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Kerr-Newman metric in deSitter background

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Abstract. In addition to the Kerr-Newman metric with cosmological constant several other metrics are presented giving Kerr-Newman type solutions of Einstein-Maxwell field equations in the background of deSitter universe. The electromagnetic field in all the solutions is assumed to be source-free. A new metric of what may be termed as an electrovac rotating de-Sitter space-time—a space-time devoid of matter but containing source-free electromagnetic field and a null fluid with twisting rays—has been presented. In the absence of the electromagnetic field, our solutions reduce to those discussed by Vaidya.

Keywords. Kerr-Newman metric; deSitter universe; electromagnetic fields.

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1. Introduction

The Kerr-Newman (Newman *et al* 1965) solution is believed to represent the ultimate state of collapsing body with rotation, mass and electric charge. Therefore considerable significance is attached to the Kerr-Newman solution. When the charge is absent, the Kerr-Newman solution reduces to the well-known Kerr solution (Kerr 1963). Patel and Trivedi (1982) considered the Kerr-Newman metric in cosmological background, the background metric being the Robertson-Walker metric. In the absence of the electromagnetic field, their solution reduces to the Vaidya solution (Vaidya 1977) which describes the Kerr metric in the background of the Robertson-Walker universe.

We know that the deSitter metric can be expressed as a particular case of the general Robertson-Walker metric. But the deSitter model has features which are geometrically distinct from those of Robertson-Walker model. Again the simple deSitter metric represents an empty and expanding universe. The deSitter space-time is an immediate generalization of Minkowski flat space-time. These facts inspired Vaidya (1984) to discuss the Kerr metric in the background of deSitter universe. The main purpose of the present article is to obtain the electromagnetic generalizations of the solutions discussed by Vaidya (1984). Our earlier paper (Patel and Trivedi 1982) will be referred to hereafter as I. In this paper we merely report the main results by avoiding the computational details and the lengthy expression for some of the quantities.

In §§2 and 3 we use the following Einstein-Maxwell field equations

$$R_{ik} - \tfrac{1}{2} g_{ik} R + \Lambda g_{ik} = -8\pi E_{ik}, \tag{1}$$

$$E_{ik} = -g^{lm} F_{il} F_{km} + \tfrac{1}{4} g_{ik} F_{lm} F^{lm}, \tag{2}$$

$$F_{ik} = A_{i,k} - A_{k,i}, \tag{3}$$

$$F^{ik}_{;k} = 0. \tag{4}$$

The various symbols occurring in the above field equations have their usual meanings. We permit the electromagnetic field tensor F_{ik} to vanish.

2. Kerr-Newman metric with Λ

Vaidya (1984) has shown that the deSitter metric can be put in the form conformal to the Einstein universe metric with negative curvature. Thus the geometry of the deSitter universe is described by the line-element

$$\mathrm{d}s^2 = \frac{R^2}{R^2 + (x^2+y^2+z^2)}\left[\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 - \frac{(x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z)^2}{R^2 + (x^2+y^2+z^2)}\right], \tag{5}$$

where R is a constant. Vaidya (1984) has given a transformation from the co-ordinates (x, y, z) to the spheroidal polar co-ordinates which transforms the metric (5) into the form

$$\mathrm{d}s^2 = \left[\cosh^2(r/R)\left(1 + \frac{a^2}{R^2}\sin^2\alpha\right)\right]^{-1}\mathrm{d}s_0^2, \tag{6}$$

where R is a constant and $\mathrm{d}s_0^2$ is given by

$$\begin{aligned}\mathrm{d}s_0^2 &= 2\,(\mathrm{d}u + a\sin^2\alpha\,\mathrm{d}\beta)\,\mathrm{d}t - (\mathrm{d}u + a\sin^2\alpha\,\mathrm{d}\beta)^2 \\ &\quad - M^2\left[\left\{1 + \frac{a^2}{R^2}\sin^2\alpha\right\}^{-1}\mathrm{d}\alpha^2 + \sin^2\alpha\,\mathrm{d}\beta^2\right].\end{aligned} \tag{7}$$

Here $u = t - r$ and

$$M^2 = (R^2 + a^2)\sinh^2(r/R) + a^2\cos^2\alpha. \tag{8}$$

Following the scheme of I, we can write down the Kerr-Newman metric in deSitter background as

$$\mathrm{d}s^2 = \left[\cosh^2(r/R)\left\{1 + \frac{a^2}{R^2}\sin^2\alpha\right\}\right]^{-1}\mathrm{d}s_1^2, \tag{9}$$

$$\mathrm{d}s_1^2 = \mathrm{d}s_0^2 - (2m\mu - 4\pi e^2\gamma)\,(\mathrm{d}u + a\sin^2\alpha\,\mathrm{d}\beta)^2, \tag{10}$$

where m, e and a are constants, $\mathrm{d}s_0^2$ is given by (7), M^2 is given by (8) and μ and γ are given by the following equations:

$$\mu M^2 = R\sinh(r/R)\cosh^3(r/R), \tag{11}$$

$$\gamma M^2 = \cosh^4(r/R). \tag{12}$$

We have verified that the metric (9) along with (10), (11) and (12) satisfies the Einstein-

Maxwell field equations (1), (2), (3) and (4) with

$$\Lambda = 3/R^2, \qquad A_i = (A, 0, Aa\sin^2\alpha, 0), \tag{13}$$

where

$$A = [eR \sinh(r/R)\cosh(r/R)]/M^2, \tag{14}$$

M^2 being given by (8). Here e is a constant of integration and the co-ordinates are named as $x^1 = u$, $x^2 = \alpha$, $x^3 = \beta$ and $x^4 = t$.

When $e = 0$, the electromagnetic field vanishes and the metric given by (9) along with (10), (11) and (12) reduces to the Kerr metric with a Λ-term in the form discussed by Vaidya (1984). The metric given by (9), (10), (11) and (12) can be transformed in the conventional Boyer-Lindquist type co-ordinates. The required transformation equations can be obtained from those given by Vaidya (1984) for the uncharged case by minor modifications. We shall not give these transformation equations here. We have verified that when $e = 0$, the Kerr-Newman metric with cosmological constant in these co-ordinates reduces to the Kerr metric with Λ term in Boyer Lindquist co-ordinates given by Demianski (1973).

When $R \to \infty$, Λ becomes zero and we recover the Kerr-Newman metric in the form

$$\begin{aligned} ds^2 &= 2(du + a\sin^2\alpha\, d\beta)\, dt - (r^2 + a^2\cos^2\alpha)(d\alpha^2 + \sin^2\alpha\, d\beta^2) \\ &\quad - \left[1 + \frac{2mr - 4\pi e^2}{r^2 + a^2\cos^2\alpha}\right](du + a\sin^2\alpha\, d\beta)^2. \end{aligned} \tag{15}$$

Here it should be noted that the Kerr-Newman metric with a Λ-term discussed by us is contained in the family of solutions of electrovac field equations with cosmological constant given by Debever *et al* (1984) (§ 2). Their co-ordinates are different from our co-ordinates. There must be a co-ordinate transformation which transform our solution into the solution given by Debever *et al* (1984). But we have not been able to find out the explicit form of this transformation. One of the advantages of our co-ordinate system is that Vaidya's results for the uncharged case become transparent on switching off the electromagnetic field in our solution. We also feel that our special form can be handled more easily than the form given by Debever *et al* (1984).

3. Anti-deSitter background

When the cosmological constant Λ is negative, say $\Lambda = -3/R^2$, $R = \text{constant}$, the space-time described by the deSitter metric is known in the literature as the anti-deSitter space-time. A surprising result is that the following axially symmetric line-element, conformal to the usual deSitter line-element, represents the anti-deSitter space time.

$$\begin{aligned} ds^2 &= \frac{R^2}{r^2\cos^2\alpha}\left[\left(1 - \frac{r^2}{b^2}\right)dt^2 - \left(1 - \frac{r^2}{b^2}\right)^{-1} dr^2 \right. \\ &\quad \left. - r^2(d\alpha^2 + \sin^2\alpha\, d\beta^2)\right]. \end{aligned} \tag{16}$$

The above metric satisfies the field equations

$$R_{ik} = \Lambda g_{ik}, \quad \Lambda = -3/R^2,$$

b^2 being undetermined constant. Vaidya (1984) discussed the Kerr metric in the background of (16). It would be interesting to obtain the Kerr-Newman metric in the anti-deSitter background (16). However we shall not derive this metric here. Instead we note a simple particular case of (16) obtained by choosing the constant $b \to \infty$. The background anti-deSitter metric then reduces to the following simple plane symmetric metric

$$ds^2 = \frac{R^2}{z^2}(dt^2 - dx^2 - dy^2 - dz^2). \tag{17}$$

The relations between the spherical polar co-ordinates and the Cartesian co-ordinates (x, y, z) are well known and hence not given here. We now give in a nutshell the Kerr-Newman metric in the anti-deSitter background (17). The explicit form of the Kerr-Newman metric in the background of (17) is given by

$$ds^2 = \frac{R^2}{r^2\cos^2\alpha}\Bigg[2(du + a\sin^2\alpha\, d\beta)\,dt - (r^2 + a^2\cos^2\alpha)(d\alpha^2 + \sin^2\alpha\, d\beta^2) - \left(1 + \frac{2mr^3 - 4\pi e^2 r^4 R^{-2}}{r^2 + a^2\cos^2\alpha}\right)(du + a\sin^2\alpha\, d\beta)^2\Bigg]. \tag{18}$$

where m and e are constants. Equation (18) gives a very simple metric for the Kerr-Newman-like gravitational field satisfying the field equations (1), (2), (3) and (4) with

$$\Lambda = -\frac{3}{R^2}, \qquad A_i = \left(\frac{er}{r^2 + a^2\cos^2\alpha}, 0, \frac{ear\sin^2\alpha}{r^2 + a^2\cos^2\alpha}, 0\right). \tag{19}$$

That Λ has to be non-zero is a condition for the existence of the background metric of anti-deSitter space-time given by (16) or (17).

The solution (18) cannot be a member of the family of solutions given by Debever *et al* (1984). In their solutions, the cosmological constant Λ is allowed to vanish. But $\Lambda \neq 0$ is a condition for the existence of our solution (18).

When $e = 0$ (i.e. in the absence of electromagnetic field), the metric (18) reduces to the Kerr-like metric in the anti-deSitter background discussed by Vaidya (1984).

4. The electrovac deSitter space-time with twisting null rays

We know that the geometry of the Reissner-Nordstrom solution is described by the line element

$$ds^2 = 2du\,dt - \left(1 + \frac{2m}{r} - \frac{4\pi e^2}{r^2} + \frac{r^2}{R^2}\right)du^2 - r^2(d\alpha^2 + \sin^2\alpha\, d\beta^2). \tag{20}$$

Here we have assumed that the cosmological constant $\Lambda = 3/R^2$ is non-zero. We also know that the Kerr-Newman solution is described under the Minkowskian space-time. Thus in the absence of the source the Kerr-Newman solution reduces to the flat space-time. This fact suggests an immediate generalization of (20) to the following metric

$$ds^2 = 2(du + g\sin\alpha d\beta)dt - H(r^2+y^2)d\Omega^2$$
$$-\left(1+\frac{2mr-4\pi e^2}{r^2+y^2}+\frac{r^2+y^2}{R^2}\right)(du+g\sin\alpha\, d\beta)^2, \tag{21}$$

with

$$g = g(\alpha),\quad y = y(\alpha),\quad H = H(\alpha),\quad \Lambda = 3/R^2 \quad \text{and}$$
$$d\Omega^2 = d\alpha^2 + \sin^2\alpha\, d\beta^2.$$

It may be noted that if $\Lambda = 0$, with $g\sin\alpha = a\sin^2\alpha$, $y = -a\cos\alpha$ and $H = 1$, (21) gives us the usual Kerr-Newman metric. Here a is a constant. It is easy to see that the metric (21) is a particular case of the Kerr-NUT metric (Vaidya *et al* 1976)

$$ds^2 = 2(du + g\sin\alpha d\beta)dt - M^2 d\Omega^2 - 2L(du + g\sin\alpha d\beta)^2, \tag{22}$$

and so we can freely use the tetrad formalism developed in that paper to work out the physics of the metric (21). Using the tetrad

$$\theta^1 = du + g\sin\alpha\, d\beta,\quad \theta^2 = M d\alpha,\quad \theta^3 = M\sin\alpha\, d\beta,$$
$$\theta^4 = dt - L\theta^1, \tag{23}$$

the tetrad components $R_{(ab)}$ of the Ricci tensor for the metric (22) are also recorded there. It should be noted that the metric (22) takes the form

$$ds^2 = 2\theta^1\theta^4 - (\theta^2)^2 - (\theta^3)^2,$$

in terms of the Cartan's frame (23). It can be easily checked that if one chooses

$$g\,d\alpha = dy,\quad H = -f/y,\quad 2f = (\partial g/\partial\alpha) + g\cot\alpha, \tag{24}$$

one finds that for the metric (21)

$$R_{(24)} = R_{(34)} = R_{(12)} = R_{(13)} = R_{(44)} = R_{(23)} = 0,$$
$$R_{(14)} = \Lambda - [4\pi e^2/(r^2+y^2)^2],$$
$$R_{(22)} = R_{(33)} = -\Lambda - \frac{4\pi e^2}{(r^2+y^2)^2} + [2 + \tfrac{4}{3}\Lambda y^2 - yG](r^2+y^2)^{-1},$$
$$R_{(11)} = -\frac{2y}{3}\Lambda\left[\frac{g^2}{f}+2y\right](r^2+y^2)^{-1} + 8\pi\frac{y}{f}e^2g^2(r^2+y^2)^{-3}. \tag{25}$$

We have verified that for the metric (21) the Maxwell equations (4) give us the electromagnetic 4 potential A_i as

$$A_i = \left(\frac{er}{r^2+y^2}, 0, \frac{erg\sin\alpha}{r^2+y^2}, 0\right). \tag{26}$$

In (24) the quantity G is defined by

$$2fG = g^2\left[\frac{1}{y^2}+\frac{\partial}{\partial y}\left(\frac{1}{f}\frac{\partial f}{\partial y}\right)\right]+2\frac{\partial f}{\partial y}-2.$$

It is clear that if we choose

$$8\pi\sigma = \frac{2y\Lambda}{3}\left[\frac{g^2}{f}+2y\right](r^2+y^2)^{-1}$$

and $$Gy-2-\tfrac{4}{3}\Lambda y^2 = 0, \tag{27}$$

we shall have $R_{ik} = \Lambda g_{ik} - 8\pi E_{ik} - 8\pi\sigma\,\xi_i\xi_k$, where ξ_i is a null vector defined by $\xi_i \mathrm{d}x^i = \mathrm{d}u + g\sin\alpha\,\mathrm{d}\beta$. We have verified that the null congruence ξ_i is geodetic and expanding. It should also be noted that the twist of the null congruence ξ_i is non-zero. Equations (24) and (27) are the three equations which determine the unknown functions of α in the metric (21).

If $R \to \infty$, $\Lambda = 0$ and (24) and (27) are satisfied by $f = -y = a\cos\alpha$, $g = a\sin\alpha$, a being a constant. The metric (21) then becomes the usual Kerr-Newman metric. Vaidya (1984) showed that the metric (21) with (24) and (27) gives a rotating deSitter space-time provided the mass parameter m and the charge parameter e are zero. Thus the metric

$$\begin{aligned}\mathrm{d}s^2 &= 2(\mathrm{d}u + g\sin\alpha\,\mathrm{d}\beta)\,\mathrm{d}t - \left(\frac{f}{-y}\right)(r^2+y^2)\,(\mathrm{d}\alpha^2+\sin^2\alpha\,\mathrm{d}\beta^2)\\ &\quad -\left[1+\frac{2mr-4\pi e^2}{r^2+y^2}+\frac{r^2+y^2}{R^2}\right](\mathrm{d}u+g\sin\alpha\,\mathrm{d}\beta)^2,\end{aligned} \tag{28}$$

with

$$\mathrm{d}y = g\,\mathrm{d}\alpha, \quad f = f(y) \quad \text{and} \quad Gy - 2 - \tfrac{4}{3}\Lambda y^2 = 0$$

becomes the Kerr-Newman metric in the cosmological background of the rotating deSitter universe. The particular case $m = 0$ of (28) gives us the electrovac deSitter space-time with twisting null rays.

In the solutions discussed by Debever *et al* (1984) the radiation density σ is zero. But for the solution discussed in this section σ is non zero. Therefore we can say that the solution of this section is not contained in the family of solutions given by Debever *et al* (1984).

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