Semiclassical approximation for relativistic potentials

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Abstract. We consider the application of semiclassical approximation to relativistic potentials for massless particles where the kinetic energy is a nontrivial, nonlocal operator. Quantization rules are derived for an arbitrary confining potential and compared to some exact results for S-waves. These results admit of a partial generalization to small l values.

Keywords. Semiclassical approximation; relativistic potentials; massless particles; quantization.

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1. Introduction

The problem of bound states of relativistic systems is a matter of great and varied interest. Such systems ought to be described by field theoretic methods such as the Bethe-Salpeter equation, but the solution, and its interpretation, is simple only for instantaneous kernels in which case it is equivalent to a potential model. Such a potential description can now be formulated quite generally (Rohrlich 1979; Thakur J (to be published)) but the problems associated with centre of mass motion is most simply accounted for in the case of two equal mass particles. In that case one is left with an internal Hamiltonian (in the CM frame)

\[ h = 2(m^2 + p^2)^{1/2} + 2V, \]

(1)

where \( 2V \) is the interaction potential which will henceforth be considered to be local and central. The eigenvalues of this Hamiltonian operator are the masses of the composite system. This operator reduces to the familiar non-relativistic (NR) Hamiltonian for large \( m \), but, for a relativistic system, the mass of the constituents may be much smaller than the masses of the composite system. This is particularly true for the excited states in a confining potential and in this sense relevant to meson spectroscopy even if the quark mass which ought to be inserted in the above expression turns out to be much larger than the current quark mass estimate of a few MeV for the \( u \) and \( d \) quarks, and a few hundred MeV for the strange quarks. Also we have objects like glueballs whose constituents are massless. For these reasons, and for mathematical simplicity, we were led to consider the massless limit (\( m = 0 \)) of (1).

We shall work in the semiclassical approximation where NR analogy suggests the following form for the wavefunction.
\[ \psi = \exp \left[ \frac{i}{\hbar} \sigma_0(r) + \sigma_1(r) + O(h) \right]. \] (2)

Then, the eigenvalue equation
\[ \hbar \psi = M \psi, \] (3)
is equivalent to
\[ h(\nabla \sigma_0, r) = M, \] (4a)
\[ \frac{1}{2} \nabla \cdot \frac{\partial h}{\partial \nabla \sigma_0} + \nabla \sigma_1 \cdot \frac{\partial h}{\partial \nabla \sigma_0} = 0. \] (4b)

But this method is practically useless for confining potentials for \( \sigma_0 \) turns out to be real both in the allowed and forbidden region and, furthermore, \( \sigma_1 \) turns out to be nearly constant so that the wavefunctions appear to be unnormalizable. For this reason it is desirable to give an alternative derivation of the quantization rules which we shall do below. We shall first consider in § 2 the quantization rules for \( S \) waves and then give a partial generalization in § 3 valid for nonzero angular momentum. Our conclusion is briefly stated in § 4.

2. Quantization rules for \( l = 0 \)

Dividing the Hamiltonian by 2, we are led to consider the eigenvalue equation
\[ p \psi + V(r) \psi = \lambda \psi. \] (5)
The operator \( p \) is best defined in momentum space. Let \( q(p) \) be the unnormalized wavefunction in momentum space, so that ignoring irrelevant factors
\[ r \psi(r) = S(r) = \int_0^\infty p \phi(p) \sin(pr) dp, \] (6)
and the eigenvalue equation reads
\[ \int_0^\infty p[\phi(p)] \sin pr dp - [\lambda - V(r)] S(r) = 0. \] (7)
Defining \( C(r) = \int_0^\infty p \phi(p) \cos pr dp, \) (8)
equation (7) can be written as
\[ -\frac{dC}{dr} - (\lambda - V(r)) S(r) = 0. \] (9)

In the equation above \( C(r) \) is determined if \( S(r) \) is known for \( C(r) \) is the cosine transform of \( p \phi(p) \) which, by Fourier inversion theorem, is the sine transform of \( S(r) \). It is, therefore, a genuine and well-posed equation for a single function \( S(r) \). But the point of transition to (9) is that we can treat \( C(r) \) and \( S(r) \) as independent functions, more
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precisely, as the real and imaginary parts of a complex valued function of \( r \). Thus we observe that (9) is the real part of the equation

\[
-\frac{dE}{dr} + i(\lambda - V(r))E(r) = -if(r),
\]

(10)

for some real \( f(r) \) which may depend on \( \lambda \). (Here \( E(r) = C(r) + iS(r) \).) The occurrence of the arbitrary function \( f(r) \) which may depend on \( \lambda \) is related to the fact that (10) is a much more general equation than (7). It is, in fact, a pair of real linear differential equations for two independent functions \( C(r) \) and \( S(r) \). All trace of the inherent interdependence of \( S(r) \) and \( C(r) \) via (6) and (8) has been lost. The solutions of (10) form a much larger class among which, however, we may find the solutions of (7). Now, it is easy to solve (10) for \( E(r) \) in terms of \( f(r) \) and obtain the general solution for the wavefunction \( S(r) \). It is

\[
S(r) = C \sin[\Phi(r) + \alpha] + \int_0^r dr' f(r') \cos[\Phi(r) - \Phi(r')],
\]

(11)

where \( \Phi(r) = \int_0^r (\lambda - V(r')) dr' \).

(12)

As a solution of (7), this is merely a representation since it contains the arbitrary function \( f(r) \) which may even depend on \( \lambda \). Nevertheless, it is useful for us because we shall see that the wavefunction and the eigenvalues are independent of \( f(r) \) in the approximation in which it is regarded as slowly varying. Keeping \( f(r) \) undetermined for the present, we notice that the wavefunction vanishes as \( r \to \infty \) if (and only if)

\[
C = -\int_0^\infty f(r) \sin(\Phi(r) + \alpha) dr,
\]

(13)

and

\[
0 = \int_0^\infty f(r) \cos(\Phi(r) + \alpha) dr.
\]

(14)

The last equation would determine \( \lambda \) if \( f(r) \) and \( \alpha \) were known. If the potential is bounded at the origin then the form of the wavefunction near \( r = 0 \) requires \( \alpha = 0 \) and we shall assume this unless otherwise stated. \( f(r) \) can be determined, in principle, for it satisfies an integral equation by virtue of the fact that \( S(r) \) and \( C(r) \) (which is obtained from the same solution of (10)) are related to each other via (6) and (8) being the sine and cosine transform of one and the same function \( p\phi(p) \). Explicitly, this integral equation is

\[
\int_0^\infty dr' f(r') \int_0^r dr \sin(\Phi(r) - \Phi(r') + pr) = 0,
\]

(15)

but solving this exactly is clearly a hopeless task. Rather we notice that with

\[
S(r) = -\int_0^r f(r') \cos(\Phi(r) - \Phi(r')) dr',
\]

(16)

in view of (11), (13) and (14), we have
\[ S''(r) = -\left(\Phi'(r)\right)^2 S(r) + \Phi''(r)C(r) + f'(r), \]  

(17)

where \( C(r) \) is derived from the same solution of (10) and is obtained by replacing cosine by minus sine in (16). Equation (17) is apparently an inhomogeneous one, but it is important to note that if we had a true solution of (5) or (7), then this will uniquely determine the form of \( f(r) \) in (10). But the point of the representation (16) is that we can assume \( f(r) \) to be slowly varying and, in fact, set \( f(r) = -1 \) (for convenience). In the allowed region, we expect \( S(r) \) and \( C(r) \) to have the same order of magnitude, so for constant \( \Phi'(r) \), \( S(r) \) will be an eigenfunction of (5) for we can formally represent the operator \( p \) by \((-d^2/dr^2)^{1/2}\) and an eigenfunction of a positive Hermitean operator \( A \) is also an eigenfunction of \((A)^{1/2}\). From this, we can infer that for large \( \lambda \) and for slowly varying potentials, for which the second term above may be neglected in comparison to the first, \( S(r) \) will be an approximate solution of (5). Actually, as we shall show below, the second term includes the dominant term taking account of the variation of the potential in (5). The presence of this term is, in fact, essential for obtaining a normalizable solution. At large distances the wavefunction \( S(r) \) has the form

\[ S(r) \rightarrow \frac{V'(r)}{(V(r) - \lambda)^2}, \]

(18)

and the wavefunction is square integrable for all confining potentials. The quantization rule can be obtained from (14). The integral can be evaluated by the method of stationary phase and is dominated by contributions from around the turning point \( r_0 \) defined by \( V(r_0) = \lambda \). We use the standard formula

\[ \int_0^\infty \cos \Phi(r) \, dr \approx \left( \frac{2\pi}{V'(r_0)} \right)^{1/2} \cos \left( \Phi(r_0) - \frac{\pi}{4} \right). \]

(19)

derived in the stationary phase approximation by expanding the integrand about the turning point. Equating this to zero, we get

\[ \int_0^{r_0} (\lambda - V(r)) \, dr = (n + \frac{1}{2})\pi. \]

(20)

These results can be compared to some exact results. For the simple potential \( V(r) = r^2 \), the eigenvalue equation can be solved in momentum space. The eigenfunctions are the Airy functions which we shall write in this case as

\[ p \phi(p) = \int_0^\infty \cos \left( \int_0^w V(w) \, dw + (p - \lambda)u \right) \, du, \]

(21)

and the eigenvalue is determined from the condition that this vanish at \( p = 0 \)

\[ 0 = \int_0^\infty \cos \left( \int_0^w V(w) \, dw - \lambda u \right) \, du, \]

(22)

and agrees exactly with (14) for \( \alpha = 0 \). More generally, we can solve exactly the eigenvalue equation for any potential \( V(r) \) which is a polynomial in \( r^2 \) of degree \( n \), say. The resulting differential equation of order \( 2n \) is of the Laplace type (Spain and Smith 1970) and has exactly \( n \) solutions which vanish as \( p \rightarrow \infty \) and (21) is one of them. The
general solution is a linear combination of those and the eigenvalues are determined from the condition that the wave function \( \langle \phi | (p\phi(p)) \rangle \) and its first \( n-1 \) even order derivatives vanish at the origin \( p = 0 \) as required by the Hermiticity of the Hamiltonian. An analysis of these equations shows that although (21) differs from the true wavefunction significantly for \( p \to \infty \), it is the true wavefunction to within terms which are suppressed by some power of \( \lambda \) at \( p = 0 \) and (22) is the correct eigenvalue equation in the same sense (i.e. correct to \( O(\lambda^{-\beta}) \) for some \( \beta > 0 \)). The correct asymptotic behaviour of the wavefunction in coordinate space for the polynomial potential turns out to be \( r^{-2n-3} \) which agrees with (18) for \( n = 1 \). For \( n = 2 \), the coefficient of the leading term \( (r^{-7}) \) is suppressed by some power of \( \lambda \) and the dominant term appears to be \( r^{-9} \) in agreement with (18) and this probably is true generally.

These comparisons make it plausible that \( f(r) = \text{constant} \) represents a reasonable approximation to the true wavefunction in (16). We shall now show directly that this is a well-defined semiclassical approximation by identifying and isolating terms whose neglect, with \textit{a priori} justification, leads to this choice. First, we note that the eigenvalue equation (5) can also be written in the “squared version” as

\[
p^2\psi + [p, V(r)]\psi - (\lambda - V(r))^2\psi = 0, \tag{23}
\]

which is like a Schrödinger equation with a complex, energy-dependent, non-local potential. Because of the wrong sign of the real part of the potential, normalizable solutions exist only because of the strong absorption provided by the purely imaginary second term. We can use the identity (see Appendix)

\[
[p, V(r)] = -\frac{i}{2} (\nabla V: \hat{p} + \hat{p} \cdot \nabla V) + \frac{1}{2} [p, [\hat{p}_x, V(r)]] , \tag{24}
\]

where \( \hat{p} = p/p \) (and \( \hat{p}_x \) are its components). Dimensional estimates show that the last term is \( O(1/n^2) \) relative to the terms to be retained and may be neglected. (The method of obtaining these and other estimates is explained in the Appendix). The identity

\[
\hat{p} = \hat{f}(\hat{r}, \hat{p}) = \frac{1}{r} \times \frac{1}{p} L , \tag{25}
\]

then shows that to a relative accuracy of \( O(l/n) \) we can identify \( [p, V(r)] \approx -i V'(r) \). This term is already included in (17), the effect of \(-i\) being the change of the wave function \( S(r) \) to \( C(r) \). Equation (23) thus reduces to (17) when the triple commutator (which must be small in the classical limit for slowly varying potentials) is neglected, the operator \( \hat{p} \) is replaced by the operator \( \hat{r} \) for \( S \)-waves and the effect of \(-i\) in the term which depends on the derivative of the potential is taken into account by changing the sine wave into cosine wave \( (S(r) \to C(r)) \). The approximation is thus physically justified. Equation (23) can also be used to generalize the above solutions to potentials which are unbounded at the origin. For this it is necessary to get a solution of (23) directly out to some distance where the singularity has been tamed. After that we take the approximate solution (11) with the phase shift \( \alpha \) taken from the exact solution of the singular part. Because of this (20) is modified by a term \( \alpha \) on the left. \( \alpha \) can also be estimated in various ways if an exact solution is not available.
3. Generalization to non-zero angular momentum

The case $l \neq 0$ is fundamentally different from the $S$-wave case in that the Hamilton–Jacobi equation (4a) shows that $\sigma_0$ becomes imaginary in some region around the turning point before becoming real again. This implies the existence of a potential barrier in that region. The height and the width of this barrier depend on $l$ and vanish with it. For that reason, we can hope to extend the above formalism to non-zero $l$ at least where $l$ can be considered small in some sense. For $l \neq 0$ the eigenvalue equation is a generalization of (7)

$$\int_0^\infty p(\phi(p)) f_l(pr) \, dp = (\lambda - V(r)) \int_0^\infty p\phi(p) f_l(pr) \, dp,$$

where $f_l(x) = x j_l(x)$, $(g_l(x) = -x n_l(x))$ are the spherical Bessel functions multiplied by $x(\pi - x)$. For large $r$ this equation is identical with (7) except for an unimportant phase shift. Thus although a direct representation on the lines of (11) is not possible, we can attempt a representation by analogy:

$$S_l(r) = r\psi_l(r) = f_l(\Phi(r)) \int_r^\infty g_l'(\Phi(r')) f_l(r') \, dr'$$

$$+ g_l(\Phi(r)) \int_0^r f_l'(\Phi(r')) f_l(r') \, dr',$$

subject to the condition

$$\int_0^\infty f_l'(\Phi(r)) f_l(r) \, dr = 0,$$

(prime on a function denotes differentiation with respect to its argument). The function $f_l(r)$ must vanish sufficiently rapidly as $r \to 0$ so that the integrals converge and in analogy with the $l = 0$ case, we expect that $f_l(r) \to 1$ at large distances, but otherwise it is not determined. A simple interpolating form for $f_l(r)$ is

$$f_l(r) = x^{l+1}/(1 + x^{l+1}),$$

where $x = |\Phi(r)|$. Direct differentiation gives

$$-S_l + \frac{l(l+1)}{r^2} S_l = (\Phi')^2 S_l - \Phi'' C_l - \frac{l(l+1)}{r^2} \left[ \left( \frac{r\Phi'(r)}{\Phi(r)} \right)^2 - 1 \right] S_l - f'(r),$$

where $C_l$ which is the generalization of $C(r)$ is obtained by replacing $f_l$, $g_l$ by $f_l'$, $g_l'$ in front of the integrals in (27). The various terms in this equation should be compared to the exact equation (23) which is valid for all angular momenta. The second term on the right of (30) is the analogue of the second term in (23) and is obtained from it by the same steps as for $S$-waves (neglect of triple commutator, changing $\hat{p}$ to $\hat{r}$ and $-i\Sigma_l(r)$ to $C_l(r)$). This is justified only for small angular momenta. The last term is apparently an inhomogeneous term but with the chosen form of $f(r)$ it corresponds to a short-ranged potential singular (like $r^{-1}$) at the origin but contributing a phase shift which is $O(\lambda^{-1})$ and hence negligible. The most dangerous term is the third term on the right of (30). It is
bounded if the potential $V(r) \sim r^2$ near the origin and has the relative magnitude $l(l+1)\lambda^{-2} - r^n$ compared to the first term if $V(r) \sim r^n$ at large distances. On the other hand if $V(r) \sim r^\beta$ with $0 < \beta < 2$ near the origin, then this term is unbounded at the origin and contributes a phase shift $O(l(l+1)/\lambda^2)$ if cut off at some distance $\sim 1$. If these are negligible, then (27) is the approximate semi-classical eigenfunction that we require. Evaluating (28) by the method of stationary phase in which case the integral is dominated by contributions from around the turning point $r_0$ given by $V(r_0) = \lambda$ where we may write $f(r) \approx 1$ and use the asymptotic form of Bessel functions appropriate for $l \ll \Phi(r_0)$, we get the quantization rule

$$\int_0^{r_0} (\lambda - V(r)) \, dr = (n + \frac{1}{2} l + \frac{3}{4})\pi,$$  \hspace{1cm} (31)$$

which differs in this limit from (20) exactly by the phase shift produced by the centrifugal potential as compared to the $l = 0$ case. There are no exact results in this case but (31) agrees with the result of the application of ordinary WKB approximation in momentum space to the potential $V(r) = r^2$ for the limiting case $l \ll n$ relevant here. The opposite case $l \gg n$, in particular the case $l \gg n$ (circular orbits) is excluded from the domain of validity of (31).

Since $\Phi(r)$ has a positive maximum at $r = r_0$ and then a continuous decrease to $-\infty$ for confining potentials, it vanishes at some point far out in the classically forbidden region. This leads to a singularity of the approximate wavefunction there for $l \neq 0$. It is not possible by this method to have globally smooth wavefunctions which lead to correct eigenvalues. The best choice for wavefunction appears to be one which corresponds to the redefinition

$$\Phi(r) = \int_0^r (\lambda - V(r')) \, dr',$$

in (25) but this leads to an error of $O(1)$ in the eigenvalues (31) because it corresponds to a wavefunction which is not sufficiently smooth around the turning point.

4. Conclusion

It appears possible to apply semiclassical approximation to relativistic potentials, despite the fact that for confining potentials the equivalent Schrödinger potential is negative unbounded as displayed in (23). The final result as given in (20) and (31) is extremely simple but is valid only for $0 \leq l \ll n$ and therefore excludes the case of circular orbits. A result that appears to be valid in both limiting cases is the expected one

$$\int \left( (\lambda - V(r))^2 - \frac{(l + \frac{1}{2})^2}{r^2} \right)^{1/2} \, dr = (n + \frac{1}{2})\pi,$$  \hspace{1cm} (33)$$

with limits being determined by the classical turning points. Equation (33) is reasonable for large $l$ (circular orbits) in which limit the potential barrier referred to earlier may be considered to extend to infinity producing an NR-like situation. Whether the formula also holds generally in the semiclassical approximation is not known in the absence of
any proof. A proof of the formula (33) can also help us to deduce the formula for the general case \((m \neq 0)\) covered by (1).

Appendix A

The identity (24) is derived as follows. Writing \(p = \hat{\beta} \cdot p = p \cdot \hat{\beta} = \frac{1}{2} (\hat{\beta} \cdot p + p \cdot \hat{\beta})\), we find

\[
[p, V(r)] = -\frac{i}{2} (\nabla V \cdot \hat{\beta} + \hat{\beta} \cdot \nabla V) + \frac{1}{2} p \cdot [\hat{\beta}, V(r)] + \frac{1}{2} [\hat{\beta}, V(r)] \cdot p. \tag{A.1}
\]

But we have the identity \(\hat{\beta} [\hat{\beta} \cdot f(r)] + [\hat{\beta}, f(r)] \cdot \hat{\beta} = 0\) since \(\hat{\beta}^2 = 1\) and hence commutes with everything. From this we find

\[
\frac{1}{2} p \cdot [\hat{\beta}, V(r)] = \frac{1}{4} p [\hat{\beta} \cdot [\hat{\beta}, V(r)]], \tag{A.2}
\]

and (24) then follows. Equation (25) follows from the identity

\[
r \times (r \times p) = r(r \cdot p) - r^2 p. \tag{A.3}
\]

In the text we have given estimates for various quantities. The principle behind this is the following. Terms which are bounded are estimated by evaluating them near the turning point. For instance, we estimate the triple commutator in (24) to have the magnitude \(\sim V''(r)/p^2\). Noting that \(rp \sim n\) because of the quantization condition, we estimate the triple commutator to have the relative magnitude \(G(1/n^2)\) with respect to the term \(\sim V'(r)\) retained in (24). When terms which are unbounded occur in the equation, their influence cannot be estimated by this method. Now the singularity can occur only at the origin, so its influence can be estimated by the method mentioned at the end of \(\S 2\). For example, the term \(f'(r)\) in (30) is equivalent to a short-ranged potential with a \(r^{-1}\) singularity at the origin. Near the origin, (30) is like a Schrödinger equation with \(\lambda\) acting as the wavenumber. We can therefore estimate the phase shifts by methods familiar in NR quantum mechanics. This leads to the estimate \(O(\lambda^{-1})\) for the phase shift which can, therefore, be neglected.

References

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