr}\,[(A\otimes B)\sigma_m\otimes\sigma_{m'}] = \mathrm{Tr}\,[A\sigma_m B\sigma_{m'}]$$

as is obvious in the context of definitions of the states in (26). The relativistic expression for $F^V_{m'm}$ is explicitly given in Appendix B. In the nonrelativistic limits with equations (16) to (19), we explicitly have,

$$F^V_{m'm} = \delta_{m'm}\left[1-\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)\mathbf{q}^2+\frac{K^2\sin^2\theta}{m_1m_2}\right]$$
$$+\left(\frac{1}{4m_1^2}+\frac{1}{4m_2^2}+\frac{1}{m_1m_2}\right)(q_{m'}K_m - K_{m'}q_m) - \frac{1}{2m_1m_2}q_{m'}q_m. \quad (52b)$$

We then obtain, parallel to equation (39)

$$\langle n'l'|P^J|nl\rangle \equiv (2\pi)^3\int \tilde{u}_{n'l'}(K'_1)\,g^*_{m'}(l',J,J_z,\hat{k}'_1)\,F^V_{m'm}(\mathbf{k}'_1,\mathbf{k}_1)$$
$$\times\, g_m(l,J,J_z,\hat{k}_1)\,\tilde{u}_{nl}(K_1)\,\tilde{V}_c(\mathbf{q})\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}'_1. \quad (53)$$

In (53), summation over m and m' is understood; however, J_z remains fixed, although, as earlier, from rotational invariance the expressions are *independent* of J_z.

We next substitute, for convenience,

$$F^V_V(\mathbf{k}'_1,\mathbf{k}_1) = \frac{4\pi}{2J+1}\sum_{J_z}\sum_{m',m} g^*_{m'}(l',J,J_z,\hat{k}'_1)\,F^V_{m'm}(\mathbf{k}'_1,\mathbf{k}_1)g_m(l,J,J_z,\hat{k}_1)$$
$$\equiv \sum A^V_{m'm}\,F^V_{m'm}(\mathbf{k}'_1,\mathbf{k}_1). \quad (54)$$

The expression for $A^V_{m'm}$ for different l and J values are given in Appendix A. We then have, for triplet matrix elements,

$$\langle n'l'|P^J_V|nl\rangle = \frac{1}{4\pi^2}\int f_{n'l'}(K'_1)f_{nl}(K_1)(RK'_1)^{l'}(RK_1)^{l}\,F^V_V(K,Q,\cos\theta)$$
$$\times\,\tilde{V}^V_c(Q)\exp(-R^2K^2-\tfrac{1}{4}R^2Q^2)K^2\,\mathrm{d}K\,Q^2\,\mathrm{d}Q\,\mathrm{d}(\cos\theta). \quad (55)$$

This equation is the parallel of equation (41). In $F^V_V(\mathbf{k}'_1,\mathbf{k}_1)$ the superscript V represents the interaction is due to vector exchange and subscript V represents the state is spin 1 state. The complete expression for $F^V_V(\mathbf{k}'_1,\mathbf{k}_1)$ is given in Appendix B. As before, it will have a *spurious* infrared divergence, which can be tackled in the same way. Here, instead of (42), we shall have to make the substitution.

$$F_V(K,Q,\cos\theta) = f_{n'l'}(K'_1)f_{nl}(K_1)(RK'_1)^{l'}(RK_1)^{l}\,F^V_V(K,Q,\cos\theta) \quad (56)$$

and proceed in the same manner parallel to equations (42) to (50). In particular we may note that P_0 of equation (50) remains unaltered; however the other expressions need to be appropriately modified.

2.4 *Matrix element for the Hamiltonian for scalar exchange*

For the sake of completeness, we supplement an interaction potential due to exchange of a scalar particle to the earlier described interaction potential due to vector exchange

(Gupta *et al* 1982). We then have parallel to (38), the contribution of P_s^J from scalar interaction obtained with $F_{PS}^V(\mathbf{k}_1', \mathbf{k}_1)\tilde{V}_c^V(Q)$ being replaced by $F_{PS}^S(\mathbf{k}_1', \mathbf{k}_1)\tilde{V}_c^S(Q)$. Here,

$$F_{PS}^S(\mathbf{k}_1', \mathbf{k}_1) = -\tfrac{1}{2}\mathrm{Tr}\,[\bar{u}(\mathbf{k}_1')u(\mathbf{k}_1)\bar{v}(-\mathbf{k}_1)v(-\mathbf{k}_1')], \tag{57}$$

which is more explicitly given in Appendix B. We may note that the negative sign arises due to the normalization of spinors. Now parallel to (19) we have the nonrelativistic limit of (57) as

$$F_{PS}^S(\mathbf{k}_1', \mathbf{k}_1) = 1 - \left(\frac{1}{2m_1^2} + \frac{1}{2m_2^2}\right)K^2. \tag{58}$$

Also $F_V(K, Q, \cos\theta)$ and $F_{PS}^V(K, Q, \cos\theta)$ in equation (42) are replaced by $F_S(K, Q, \cos\theta)$ and $F_{PS}^S(K, Q, \cos\theta)$ respectively. Further, in (45) and (46), P_V^J and ΔP_V^J will be replaced by P_S^J and ΔP_S^J respectively. The potential $\tilde{V}_c^V(Q)$ for vector exchange interaction will be replaced by $\tilde{V}_C^s(Q)$ in this case. For spin 1 states, parallel to (52) there will be $F_{m'm}^s(\mathbf{k}_1', \mathbf{k}_1)$ written as

$$F_{m'm}^s(\mathbf{k}_1', \mathbf{k}_1) = -\tfrac{1}{2}\mathrm{Tr}\,[\bar{u}(\mathbf{k}_1')u(\mathbf{k}_1)\sigma_m\bar{v}(-\mathbf{k}_1)v(-\mathbf{k}_1')\sigma_{m'}] \tag{59}$$

which is more explicitly given in Appendix B. However, we give the expression for the nonrelativistic limit of (59) as,

$$F_{m'm}^s(\mathbf{k}_1', \mathbf{k}_1) = \delta_{m'm}\left(1 - \left(\frac{1}{2m_1^2} + \frac{1}{2m_2^2}\right)K^2\right) - \left(\frac{1}{4m_1^2} + \frac{1}{4m_2^2}\right)\varepsilon_{imm'}(\mathbf{q}\times\mathbf{k})_i. \tag{60}$$

Parallel to (53) we have now,

$$\langle n'l'|P_S^J|nl\rangle = (2\pi)^3\int \tilde{u}_{n'l'}(K_1')g_{m'}(l', J, J_z, \hat{k}_1')F_{m'm}^s(\mathbf{k}_1', \mathbf{k}_1) \times g_m(l, J, J_z, \hat{k}_1)\tilde{u}_{nl}(K_1)\tilde{V}_c^s(\mathbf{q})\,\mathrm{d}\mathbf{k}_1'\,\mathrm{d}\mathbf{k}_1. \tag{61}$$

In equation (54) $F_{m'm}^V(\mathbf{k}_1', \mathbf{k}_1)$ will be replaced by $F_{m'm}^s(\mathbf{k}_1', \mathbf{k}_1)$. There will be another expression $F^s(K, Q, \cos\theta)$ parallel to (56) due to scalar exchange.

For spin zero states now the matrix element for total potential is written as

$$\begin{aligned}\langle n'|P|n\rangle &= \langle n'|P_V^J|n\rangle + \langle n'|P_S^J|n\rangle\\ &= \frac{1}{4\pi^2}\int f_{n'J}(K_1')f_{nJ}(K)(R^2KK')^l P_J(\cos\theta_1)\\ &\quad\times[F_{PS}^V(K, Q, \cos\theta)\tilde{V}_c^V(Q) + F_{PS}^S(K, Q, \cos\theta)\tilde{V}_c^S(Q)]\\ &\quad\times\exp(-R^2K^2 - \tfrac{1}{4}R^2Q^2)K^2\mathrm{d}K\,Q^2\mathrm{d}Q\,\mathrm{d}(\cos\theta).\end{aligned} \tag{62}$$

Similarly for spin 1 case, the matrix elements for the total potential is given by,

$$\langle n'|P^J|n\rangle = \langle n'l'|P_V^J|nl\rangle + \langle n'l'|P_S^J|nl\rangle$$

$$= \frac{1}{4\pi^2} \int f_{n'l'}(K_1')f_{nl}(K_1)(RK_1')^{l'}(RK_1)^{l}[F_V^V(K, Q, \cos\theta)$$
$$\times \tilde{V}_c^V(Q) + F_V^S(K, Q, \cos\theta)\, V_c^S(Q)]\exp(-R^2K^2 - \tfrac{1}{4}R^2Q^2)$$
$$\times K^2 \mathrm{d}K\, Q^2 \mathrm{d}Q\, \mathrm{d}(\cos\theta). \qquad (63)$$

3. Approximation schemes

We have given here the general framework for the sake of completeness. $S = 0$ and $S = 1$ states do not mix due to CP invariance. Then, with specific values of J^P, we can construct the matrix elements of H corresponding to a *finite* subset of the basis, and diagonalize the Hamiltonian. For a large enough basis, we expect progressive convergence for the masses of the low-lying levels. However, we may note that R^2 is an arbitrary parameter of the basis, and the dynamics, including the mass levels, has to be independent of this. Further, a small subset of the basis may yield more *rapid* convergence to the actual wave functions for a suitable value of R^2 as compared to other values. Hence, we shall obviously adopt the variational procedure to optimize the basis in the context of the eigenvalues; in fact, for each finite choice of elements of the basis, we shall first do so. The rate of convergence of the results when we change the number of elements of the basis will broadly indicate the degree of reliability of the corresponding values quoted. We shall now develop an approximation which is relativistic, but depends on the dominance of the values of the matrix elements on small values of the variable Q. Such an approximation may be sometimes reasonable since $\tilde{V}(Q)$ has terms like $1/Q^2$ or $1/Q^4$.

(a) *Small Q dominance*

To make this approximation, we first note that $F(K, Q, \cos\theta)$ in (46) is effective always even in $\cos\theta$. Hence we can use the expansion in Q^2 yielding

$$F(K, Q, \cos\theta) - F(K, 0, \cos\theta) = -Q^2 F_1(K, \cos\theta). \qquad (64)$$

One *then* has from (46)

$$\langle n'|\Delta P^J|n\rangle = \langle n'|I_1|n\rangle\, V_1, \qquad (65)$$

where

$$V_1 = V_1(R) = \frac{R^2}{2\pi^2} \int (-Q^2\tilde{V}(Q))\exp(-R^2Q^2/4)\,Q^2\mathrm{d}Q, \qquad (66)$$

and

$$\langle n'|I_1^J|n\rangle = \frac{1}{2R^2} \int F_1(K, \cos\theta)\exp(-R^2K^2)K^2\mathrm{d}K\,\mathrm{d}(\cos\theta). \qquad (67)$$

In (66) for the Cornell potential, or Martin potential, the momentum space integration has no spurious divergence. In case we find this to be there, we can use equation (7) and write the parallel of equation (50). As mentioned, the method here includes relativistic corrections, but depends crucially on the dominance of the matrix elements for small Q. One can still make a nonrelativistic approximation for

$F_1(K, \cos\theta)$ in (64) and derive the conventional results of Schrödinger equation, including the corrections mentioned in equation (16) to (19).

We may now consider e.g. the Cornell potential (Eichten *et al* 1978) given as

$$V_c(r) = -\frac{K_0}{r} + ar. \tag{68}$$

From (7) we have

$$\tilde{V}_c(Q) = -\left[\frac{4\pi K_0}{Q^2} + \frac{8\pi a}{Q^4}\right]. \tag{69}$$

Also, we get

$$V_0|_{\text{Cornell}} = \frac{2}{\sqrt{\pi}\,R}[-K_0 + aR^2], \tag{70}$$

and

$$V_1|_{\text{Cornell}} = \frac{4}{\sqrt{\pi}\,R^3}[K_0 + aR^2]. \tag{71}$$

For the Martin potential (Martin 1980) given as

$$V_c(r) = A + Br^{\alpha}, \tag{72}$$

we similarly get

$$V_0|_{\text{Martin}} = A + B\cdot\frac{2R^{\alpha}}{\sqrt{\pi}}\Gamma\left(\frac{3+\alpha}{2}\right), \tag{73}$$

and

$$V_1|_{\text{Martin}} = B\cdot\frac{2}{\sqrt{\pi}}\alpha(\alpha+1)R^{\alpha}\Gamma\left(\frac{\alpha+1}{3}\right). \tag{74}$$

We next note that for the Richardson potential (Richardson 1979) from (5) and (6) we get

$$\tilde{V}_c(Q) = -\frac{4}{3}\cdot\frac{48\pi^2}{(33-2n_f)}\cdot\frac{1}{Q^2\ln(1+Q^2/\Lambda^2)}. \tag{75}$$

For small Q, we clearly have

$$\tilde{V}_c(Q) = -\frac{4}{3}\cdot\frac{48\pi^2}{(33-2n_f)}\cdot\frac{\Lambda^2}{Q^4}, \tag{76}$$

such that from (69) this part behaves in coordinate space like ar with

$$a = \frac{4}{3}\cdot\frac{6\pi\Lambda^2}{(33-2n_f)}.$$

Separating this, we thus obtain from (49) the value of V_0 as

$$V_0|_{\text{Richardson}} = \frac{2R}{\sqrt{\pi}}\cdot\frac{4}{3}\cdot\frac{6\pi\Lambda^2}{(33-2n_f)} + I_{\text{Richardson}}, \tag{77}$$

where from (49) and (70)

$$I_{\text{Richardson}} = \frac{1}{2\pi^2}\cdot\left(-\frac{4}{3}\right)\cdot\frac{48\pi^2}{(33-2n_f)}\cdot\int\left\{\frac{1}{\ln(1+Q^2/\Lambda^2)}-\frac{\Lambda^2}{Q^2}\right\} \times \exp(-R^2Q^2/4)Q\,\mathrm{d}Q. \tag{78}$$

(b) *Nonrelativistic limit*

The nonrelativistic expressions for $F^{V}_{V}(K, Q, \cos\theta)$, $F^{V}_{PS}(K, Q, \cos\theta)$, $F^{S}_{V}(K, Q, \cos\theta)$ and $F^{S}_{PS}(K, Q, \cos\theta)$ are explicitly given in Appendix B. We can easily compare our expressions due to hyperfine, spin-orbit and tensor interactions with the earlier standard calculations (Rosner 1981). From equation (40b) and (74), the splitting due to the hyperfine interaction is

$$\Delta H|_{\text{hyperfine}} = -\frac{2Q^2}{3m^2}\tilde{V}_c(Q). \tag{79}$$

In our usual Fourier transform technique

$$\Delta H|_{\text{hyperfine}} = \frac{2}{3m^2}\nabla^2 V(r). \tag{80}$$

From equation (B.13), (B.17) and (B.19) in Appendix B we can compare the expressions due to dyadic, hyperfine, spin-orbit and tensor interactions for different ^3P_J states.

The expressions due to the dyadic term which arises due to the four-component Dirac spinor is the same for all the ^3P_J states. It does not give rise to any splitting and is given by

$$\Delta F^{V}_{J}(K, Q, \cos\theta)|_{\text{Dyadic}} = 1 + \frac{K^2\sin^2\theta}{m^2} - \frac{Q^2}{4m^2}. \tag{81}$$

The expressions due to spin-orbit interaction for all the ^3P_J states are given by,

$$R^2K_1K_1'\Delta F^{V}_{J}(K, Q, \cos\theta)\Big|^{\text{s.o.}}_{^3P_0} = \frac{3}{2m^2}R^2K^2Q^2\sin^2\theta, \tag{82}$$

$$R^2K_1K_1'\Delta F^{V}_{J}(K, Q, \cos\theta)\Big|^{\text{s.o.}}_{P_1} = \frac{3}{4m^2}R^2K^2Q^2\sin^2\theta, \tag{83}$$

and

$$R^2K_1K_1'\Delta F^{V}_{J}(K, Q, \cos\theta)\Big|^{\text{s.o.}}_{^3P_2} = -\frac{3}{4m^2}R^2K^2Q^2\sin^2\theta. \tag{84}$$

(c) *Extreme relativistic limit*

In extreme relativistic limit we make a very crude approximation (Bardeen *et al* 1975) by taking quark mass to be zero. In this approximation, the expression for p^0, $f(\mathbf{K})$ and $g(\mathbf{K})$ in equations (9a) and (9b) get simplified as $p^0 = |(\mathbf{K})|$, $f(\mathbf{K}) = 1/\sqrt{2}$ and $g(K) = |\mathbf{K}|/\sqrt{2}$.

To get the matrix elements for kinetic energy, we take $p_1^0 = p_2^0 = p^0 = K$ in equation (51) and easily evaluate the integral. The simplified expressions in this approximation scheme for $F^{V}_{PS}(\mathbf{k}_1, \mathbf{k}_1')$, $F^{S}_{PS}(\mathbf{k}_1, \mathbf{k}_1')$, $F^{V}_{V}(\mathbf{k}_1, \mathbf{k}_1')$ and $F^{S}_{V}(\mathbf{k}_1, \mathbf{k}_1')$, can be found out using (B.2a), (B.5a), (B.10a), (B.12a), (B.16a), (B.23a), (B.25a) and (B.29a) of Appendix B.

4. Applications

We now apply our model and the approximation scheme to calculate different mass levels and compare them with the experiments wherever available. To do this, as mentioned earlier, we test the convergence of the minimum energy of a particular state with respect to the number of matrix elements. To test the reliability of our calculations we have applied our scheme to some known calculations, that of Martin (1980) and Bhaduri and Brack (1982). Next, we optimize our potential parameters with respect to the exact relativistic calculations and use the same parameters for nonrelativistic calculations of heavy quark systems.

4.1 *Illustrations for different potentials*

(i) *Martin potential*

As described our scheme can be applicable to any potential. To test the reliability of our scheme we do some known calculations. Here, we take that of Martin (1980) with the potential given in equation (72), and using our scheme calculate the corresponding spin average mass levels and find that they are similar to that of Martin's as given in table 1.

(ii) *Bhaduri's model*

Next, we compare calculations in our scheme with Bhaduri's model (Bhaduri and Brack 1982) and find that they also agree, which may be seen from table 2. We may remark that

Table 1. Comparison of our calculation with Martin's potential with that of Martin's parameters are: $A = -8{\cdot}064$, $B = 6{\cdot}681$, $\alpha = 0{\cdot}1$, $m_c = 1{\cdot}8$ GeV, $m_b = 5{\cdot}174$ GeV and $m_s = 0{\cdot}518$ GeV (Martin 1980).

States	Our calculation (in GeV)	Martin's calculation (in GeV)
J/ψ	3·086	3·095
ψ'	3·691	3·687
ψ''	4·050	4·032
ψ'''	4·307	4·307
Average P-state	3·510	3·502
Average D-state	3·793	3·787
Υ	9·46	9·46
Υ'	10·040	10·025
Υ''	10·38	10·36
Υ'''	10·62	10·60
Υ^{IV}	10·85	10·76
$1P$	9·871	9·851
$2P$	10·250	10·242
ϕ	0·99	1·02
ϕ'	1·626	1·634
$E(1^{++})$	1·43	1·42

Table 2. Comparison of Bhaduri's results with our calculation, with same parameters (Bhaduri and Brack 1982). $m_q = 0.5$, $a_0 = 0.054$ in GeV units.

States	Our calculation	Bhaduri's calculation
$1S$	0·334	0·331
$2S$	0·584	0·579
$3S$	0·788	0·772
$4S$	0·975	0·961
$1P$	0·480	0·476
$1d$	0·606	0·602
$1f$	0·721	0·715
$1g$	0·827	0·821

Bhaduri showed that non-relativistic Schrödinger's equation with some effective mass and effective potential can simulate the solutions of relativistic Dirac equation of a massless particle with a known scalar potential, taken as $V = a_0 r$. We have here shown the same by variational method.

These two calculations give quite a good amount of reliability and confidence in our calculations.

4.2 *Heavy quark: Relativistic effects (present model)*

With the test of reliability of our scheme as mentioned in § 4.1 we do the calculations for relativistic heavy quark systems. For this, we choose a Cornell (Eichten *et al* 1978) type of potential (68) for the interaction due to the exchange of a vector particle. We recognize that in all applications, a scalar potential is needed. Thus we supplement the above potential with a purely linear confining potential and a constant to scalar exchange potential as has been taken by (Gupta *et al* 1982), written as

$$V_c^S(r) = a_{0s} r - A_c, \tag{85}$$

which in momentum space becomes

$$\tilde{V}_c^S(Q) = -\frac{8\pi a_{0s}}{Q^4} - (2\pi)^3 \cdot A_c \delta(\mathbf{q}). \tag{86}$$

The calculated results correspond to this sum of the two potentials, and, thus we optimize the four-potential parameters along with the quark mass to get the lowest lying levels of charmonium. Also, we calculate the masses of the excited states and ${}^1P_1$ state, as given in table 3 along with the experimental data (Particle Data Group 1984). We may note here that our prediction for the mass of ${}^1P_1$ state agrees with the limit given by the Crystal Ball Data (Porter 1982). Further, we calculate the wave function at the origin and use them along with the Van Royen and Weiskopf (1967) formula, to calculate the leptonic widths as well as the photonic widths of ${}^1S_0$ and ${}^3S_1$ states. We also compare our calculations against the experimental data (Particle Data Group 1984) whenever available as in table 4. We may note here that the agreement appears to be reasonably good. We would like to note here that our estimation of $\Gamma(\eta_c \to \gamma\gamma)$ agrees with that of Rosner (1981). Also, our calculations for $\Gamma(J/\psi \to 3\gamma)$ agrees well with the

Table 3. $c\bar{c}$ spectrum with $m_c = 1.648$, $a_0 = a_{0s} = 0.11$, $K = 0.46$, $A_c = 0.69$.

State	Calculated mass in GeV	R^2 in GeV^{-2}	Experimental values in GeV
$1\,^1S_0\,(\eta_c)$	2.993	1.5	2.981 ± 0.006
$1\,^3S_1\,(\psi)$	3.109	2.0	3.097 ± 0.0001
$1\,^3P_0\,(\chi_0)$	3.410	2.0	3.415 ± 0.0001
$1\,^3P_1\,(\chi_1)$	3.500	3.0	3.510 ± 0.00006
$1\,^3P_2\,(\chi_2)$	3.544	4.0	3.5558 ± 0.00006
$1\,^1P_1\,(\;)$	3.546	5.0	
$2\,^1S_0\,(\eta_c')$	3.636	3.0	3.590
$2\,^3S_1\,(\psi')$	3.710	3.5	3.686 ± 0.0001

Table 4. Electromagnetic width.

Decay mode	Calculated width (in keV)	Experimental width (in keV)
$\Gamma(\psi \to e^+e^-)$	4.992	4.662 ± 0.756
$\Gamma(\Upsilon \to e^+e^-)$	1.208	1.176 ± 0.462
$\Gamma(\psi' \to e^+e^-)$	2.855	1.36 – 2.55
$\Gamma(\Upsilon' \to e^+e^-)$	0.4040	0.51 ± 0.18
$\Gamma(\eta_c \to \gamma\gamma)$	6.656	
$\Gamma(\eta_b \to \gamma\gamma)$	1.610	
$\Gamma(\psi \to \gamma\gamma\gamma)$	3.709×10^{-3}	$\leqslant 3.5 \times 10^{-3}$

experiment (Rosner 1981; Partridge *et al* 1980). The leptonic width formula (Van Royen and Weiskopf 1967) we use is written as

$$\Gamma(V \to e^+e^-) = \frac{16\pi\alpha^2 e_q^2}{M^2(q\bar{q})} . |\psi(0)|^2. \tag{88}$$

$\psi(0)$ is the wave function at the origin given by $\psi(0) = \Sigma C_n U_{n00}(0)$. $U_{n00}(0)$ is calculated from (24) and C_{ns}' are eigenvectors of $\langle n'|H^J|n\rangle$. The photonic width formula (Rosner 1981) for $^1S_0$ and $^3S_1$ states are respectively written as

$$\Gamma(^1S_0 \to 2\gamma) = \tfrac{4}{3}\,\Gamma(^3S_1 \to e^+e^-), \tag{89}$$

and

$$\Gamma(^3S_1 \to 3\gamma) = \frac{2^8(\pi^2 - 9)}{3^7} \cdot \alpha\Gamma(^3S_1 \to e^+e^-). \tag{90}$$

Observing a reasonable agreement of our calculations with the experimental data (Particle Data Group 1984) for charmonium family, we now proceed in a similar manner to do our calculations and the optimization for upsilon family. We note here that we need here a different set of potential parameters (Gupta and Radford 1982) and the quark mass for the optimization of lowest lying states. Then as in the case of charmonium family we make our calculations for excited states and compare them with experiments wherever available as in table 5. Further we also calculate the electromagnetic widths for Υ-family using (87)–(89) and compare them with the experiment

Table 5. bb^- spectrum with $m_b = 5{\cdot}005$, $a_0 = 0{\cdot}16$, $K = 0{\cdot}288$, $a_{0s} = 0{\cdot}187$, $A_c = 0{\cdot}954$.

State	Calculated mass (in GeV)	R^2 (in GeV^{-2})	Experimental mass (in GeV)
$1\,^1S_0$	9·410	0·6	
$1\,^3S_1$	9·460	0·6	$9{\cdot}460 \pm 0{\cdot}0003$
$1\,^3P_0$	9·869	1·1	$9{\cdot}8729 \pm 0{\cdot}0058$
$1\,^3P_1$	9·896	0·6	$9{\cdot}8945 \pm 0{\cdot}0035$
$1\,^3P_2$	9·912	0·6	$9{\cdot}9146 \pm 0{\cdot}0024$
$2\,^1S_0$	10·04	1·1	
$2\,^3S_1$	10·07	1·1	$10{\cdot}0234 \pm 0{\cdot}0003$
$2\,^3P_0$	10·35	1·5	
$2\,^3P_1$	10·37	1·5	$10{\cdot}2537 \pm 0{\cdot}0034$
$2\,^3P_2$	10·38	1·5	$10{\cdot}271 \pm 0{\cdot}0024$

wherever available as in table 4. We may note here that our calculation agrees reasonably with the experimental data.

Next, with the parameters as above, we do the nonrelativistic calculations for heavy quark systems i.e. for charmonium and upsilon systems, to estimate the magnitude of the fully relativistic corrections. We find that these corrections over nonrelativistic calculations are of the order of 50 MeV for charmonium family and 20 MeV for the upsilon family, which is similar to the magnitude of the splittings, and are *not* the same for all sublevels. Thus, with this we do feel that the relativistic corrections for the light quark systems will be quite relevant.

5. Discussion

We have derived the effective Hamiltonian in field theory. Next, we have used it retaining all relativistic effects. Further, we have done the diagonalization of the expectation values of the Hamiltonian instead of solving differential equations, for obtaining mass levels of mesons. For this, we have taken different phenomenological potentials as the quark-antiquark potential is not yet known. In our calculations, we notice a significant effect due to the totally relativistic calculations, over the Fermi-Breit approximation.

We may also find out a major difference of our model in contrast to most of the models. The potential parameters are taken to be fixed except for a variation dependent when we go from charmonium to upsilon system, as expected on the basis of renormalization group equation (e.g. Gupta and Radford 1982). However the radial wave function has been optimized separately for each radial or angular quantum number.

Although good agreement was obtained for heavy quark systems, this was not possible for light quark systems. This is surprising, since our calculations include relativistic effects. We attribute this to our inadequate knowledge of the structure of dynamics at low energies for quantum chromodynamics (QCD). In case we derive a potential from QCD, our relativistic format for bound state will be useful. For this purpose, in Appendix B, we have included the expressions for the relevant contributions when quarks are massless.

An alternative way of looking at the bound state is also possible in terms of Lagrangian density (Misra 1985) which is a covariant formalism and an equivalent approach through Hamiltonian density (Misra and Panda 1986). Further attempts in this direction are in progress. Also the present technique can be applied to the positronium system. However, since this method is numerical, and exact analytical methods exist, such an exercise will be merely a verification in addition to those quoted in §4.

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Appendix A. Angular momentum coefficients

From equation (7) we have,

$$g_m(l, J, J_z, \hat{k}_1)\sigma_m = \langle lm' 1 m'_1 | J J_z \rangle \mathscr{Y}_{lm'} \sigma^P_{m'_1}. \tag{A.1}$$

For $J = l - 1$,

$$g_1(l, J, J_z, \hat{k}_1) = (-i)^l \cdot \frac{1}{\sqrt{2}} \cdot \left\{ Y_{l, J_z+1}(\hat{k}_1) \left(\frac{(l+J_z+1)(l+J_z)}{2l(2l+1)} \right)^{1/2} - Y_{l, J_z-1}(\hat{k}_1) \left(\frac{(l-J_z)(l-J_z+1)}{2l(2l+1)} \right)^{1/2} \right\}. \tag{A.2}$$

$$g_2(l, J, J_z, \hat{k}_1) = (-i)^l \cdot \frac{-i}{\sqrt{2}} \left\{ Y_{l, J_z+1}(\hat{k}_1) \left(\frac{(l+J_z+1)(l+J_z)}{2l(2l+1)} \right)^{1/2} + Y_{l, J_z-1}(\hat{k}_1) \left(\frac{(l-J_z)(l-J_z+1)}{2l(2l+1)} \right)^{1/2} \right\}, \tag{A.3}$$

$$g_3(l, J, J_z, \hat{k}_1) = (-i)^l Y_{l, J_z-1}(\hat{k}_1) \left(\frac{(l^2 - J_z^2)}{l(2l+1)} \right)^{1/2}. \tag{A.4}$$

For $J = 1$

$$g_1(l, J, J_z, \hat{k}_1) = (-i)^l \cdot \frac{1}{\sqrt{2}} \cdot \left\{ Y_{l, J_z+1}(\hat{k}_1) \left(\frac{(l-J_z)(l+J_z+1)}{2l(l+1)} \right)^{1/2} - Y_{l, J_z-1}(\hat{k}_1) \left(\frac{(l+J_z)(l+J_z+1)}{2l(l+1)} \right)^{1/2} \right\}, \tag{A.5}$$

$$g_2(l, J, J_z, \hat{k}_1) = (-i)^l \cdot \frac{-i}{\sqrt{2}} \left\{ Y_{l, J_z+1}(\hat{k}_1) \left(\frac{(l-J_z)(l+J_z+1)}{2l(l+1)} \right)^{1/2} - Y_{l, J_z-1}(\hat{k}_1) \left(\frac{(l+J_z)(l-J_z+1)}{2l(l+1)} \right)^{1/2} \right\}, \tag{A.6}$$

$$g_3(l,J,J_z,\hat{k}_1)=(-i)^l Y_{l,J_z}(\hat{k}_1)\frac{J_z}{\sqrt{l(l+1)}}. \tag{A.7}$$

For $J=l+1$,

$$g_1(l,J,J_z,\hat{k}_1)=(-i)^l\cdot\frac{1}{\sqrt{2}}\cdot\left\{Y_{l,J_z+1}(\hat{k}_1)\left(\frac{(l-J_z)(l-J_z+1)}{(2l+1)(2l+2)}\right)^{1/2}\right.$$
$$\left.-Y_{l,J_z-1}(\hat{k}_1)\left(\frac{(l+J_z)(l+J_z+1)}{(2l+1)(2l+2)}\right)^{1/2}\right\}. \tag{A.8}$$

$$g_2(l,J,J_z,\hat{k}_1)=(-i)^l\cdot\frac{-i}{\sqrt{2}}\left\{Y_{l,J_z+1}(\hat{k}_1)\left(\frac{(l-J_z)(l-J_z+1)}{(2l+1)(2l+2)}\right)^{1/2}\right.$$
$$\left.+Y_{l,J_z-1}(\hat{k}_1)\left(\frac{(l+J_z)(l+J_z+1)}{(2l+1)(2l+2)}\right)^{1/2}\right\}, \tag{A.9}$$

$$g_3(l,J,J_z,\hat{k}_1)=(-i)^l Y_{l,J_z}(\hat{k}_1)\left(\frac{(l-J_z+1)(l+J_z+1)}{(2l+1)(l+1)}\right)^{1/2}. \tag{A.10}$$

From equation (54), we have

$$A^V_{m'm}(\hat{k}'_1,\hat{k}_1)=\frac{4\pi}{2J+1}\sum_{J_z} g^*_{m'}(l',J,J_z,\hat{k}')g_m(l,J,J_z,\hat{k}_1). \tag{A.11}$$

For $l=0$ and $J=1$, i.e. for ${}^3S_1$ state, we have

$$A^V_{m'm}(\hat{k}'_1,\hat{k}_1)=\frac{1}{3}\delta_{m'm}. \tag{A.12}$$

For $l=1$ and $J=0$, i.e. for ${}^3P_0$ state we have,

$$A^V_{m'm}(\hat{k}'_1,\hat{k}_1)=\hat{k}'_{1m'}\hat{k}_{1m}. \tag{A.13}$$

For $l=1$ and $J=1$, i.e. for ${}^3P_1$ state we have,

$$A^V_{m'm}(\hat{k}'_1\cdot\hat{k}_1)=\tfrac{1}{2}(\hat{k}'_1,\hat{k}_1)\delta_{m'm}-\tfrac{1}{2}\hat{k}_{1m'}\hat{k}'_{1m}. \tag{A.14}$$

For $l=1$ and $J=2$, i.e. for ${}^3P_2$ state we have

$$A^V_{m'm}(\hat{k}'_1,\hat{k}_1)=[\tfrac{3}{10}(\hat{k}'_1\cdot\hat{k}_1)\delta_{m'm}-\tfrac{1}{5}\hat{k}'_{1m'}\hat{k}_{1m}+\tfrac{3}{10}\hat{k}'_{1m}\hat{k}_{1m'}]. \tag{A.15}$$

Appendix B. Relativistic corrections

From equation (40) we have

$$F^V_{PS}(\mathbf{k}'_1,\mathbf{k}_1)=\tfrac{1}{2}\mathrm{Tr}\,[\bar{u}(\mathbf{k}'_1)\gamma^\mu u(\mathbf{k}_1)\bar{v}(-\mathbf{k}_1)\gamma_\mu v(-\mathbf{k}'_1)]$$
$$+\frac{1}{2|\mathbf{q}|^2}\mathrm{Tr}\,[\bar{u}(\mathbf{k}'_1)\gamma\cdot\bar{\mathbf{q}}u(\mathbf{k}_1)\bar{v}(-\mathbf{k}_1)\gamma\cdot\mathbf{q}\,v(-\mathbf{k}'_1)] \tag{B.1}$$
$$=f'_1f_1f'_2f_2+(\mathbf{k}'_1\cdot\mathbf{k}_1)(f'_1f_1g'_2g_2+f'_2f_2g'_1g_1)$$

$$+2(f_1 f_2 g'_1 g'_2 K_1'^2 + f'_1 f'_2 g_1 g_2 K_1^2) + g'_1 g'_2 g_1 g_2 K_1'^2 K_1^2$$

$$-\frac{2}{|\mathbf{q}|^2}(f'_1 f_2 g_1 g'_2 + f_1 f'_2 g'_1 g_2)(\mathbf{k}'_1\cdot\mathbf{q})(\mathbf{k}_1\cdot\mathbf{q}), \tag{B.2}$$

which for quark mass being zero becomes,

$$F^V_{PS}(\mathbf{k}'_1, \mathbf{k}_1) = \tfrac{3}{2} + \tfrac{1}{4}\cos\theta_1 - \frac{1}{K'_1 K_1}\left(K^2\cos^2\theta - \frac{Q^2}{4}\right), \tag{B.2a}$$

where $\cos\theta_1 = (\mathbf{k}'_1\cdot\mathbf{k}_1)/K'_1 K_1$.

From equation (52) we have,

$$F^V_{m'm}(\mathbf{k}'_1, \mathbf{k}_1) = \tfrac{1}{2}\mathrm{Tr}\,[\bar{u}(\mathbf{k}'_1)\gamma^\mu u(\mathbf{k}_1)\sigma_m \bar{v}(-\mathbf{k}_1)\gamma_\mu v(-\mathbf{k}'_1)\sigma_{m'}]$$

$$+\frac{1}{2|\mathbf{q}|^2}\mathrm{Tr}\,[\bar{u}(\mathbf{k}'_1)\gamma\cdot\mathbf{q}\, u(\mathbf{k}_1)\sigma_m \bar{v}(-\mathbf{k}_1)\gamma\cdot\mathbf{q}\, v(-\mathbf{k}'_1)\sigma_{m'}] \tag{B.3}$$

$$= \Bigg[(f'_1 f_1 + g'_1 g_1(\mathbf{k}'_1\cdot\mathbf{k}_1))(f'_2 f_2 + g'_2 g_2(\mathbf{k}'_1\cdot\mathbf{k}_1))$$

$$+\left(K_1^2 - \frac{(\mathbf{k}_1\cdot\mathbf{q})^2}{|\mathbf{q}|^2}\right)(f_1 f_2 g'_1 g'_2 + f'_1 f'_2 g_1 g_2 + f_1 f'_2 g'_1 g_2$$

$$- f'_1 f_2 g_1 g'_2)\Bigg]\delta_{m'm} + \Bigg[-2f_1 f_2 g'_1 g'_2 - 2(f'_1 f'_2 g_1 g_2$$

$$-f_1 f_2 g'_1 g'_2)\frac{\mathbf{k}_1\cdot\mathbf{q}}{|\mathbf{q}|^2} - (f'_1 f_2 g_1 g'_2 + f_1 f'_2 g'_1 g_2)$$

$$-(f_1 f'_1 g_2 g'_2 + f_2 f'_2 g_1 g'_1) - 2g_1 g_2 g'_1 g'_2(\mathbf{k}'_1\cdot\mathbf{k}_1)\Bigg]\varepsilon_{imm'}(\mathbf{q}\times\mathbf{k}'_1)_i$$

$$+[(f'_1 f_2 g_1 g'_2 + f_1 f'_2 g'_1 g_2)(\mathbf{k}'_1\cdot\mathbf{k}_1) - f_1 f_2 g'_1 g'_2 K_1'^2$$

$$- f'_1 f'_2 g_1 g_2 K_1^2]\delta_{m'm} - [(f'_1 f_2 g_1 g'_2 + f_1 f'_2 g'_1 g_2)((\mathbf{k}'_1\cdot\mathbf{k}_1)\delta_{m'm}$$

$$- K'_{1m'}K_{1m} - K_{1m'}K'_{1m}) + f_1 f_2 g'_1 g'_2(2K'_{1m}K'_{1m'} - K_1'^2\delta_{m'm})$$

$$+ f'_1 f'_2 g_1 g_2(2K_{1m'}K_{1m} - K_1^2\delta_{m'm})]$$

$$+\Bigg[g_1 g_2 g'_1 g'_2\{2(\mathbf{q}\times\mathbf{K}_1)_m(\mathbf{q}\times\mathbf{K}_1)_{m'} - (\mathbf{q}\times\mathbf{K}_1)^2\delta_{m'm}$$

$$+(f'_1 g_1 - f_1 g'_1)(f_2 g'_2 - f'_2 g_2)\frac{1}{|\mathbf{q}|^2}(2(\mathbf{q}\times\mathbf{K}_1)_m(\mathbf{q}\times\mathbf{K}_1)_{m'}$$

$$-(\mathbf{q}\times\mathbf{K}_1)^2\delta_{mm'})\Bigg]. \tag{B.4}$$

From equation (57) we get,

$$F^S_{PS}(\mathbf{k}'_1, \mathbf{k}_1) = -\tfrac{1}{2}\mathrm{Tr}\,[\bar{u}(\mathbf{k}'_1)u(\mathbf{k}_1)\bar{v}(-\mathbf{k}_1)v(-\mathbf{k}'_1)]$$

$$= f'_1 f_1 f'_2 f_2 - (f'_1 f_1 g_2 g'_2 + f'_2 f_2 g'_1 g_1)(\mathbf{k}'_1\cdot\mathbf{k}_1)$$

$$+ g'_1 g_1 g'_2 g_2 K_1'^2 K_1^2, \tag{B.5}$$

which for quarks being massless becomes

$$F^{S}_{PS}(\mathbf{k}_1', \mathbf{k}_1) = \tfrac{1}{2}(1-\cos\theta_1). \tag{B.5a}$$

From equation (59),

$$\begin{aligned} F^{S}_{m'm}(\mathbf{k}_1', \mathbf{k}_1) &= -\tfrac{1}{2}\mathrm{Tr}\,[\bar{u}(\mathbf{k}_1')u(\mathbf{k}_1)\sigma_m\bar{v}(-\mathbf{k}_1)v(-\mathbf{k}_1')\sigma_{m'}] \\ &= -[(f_1'f_1 - g_1'g_1(\mathbf{k}_1'\cdot\mathbf{k}_1)(g_2'g_2(\mathbf{k}_1'\cdot\mathbf{k}_1) - f_2'f_2)\delta_{m'm} \\ &\quad + \{(f_1'f_1 - g_1'g_1(\mathbf{k}_1'\cdot\mathbf{k}_1))g_2'g_2 + (f_2'f_2 - g_2'g_2(\mathbf{k}_1'\cdot\mathbf{k}_1))g_1'g_1\} \\ &\quad - \varepsilon_{imm'}(\mathbf{k}_1'\times\mathbf{k}_1)_i - g_1'g_1g_2'g_2\{2(\mathbf{k}_1'\times\mathbf{k}_1)_m(\mathbf{k}_1'\times\mathbf{k}_1)_{m'} \\ &\quad - (\mathbf{k}_1'\times\mathbf{k}_1)^2\delta_{m'm}\}]. \end{aligned} \tag{B.6}$$

We have

$$F^{V}({}^1S_0) = F^{V}_{PS}(\mathbf{k}_1', \mathbf{k}_1) \tag{B.7}$$

and

$$F^{V}({}^1P_1) = (\mathbf{k}_1'\cdot\mathbf{k}_1)\,F^{V}_{PS}(\mathbf{k}_1', \mathbf{k}_1). \tag{B.8}$$

From equation (54),

$$F^{V}_{V}(\mathbf{k}_1', \mathbf{k}_1) = \sum F^{V}_{m'm}(\mathbf{k}_1', \mathbf{k}_1)\,A^{V}_{m'm}(\hat{k}_1', \hat{k}_1). \tag{B.9}$$

For $l = 0$ and $J = 1$, we have ${}^3S_1$ state.

From (B.4) and (A.12) we get

$$\begin{aligned} F^{V}({}^3S_1) &= f_1'f_1f_2'f_2 + (f_1'f_1g_2'g_2 + f_2'f_2g_1'g_1)(\mathbf{k}_1'\cdot\mathbf{k}_1) \\ &\quad + g_1'g_1g_2'g_2((\mathbf{k}_1'\cdot\mathbf{k}_1)^2 - \tfrac{1}{3}(\mathbf{k}_1'\times\mathbf{k}_1)^2) \\ &\quad + \tfrac{2}{3}f_1'f_2'g_1g_2\left(K^2 - 2K^2\cos^2\theta - \frac{Q^2}{4} + KQ\cos\theta\right) \\ &\quad + \tfrac{2}{3}f_1f_2g_1'g_2'\left(K^2 - 2K^2\cos^2\theta - \frac{Q^2}{4} - KQ\cos\theta\right) \\ &\quad + \tfrac{2}{3}(f_1'f_2g_1g_2' + f_1f_2'g_1'g_2)\left(K^2 + K^2\sin^2\theta - \frac{Q^2}{4}\right), \end{aligned} \tag{B.10}$$

which for quarks being massless becomes

$$\begin{aligned} F^{V}({}^3S_1) &= \tfrac{1}{4} + \tfrac{1}{2}\cos\theta_1 + \tfrac{1}{4}(\cos^2\theta_1 - \tfrac{1}{3}\sin^2\theta_1) \\ &\quad + \frac{1}{6K_1'^2}\left(K^2 - 2K^2\cos^2\theta - \frac{Q^2}{4} - KQ\cos\theta\right) \\ &\quad + \frac{1}{6K_1^2}\left(K^2 - 2K^2\cos^2\theta - \frac{Q^2}{4} + KQ\cos\theta\right) \\ &\quad + \frac{1}{3K_1'K_1}\left(K^2 + K^2\sin^2\theta - \frac{Q^2}{4}\right). \end{aligned} \tag{B.10a}$$

In the non-relativistic limit and equal mass case,

$$F^V(^3S_1) = 1 + \frac{K^2}{m^2} - \frac{K^2\cos^2\theta}{m^2} - \frac{5}{12m^2}Q^2. \tag{B.11}$$

For $l = 1$ and $J = 0$, we have $^3P_0$ state. From (B.4) and (A.13) we have

$$\begin{aligned} F^V = (^3P_0) = {} & f_1' f_1 f_2' f_2 (\mathbf{k}_1' \cdot \mathbf{k}_1) + (f_1' f_1 g_2' g_2 + f_2' f_2 g_1' g_1 + 2 f_1' f_2 g_1 g_2' \\ & + 2 f_1 f_2' g_1' g_2)(K_1' K_1)^2 + g_1' g_1 g_2' g_2 (K_1' K_1)^2 (\mathbf{k}_1' \cdot \mathbf{k}_1) \\ & - 2(f_1' f_2' g_1 g_2 K_1^2 + f_1 f_2 g_1' g_2' K_1'^2)(\hat{q} \cdot \mathbf{k}_1)(\hat{q} \cdot \mathbf{k}_1'). \end{aligned} \tag{B.12}$$

For quarks being massless, this becomes,

$$\begin{aligned} F^V(^3P_0) = {} & \left\{\frac{1}{4}(1+\cos\theta_1)^2 + \frac{2K^2\sin^2\theta}{4}\left(\frac{1}{K_1^2} + \frac{1}{K_1'^2}\right)\right\}\left(K^2 - \frac{Q^2}{4}\right) \\ & + \left(\frac{1}{2K_1'^2} + \frac{1}{K_1' K_1}\right) K^2 Q^2 \sin^2\theta + \frac{1}{2}\left(\frac{1}{K_1'^2} - \frac{1}{K_1^2}\right) \\ & \times \left(K^3 Q \sin^2\theta\cos\theta - \frac{K^2 Q^2 \sin^2\theta}{2}\right) \\ & + \frac{1}{4K_1'^2 K_1^2}\left(K^2 - \frac{Q^2}{4}\right) K^2 Q^2 \sin^2\theta \\ & - \left(K^2 - \frac{Q^2}{4}\right) + \frac{1}{2K_1' K_1}\{(\mathbf{k}_1' \cdot \mathbf{k}_1)^2 + K_1'^2 K_1^2\}. \end{aligned} \tag{B.12a}$$

In the non-relativistic limit and equal mass case

$$\begin{aligned} F^V(^3P_0) = {} & K^2 + \frac{K^4}{m^2} - \frac{K^4\cos^2\theta}{m^2} - \frac{Q^2}{4} + \frac{K^2 Q^2}{m^2} \\ & - \frac{7K^2 Q^2 \cos^2\theta}{4m^2} + \frac{3Q^4}{16m^2}. \end{aligned} \tag{B.13}$$

For $l = 1$, $S = 1$ and $J = 1$, we have for $^3P_1$ state,

$$F^V(^3P_1) = K_1' K_1 F_V^V(\mathbf{k}_1', \mathbf{k}_1).$$

From (B.4) and (A.14) we then have,

$$F^V(^3P_1) = \tfrac{3}{2}(\mathbf{k}_1' \cdot \mathbf{k}_1) F^V(^3S_1) - \tfrac{1}{2} F_V^e(^3P_0), \tag{B.14}$$

where we have taken,

$$F_V^e(^3P_0) = \sum_{m',m} k'_{1m} k_{1m'} F^V_{m'm}(\mathbf{k}_1', \mathbf{k}_1). \tag{B.15}$$

Hence,

$$\begin{aligned} F_V^e(^3P_0) = {} & [(f_1' f_1 + g_1' g_1 (\mathbf{k}_1' \cdot \mathbf{k}_1))(f_2' f_2 + g_2' g_2 (\mathbf{k}_1' \cdot \mathbf{k}_1)) \\ & + 2K^2 \sin^2\theta (f_1 f_2 g_1' g_2' + f_1' f_2' g_1 g_2)](\mathbf{k}_1' \cdot \mathbf{k}_1) \end{aligned}$$

$$-\Big[(2f_1f_2g'_1g'_2 + f'_1f_2g_1g'_2 + f_1f'_2g'_1g_2 + f_2f'_2g_1g'_1$$
$$+ f_1f'_1g_2g'_2)Q^2K^2\sin^2\theta + 2(f'_1f'_2g_1g_2$$
$$- f_1f_2g'_1g'_2)\left(K^3Q\sin^2\theta\cos\theta - \frac{K^2Q^2\sin^2\theta}{2}\right)$$
$$+ 2g_1g_2g'_1g'_2\left(K^2 - \frac{Q^2}{4}\right)K^2Q^2\sin^2\theta\Big]$$
$$-2\left(K^2 - \frac{Q^2}{4}\right)\{f_1f_2g'_1g'_2K'^2_1 + f'_1f'_2g_1g_2K^2_1\}$$
$$+ (f'_1f_2g_1g'_2 + f_1f'_2g'_1g_2)((\mathbf{k}'_1\cdot\mathbf{k}_1)^2 + K_1K'^2_1)$$
$$- g_1g_2g'_1g'_2\left(K^2 - \frac{Q^2}{4}\right)K^2Q^2\sin^2\theta, \tag{B.16}$$

which for quarks being massless becomes,

$$F^e_V(^3P_0) = \left\{\frac{1}{4}(1+\cos\theta_1)^2 + \frac{K^2\sin^2\theta}{2}\left(\frac{1}{K^2_1} + \frac{1}{K'^2_1}\right)\right\}\left(K^2 - \frac{Q^2}{4}\right)$$
$$-\left\{\left(\frac{1}{2K'^2_1} + \frac{1}{K'_1K_1}\right)K^2Q^2\sin^2\theta + \frac{1}{2}\left(\frac{1}{K'^2_1} - \frac{1}{K^2_1}\right)\right.$$
$$\times\left(K^3Q\sin^2\theta\cos\theta - \frac{K^2Q^2\sin^2\theta}{2}\right)$$
$$\left. + \frac{3}{4K^2_1K'^2_1}\left(K^2 - \frac{Q^2}{4}\right)K^2Q^2\sin^2\theta\right\} - \left(K^2 - \frac{Q^2}{4}\right)$$
$$+ \frac{1}{2K'_1K_1}\{(\mathbf{k}'_1\cdot\mathbf{k}_1)^2 + K'^2_1K^2_1\}. \tag{B.16a}$$

In the non-relativistic limit and equal mass case,

$$F^V(^3P_1) = K^2 + \frac{K^4}{m^2} - \frac{K^4\cos^2\theta}{m^2} - \frac{Q^2}{4} + \frac{K^2Q^2}{6m^2}$$
$$- \frac{3K^2Q^2\cos^2\theta}{4m^2} + \frac{Q^4}{16m^2}. \tag{B.17}$$

For $l = 1$, $S = 1$ and $J = 2$, we have $^3P_2$ state.

From (B.4) and (A.15) we get

$$F^V(^3P_2) = \tfrac{9}{10}(\mathbf{k}'_1\cdot\mathbf{k}_1)F^V(^3S_1) - \tfrac{1}{15}F^V(^3P_0) + \tfrac{3}{10}F^e_V(^3P_0). \tag{B.18}$$

In the non-relativistic limit and equal mass case

$$F^V(^3P_2) = K^2 + \frac{K^4}{m^2} - \frac{K^4\cos^2\theta}{m^2} - \frac{Q^2}{4} - \frac{7K^2Q^2}{5m^2}$$
$$+ \frac{19K^2Q^2\cos^2\theta}{20m^2} + \frac{9Q^4}{80m^2}. \tag{B.19}$$

Similarly for scalar exchange,

$$F^S({}^1S_0) = F^S_{PS}(\mathbf{k}'_1, \mathbf{k}_1) \tag{B.20}$$

and

$$F^S({}^1P_1) = (\mathbf{k}'_1 \cdot \mathbf{k}_1)\, F^S({}^1S_0). \tag{B.21}$$

Also, we have

$$F^S_V(\mathbf{k}'_1, \mathbf{k}_1) = \sum_{m'm} A^V_{m'm}(\mathbf{k}'_1, \mathbf{k}_1)\, F^S_{m',m}(\mathbf{k}'_1, \mathbf{k}_1) \tag{B.22}$$

Combining (B.6) and (A.12) we get,

$$\begin{aligned} F^S({}^3S_1) = {} & f'_1 f_1 f'_2 f_2 - (f'_2 f_2 g'_1 g_1 + f'_1 f_1 g'_2 g_2)(\mathbf{k}'_1 \cdot \mathbf{k}_1) \\ & + g'_1 g_1 g'_2 g_2 (\mathbf{k}'_1 \cdot \mathbf{k}_1)^2 - \tfrac{1}{3} g'_1 g_1 g'_2 g_2 K^2 Q^2 \sin^2\theta, \end{aligned} \tag{B.23}$$

which for quarks being massless becomes,

$$F^S({}^3S_1) = \tfrac{1}{4} - \tfrac{1}{2}\cos\theta_1 + \tfrac{1}{4}\left(\cos^2\theta_1 - \tfrac{1}{3}\sin^2\theta_1\right). \tag{B.23a}$$

In the non-relativistic limit

$$F^S({}^3S_1) = 1 - \frac{K^2}{m^2}. \tag{B.24}$$

From (B.6) and (A.13),

$$\begin{aligned} F^S({}^3P_0) = {} & [\{(f'_1 f_1 - g'_1 g_1 (\mathbf{k}'_1 \cdot \mathbf{k}_1))(g'_2 g_2 (\mathbf{k}'_1 \cdot \mathbf{k}_1) - f'_2 f_2)\}(\mathbf{k}'_1 \cdot \mathbf{k}_1) \\ & + \{(f'_1 f_1 - g'_1 g_1 (\mathbf{k}'_1 \cdot \mathbf{k}_1)) g'_2 g_2 \\ & + (f'_2 f_2 - g'_2 g_2 (\mathbf{k}'_1 \cdot \mathbf{k}_1))\} K^2 Q^2 \sin^2\theta \\ & + g'_1 g_1 g'_2 g_2 (\mathbf{k}'_1 \cdot \mathbf{k}_1) K^2 Q^2 \sin^2\theta], \end{aligned} \tag{B.25}$$

which for quark mass being zero becomes,

$$\begin{aligned} F^S({}^3P_0) = -\Bigg[& -\tfrac{1}{4}(1 - \cos\theta_1)^2 \left(K^2 - \frac{Q^2}{4}\right) \\ & + \tfrac{1}{2} K'_1 K_1 \sin^2\theta \left(1 - \frac{\cos\theta_1}{2}\right)\Bigg]. \end{aligned} \tag{B.25a}$$

In the non-relativistic limit

$$F^S({}^3P_0) = K^2 - \frac{K^4}{m^2} - \frac{Q^2}{4} - \frac{K^2 Q^2}{4m^2} + \frac{K^2 Q^2 \cos^2\theta}{2m^2}. \tag{B.26}$$

From (B.6) and (A.14) we get

$$F^S({}^3P_1) = \tfrac{3}{2}(\mathbf{k}'_1 \cdot \mathbf{k}_1)\; F^S({}^3S_1) - \tfrac{1}{2} F^e_s({}^3P_0), \tag{B.27}$$

where

$$F_S^e(^3P_0) = \sum_{m',m} k'_{1m} k_{1m'} F^S_{m'm}(\mathbf{k}'_1, \mathbf{k}_1). \tag{B.28}$$

$$\begin{aligned} &= -[\{(f'_1 f_1 - g'_1 g_1 (\mathbf{k}'_1 \cdot \mathbf{k}_1))(g'_2 g_2 (\mathbf{k}'_1 \cdot \mathbf{k}_1) - f'_2 f_2)\}(\mathbf{k}'_1 \cdot \mathbf{k}_1) \\ &\quad - \{(f'_1 f_1 - g'_1 g_1 (\mathbf{k}'_1 \cdot \mathbf{k}_1)) g'_2 g_2 + (f'_2 f_2 - g'_2 g_2 (\mathbf{k}'_1 \cdot \mathbf{k}_1)) g'_1 g_1\} \\ &\quad \times K^2 Q^2 \sin^2\theta + g'_1 g_1 g'_2 g_2 (\mathbf{k}'_1 \cdot \mathbf{k}_1) K^2 Q^2 \sin^2\theta], \end{aligned} \tag{B.29}$$

which for quark mass being zero becomes

$$\begin{aligned} F_S^e(^3P_0) = -\Bigg[&-\tfrac{1}{4}(1-\cos\theta_1)^2\left(K^2 - \frac{Q^2}{4}\right) \\ &- \tfrac{1}{2} K'_1 K_1 \sin^2\theta_1 \,(1 - \tfrac{3}{2}\cos\theta_1)\Bigg]. \end{aligned} \tag{B.29$'$}$$

In the non-relativistic limit

$$F^S(^3P_1) = K^2 - \frac{K^4}{m^2} - \frac{Q^2}{4} + \frac{K^2 Q^2 \cos^2\theta}{4m^2}. \tag{B.30}$$

From (B.6) and (A.15) we get

$$F^S(^3P_2) = \tfrac{9}{10}(\mathbf{k}'_1 \cdot \mathbf{k}_1) F^S(^3S_1) - \tfrac{1}{5} F^S(^3P_0) + \tfrac{3}{10} F_s^e(^3P_0). \tag{B.31}$$

In the non-relativistic limit,

$$F^S(^3P_2) = K^2 - \frac{K^4}{m^2} - \frac{Q^2}{4} + \frac{K^2 Q^2}{2m^2} - \frac{K^2 Q^2 \cos^2\theta}{4m^2}. \tag{B.32}$$

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