Existence of quantum soliton for $\phi^6$-like field theories in 1 + 1 dimensions

C N KUMAR and B K PARIDA*
Institute of Physics, Sachivalaya Marg, Bhubaneswar 751 005, India
*Faculty of Physics, Regional College of Education, Ajmer 305 004, India

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Abstract. We reexamine the recent claim that the soliton of the 1 + 1 dimensional field theories does not survive quantum corrections if the adjacent minima of the potential do not have same curvature and show that it is in fact possible to choose counter terms such that the quantum correction to the soliton mass is finite.

Keywords. Solitons; 2-D scalar field model; triple-well potential.

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1. Introduction

In a recent article, Kaul and Rajaraman (1985) (KR) have considered the $\phi^6$ field theory model in 1 + 1 dimensions which has three degenerate minima and have claimed that the quantum soliton does not exist in this model. These authors have claimed that the quantum soliton will not exist in all those 1 + 1 dimensional field theory models in which the curvatures at the adjacent degenerate absolute minima are unequal. In particular they showed that in all these cases the quantum correction to the soliton mass has a $L$-dependent piece which diverges as the length $L$ of the box in which the system is quantized goes to infinity.

We find this claim rather surprising since all such neutral scalar field theories are in fact renormalizable in 1 + 1 dimensions. We have therefore taken a fresh look at this problem and find that in all these cases, contrary to the claim of KR it is in fact possible to choose the counterterms in such a way that the quantum correction to the soliton mass is finite.

The paper is organized as follows. In §2 we sketch the essential steps of KR so as to make the paper self-contained. In §3 we demonstrate as to how counterterms can be appropriately chosen to make the quantum correction to the soliton mass finite. Finally, in §4 we compute one-loop effective potential for the case of $\phi^6$-field theory with the same counterterms and show the phenomenon of spontaneous symmetry breaking.
2. Kaul-Rajaraman argument

Kaul and Rajaraman (1985) have considered in detail the following field theory model in 1 + 1 dimensions

\[ L = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} S^2(\phi), \]  

where

\[ \frac{1}{2} S^2(\phi) \equiv V(\phi) = \frac{\lambda^2}{2m^2} \phi^2 \left( \phi^2 - \frac{m^2}{\lambda} \right)^2. \]  

At the classical level, this model has three degenerate minima at \( \phi = 0 \), and \( \phi = \pm m/\sqrt{\lambda} \). The soliton solution of this model (Khare 1979) is

\[ \phi_s(x) = \frac{m}{\sqrt{2\lambda}} (1 + \tanh mx)^{1/2}, \]

and the corresponding soliton mass is

\[ M_s^{cl} = m^3/3\lambda. \]  

The other three soliton solutions can be trivially obtained by exchanging \( \phi \rightarrow -\phi \) and \( x \rightarrow -x \) and all of them have same soliton mass as given by (4).

KR have calculated the quantum correction to the soliton mass \( M_s^{cl} \) as given by equation (4) and have claimed that it is \( L \) divergent, \( L \) being the length of the box in which the system is quantized. They have argued that the origin of this \( L \)-divergence can be quite easily anticipated by considering the vacuum of the theory. In particular, they have argued that since the curvature of the potential at \( \phi = 0 \) is less than that of \( \phi = \pm m/\sqrt{\lambda} \), i.e.,

\[ \frac{d^2 V}{d\phi^2} \bigg|_{\phi = 0} = m^2, \quad \frac{d^2 V}{d\phi^2} \bigg|_{\phi = \pm m/\sqrt{\lambda}} = 4m^2. \]

Hence the zero-point energy of the vacuum at \( \phi = 0 \) will be lower than that of the vacuum at \( \phi = \pm m/\sqrt{\lambda} \). To order \( \hbar \), the difference of these vacuum energies is

\[ (E_{\text{vac}})_{\phi = \pm m/\sqrt{\lambda}} - (E_{\text{vac}})_{\phi = 0} = \frac{hL}{2} \int \frac{dk}{2\pi} \left[ (k^2 + 4m^2)^{1/2} - (k^2 + m^2)^{1/2} \right] \]

from which they have concluded that the states built around \( \phi = \pm m/\sqrt{\lambda} \) have higher energy density per unit length. As a result the quantum soliton will also have higher energy (compared to the vacuum) that would tend to infinity as \( L \rightarrow \infty \). KR have further argued that this reasoning will apply for any model in 1 + 1 dimensions for which the potential has unequal curvatures at the neighbouring degenerate minima. Thus according to them, quantum soliton will not exist in all these cases. KR have also done a detailed analysis of the one-loop correction to the soliton mass which is given by

\[ M_s^{1\text{-loop}} = \frac{\hbar}{2} \left[ \sum_\text{sol} \omega - \sum_\text{vac} \tilde{\omega} \right] + [\Delta E_{\text{ct.}}(\text{sol}) - \Delta E_{\text{ct.}}(\text{vac})] \]

where \( \omega \) are fluctuation frequencies. By anticipating the true vacuum to be at \( \phi = 0 \) they
Soliton for $\phi^6$-like field theories

They further show that

$$\frac{\hbar}{2} \sum_{\text{sol}} \omega = \frac{\hbar L}{2} \int_{0}^{\infty} \frac{dk}{2\pi} \left( k^2 + m^2 \right)^{1/2}$$

$$+ L \text{ independent terms.} \tag{9}$$

The one-loop renormalization counter term for the Lagrangian (1) is of the form

$$L_{\text{ct.}} = \frac{\hbar C}{2} \left( (S')^2 + SS'' \right), \tag{10}$$

where $C$ is the loop integral which has been chosen by KR as

$$C = \frac{1}{2\pi} \int_{0}^{\infty} \frac{dk}{(k^2 + m^2)^{1/2}}. \tag{11}$$

With this they show that one-loop radiative correction to the soliton mass is

$$M_{\text{sol}}^{1\text{-loop}} = \frac{\hbar L}{2} \int_{0}^{\infty} \frac{dk}{2\pi} \left[ (k^2 + 4m^2)^{1/2} - (k^2 + m^2)^{1/2} - \frac{3m^2}{2(k^2 + m^2)^{1/2}} \right]$$

$$+ L \text{ independent terms.} \tag{12}$$

Since the first term is $L$- divergent (even though ultraviolet finite) they conclude that the finite energy classical soliton does not survive quantum correction and hence quantum soliton does not exist.

3. Existence of quantum soliton

We do not agree with the argument of KR for the following reasons. There is no compulsion to choose the mass parameter in the loop integral $C$ as $m^2$ (equation (11)). In fact, the correct procedure would be to keep it arbitrary (say $\mu^2$) and then adjust $\mu^2$ so that even to the order $\hbar$ the difference of the vacuum energies of the neighbouring minima is zero. One should then calculate the quantum correction to the soliton mass with such choice of $L_{\text{ct.}}$ (equation (10)). We shall show below that with this procedure the quantum correction to the soliton mass is indeed $L$-independent and finite. We shall also see that this procedure applies to any potential with different curvatures at the neighbouring degenerate minima. Similar analysis was done by Lohe and O'Brien (1981). Even though the final aim in both is to achieve $L$-independent finite correction to soliton mass the procedures in the two are slightly different and in the light of KR's paper we thought it worth reporting the same.

The difference of vacuum energies (to order $\hbar$) as given by (6) is divergent. Let us try to make it finite by choosing $L_{\text{ct.}}$ as given by (10) but with the mass parameter in the
loop integral $C$ being arbitrary i.e.

$$C = \frac{1}{2\pi} \int \frac{dk}{(k^2 + \mu^2)^{1/2}}.$$  \hspace{1cm} (13)

Hence

$$(\Delta E_{ct.})_\phi = \pm m/\sqrt{\lambda} - (\Delta E_{ct.})_{\phi = 0} = -\frac{3}{4\pi} hLm^2 \int \frac{dk}{(k^2 + \mu^2)^{1/2}}.$$  \hspace{1cm} (14)

Thus the 'true' difference of vacuum energies will be

$$hL \int_0^\infty \frac{dk}{2\pi} \left[ (k^2 + 4m^2)^{1/2} - (k^2 + m^2)^{1/2} - \frac{3}{2} \frac{m^2}{(k^2 + \mu^2)^{1/2}} \right]$$  \hspace{1cm} (15)

which is zero provided.*

$$\ln \left( \frac{\mu^2}{m^2} \right) = \frac{3}{2} \ln 4 - 1.$$  \hspace{1cm} (16)

i.e. $\mu^2$ lies between $m^2$ and $4m^2$. In other words, with this choice of $\mu^2$ in the loop integral $C$, the vacuum energies are degenerate even to the order $h$.

Let us now compute the quantum correction to the soliton mass by using the same $L_{ct.}$ as given by (10), (13) and (16). On using the soliton solution (3) it is easy to compute the contribution from the counterterms to the soliton mass. Taking (10) and (13) one finds that

$$\Delta E_{ct.}(\text{sol}) - \Delta E_{ct.}(\text{vac}) = \frac{3}{2} m^2 hL \int \frac{dk}{2\pi} \frac{1}{(k^2 + \mu^2)^{1/2}}.$$  \hspace{1cm} (17)

From (7) to (9) it then follows that

$$M_{s,1}\text{-loop} = \frac{hL}{2} \int_0^\infty \frac{dk}{2\pi} \left[ (k^2 + 4m^2)^{1/2} - (k^2 + m^2)^{1/2} - \frac{3}{2} \frac{m^2}{(k^2 + \mu^2)^{1/2}} \right]$$

\[ + \text{L independent terms}. \hspace{1cm} (18)\]

Note that here the vacuum is chosen at $\phi^2 = 1$. This choice is unimportant. Even if one chooses $\phi = 0$ as the true vacuum and does calculations one will be able to obtain the same $L$-independent part with $L$ dependent part (which any way cancels out) differing by an overall sign.

In view of (15) and (16) it is then clear that $M_{s,1}\text{-loop}$ is $L$-independent and finite. In a sense this result is understandable since, with our choice of $L_{ct.}$ the two vacuum energies are degenerate to order $h$. Hence it follows that the soliton exists to order $h$. Needless to say that with this prescription, the supersymmetric soliton will also have finite quantum correction to order $h$.

This prescription is not special to this particular problem but can in fact be generalized to any model in $1 + 1$ dimensions when the potential has unequal

\* This result is obtained by following the approach of Kaul and Rajaraman (1985) for calculating soliton fluctuations contribution which is different from that of Lohel and O'Brien (1981).
curvatures at the neighbouring degenerate minima (Lohe and O'Brien 1981). In particular, if the potential has curvatures $u_+$ and $u_-$ at the two neighbouring degenerate minima (so that classical soliton exists) then it can be shown that the difference of the vacuum energies is zero provided the $L_{ct.}$ is given by (10) and (13) with $\mu^2$ given by

$$
\left( \frac{u_+}{u_+ - u_-} \right) \ln \left( \frac{u_+}{\mu^2} \right) - \left( \frac{u_-}{u_+ - u_-} \right) \ln \left( \frac{u_-}{\mu^2} \right) = 1. \tag{19}
$$

By following the above arguments it is then clear that in these cases the one-loop quantum correction to the soliton mass is $L$-independent and finite.

4. One-loop effective potential with the new counterterm

In the last section we have shown that if the $L_{ct.}$ is as given by (10), (13) and (16) then the one-loop quantum correction to the soliton mass is $L$-independent and finite. KR may criticize this choice of $L_{ct.}$ by pointing out that the mass parameter in the loop integral has to be $m^2$ (and not $\mu^2$) since at one-loop level $\phi = 0$ is the true vacuum of the theory. However, this is not necessarily so and the true vacuum of the theory crucially depends on the renormalization prescription. To demonstrate it, in this section we compute one-loop effective potential with $L_{ct.}$ as given by (10), (13) and (16) and show that $\phi = 0$ is in fact not the true vacuum of the theory.

We shall follow the elegant procedure of Coleman and Weinberg (1973) while calculating the effective potential. Consider the Lagrangian density

$$
\tilde{L}(\tilde{x}, \tilde{t}) = \frac{1}{2} (D_\mu \tilde{\phi})^2 - \frac{1}{4} S^2(\tilde{\phi}), \tag{20}
$$

where

$$
\frac{1}{4} S^2(\tilde{\phi}) \equiv V_0(\tilde{\phi}) = \frac{\lambda'}{2m^2} \tilde{\phi}^2 \left( \tilde{\phi}^2 - \frac{m^2}{\lambda'} \right)^2. \tag{21}
$$

On changing variables

$$
\phi = \frac{\sqrt{\lambda'}}{m} \tilde{\phi}, \quad x_\mu = \frac{m^2}{\lambda'} \tilde{x}_\mu, \tag{22}
$$

equation (20) takes the form

$$
L(x, t) = \frac{1}{\lambda} \left[ \frac{1}{2} (\partial_\mu \phi)^2 - V_0(\phi) \right], \tag{23}
$$

where

$$
V_0(\phi) = \frac{1}{2} \phi^2 (\phi^2 - 1)^2, \quad \lambda = \frac{\lambda'}{m^6}. \tag{24}
$$

To one-loop order the effective potential is given by

$$
V_{\text{eff}}(\phi) = \frac{1}{\lambda} V_0(\phi) + \lambda_0 V_1(\phi), \tag{25}
$$

where $V_0(\phi)$ is given by (24), the one-loop term $V_1(\phi)$ can be obtained by summing up
of infinitely many one-loop graphs. As in the work of Coleman and Weinberg (1973) it is not difficult to show that \( V_1(\phi) \) is given by

\[
V_1(\phi) = \frac{1}{8\pi} V_0^\prime(\phi) \left[ \ln \left( \frac{\Lambda^2}{V_0(\phi)} \right) + 1 \right].
\]

where \( \Lambda \) is cut off and terms that vanish as \( \Lambda^2 \to \infty \) have been dropped. The divergence in \( V_1(\phi) \) can be taken care of by adding \( L_{c.t.} \) as given by (10), (13) and (16) which in this case has the form

\[
L_{c.t.} = \frac{1}{8\pi} (1 + \ln \Lambda^2 - \frac{1}{2} \ln 4) V_0^\prime(\phi).
\]

Hence \( V_{\text{eff}} \) up to one-loop level is given by

\[
V_{\text{eff}}(\phi) = \frac{1}{\lambda} V_0(\phi) + \frac{V_0^\prime(\phi)}{8\pi} \left[ \frac{1}{2} \ln 4 - \ln V_0^\prime(\phi) \right],
\]

where \( V_0(\phi) \) is given by (23). From here we notice that

\[
V_{\text{eff}}(\phi = 0) = \frac{1}{24\pi} \ln 4, \quad V_{\text{eff}}(\phi = \pm 1) = -\frac{1}{3\pi} \ln 4,
\]

so that \( \phi = 0 \) is not the true vacuum of the theory.

Before concluding the paper, we would like to make a few remarks regarding the general question of spontaneous symmetry breaking by radiative corrections.

(i) For our problem, at the classical level the potential \( V_0(\phi) \) has three degenerate minima at \( \phi = 0, \pm 1 \) which is the typical case of first-order transition point. However the 1-loop effective potential as given by (28) show that \( \phi = 0 \) is no more the true vacuum of the theory i.e. the symmetry is spontaneously broken. The true vacuum of the theory is at \( \phi = 1 \) provided \( \lambda < \lambda_c \). While for \( \lambda > \lambda_c \) the true vacuum is around \( \phi = \pm 1 \).

(ii) In an interesting article Rajaraman and Raj Lakshmi (1981) had started with the classical potential

\[
V_0(\phi) = (\phi^2 + a^2)(\phi^2 - 1)^2 - a^2, \quad 0 < a^2 < \frac{1}{2}
\]

which has local minima at \( \phi = 0 \) and degenerate absolute minima at \( \phi = \pm 1 \). By using the renormalization conditions at \( \phi = 0 \) they calculated the corresponding one-loop effective potential and showed that the true vacuum of the theory is now at \( \phi = 0 \) provided \( \lambda > \lambda_1 \) i.e. the symmetry is restored by radiative corrections if \( \lambda > \lambda_1 \) (\( \lambda_1 \) being some constant). However, if instead one calculated \( V_{\text{eff}} \) by using the renormalization conditions at \( \phi = \pm 1 \) rather than at \( \phi = 0 \) then one finds that symmetry is not restored no matter what the value of \( \lambda \) is. This again demonstrates that the true vacuum of the theory at one-loop level depends quite crucially on the renormalization prescription.

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