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Relativistic dynamics for spin-zero and spin-half particles

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Abstract. The classical and quantum mechanics of a system of directly interacting relativistic particles is discussed. We first discuss the spin-zero case, where we basically follow Rohrlich in introducing a set of covariant centre of mass (CM) and relative variables. The relation of these to the classic formulation of Bakamjian and Thomas is also discussed. We also discuss the important case of relativistic potentials which may depend on total four-momentum squared. We then consider the quantum mechanical case of spin-half particles. The negative energy difficulty is solved by introducing a number of first class constraints which fix the spinor structure of physical solutions and ensure the existence of proper CM and relative variables. We derive the form of interactions consistent with Lorentz invariance, space inversion, time reversal and charge conjugation and with the above mentioned first class constraints and find that it is analogous to that for the non-relativistic case. Finally the relationship of the present work with some previous work is briefly discussed.

Keywords. Direct interactions; spin zero; spin half; covariant centre of mass; first class constraints.

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1. Introduction

There has been considerable progress recently in formulating a Hamiltonian theory of directly interacting relativistic particles. By direct interaction we mean an interaction between particles which does not require, *for its description*, the aid of an intervening field. Such a theory was first considered by Dirac (1949) who recognized that relativistic invariance implied that such a theory must admit the ten generators P^μ, $M^{\mu\nu}$ of the Poincaré group $\mathscr{P}$. It is also reasonable to demand that the coordinates and the momenta of real particles in a true theory should have suitable transformation properties under changes of space-time axes though this was not insisted upon by Dirac. It was shown, however, by various authors (Currie 1963; Currie *et al* 1963; Leutwyler 1965) that this was impossible except for free particles. Such a theory was also required to possess a property called 'separability' by Foldy (1961) who first considered the quantum case in the context of an expansion in powers of $1/c$. Briefly stated separability required that the generators of $\mathscr{P}$ referring to the composite system should converge in an appropriate sense to the sums of the generators for clusters into which the system divides itself as the intercluster distances become large. Separability was a strong restriction for Foldy used the so-called instant formalism of Dirac (1949) and the Hamiltonian used there is one of the generators of $\mathscr{P}$. Exact relativistic invariance is, however, so restrictive that solutions of a sufficiently general nature do not exist in this formalism.

Following the original work of Dirac (1964), it was guessed by various authors that non-trivial interactions may be possible within the constraint formalism. In such a formalism one either introduces interactions by modifying the energy momentum relation of a single particle (Komar 1978; Todorov 1980; Sazdjian 1981) or by considering the system as a whole (Rohrlich 1979a,b; King and Rohrlich 1980; Mukunda and Sudarshan 1981; Balachandran *et al* 1982a). Following Rohrlich, we shall formulate our theory here in terms of an overcomplete set of centre of mass (CM) and internal variables subject to certain constraints. These fall into two categories, one of them being a CM constraint involving all internal variables and the other ensuring that these internal variables are orthogonal to the total four-momentum. An explicit form of the CM constraint is however possible in only a few cases such as for two particles of equal mass or in the non-relativistic limit or for free particles. In the alternative procedure of using constituent particle variables, there is, of course, no centre of mass constraint but the time component of the momentum variables must be fixed by mass shell constraints which are known in precisely the above simple cases. For many problems the lack of knowledge of the CM constraint is not a hindrance and so, in general, the CM formalism is preferable. But it suffers from obvious disadvantages when electromagnetic interactions are to be considered. As we are not considering electromagnetic interactions here, we shall stick to the CM formalism.

Our chief new result is the treatment of spin-half particles. But we have also included spin-zero particles here for the sake of completeness and continuity and also because our treatment of these is somewhat different from that of Rohrlich, ours being more in the spirit of Dirac (1964). We have, furthermore, considered more general interactions than those discussed by Rohrlich and have also given explicit expressions for the CM variables. In considering spin-half particles we have to face the familiar negative energy difficulty. We solve it by imposing first class constraints that the kinetic energy of each particle be a positive definite operator within the subspace of physical states. Such a constraint is also needed to ensure the consistency of the CM constraint for the internal position variables, for it is well known that the time derivative of a position operator has a nontrivial spinor structure because of *zitterbewegung* effects. (Consistency of the formalism requires that the time derivative of the CM constraints vanish weakly in some sense. Ensuring this without the introduction of additional second class constraints is required by the demands of preserving the number of degrees of freedom). The first class constraint then amounts to saying that the theory must possess a certain invariance under rotations in the spinor space. This invariance greatly restricts the form of admissible interactions and in fact ensures that the acceptable interactions are precisely those that may be written down in the non-relativistic case except for a dependence on the total four momentum squared. (We discuss this in Appendix B). Invariance of the theory under space inversion and time reversal can be easily demonstrated. To introduce the notion of charge conjugation, it is necessary to enlarge the set of usual coordinates, momenta and spins by the addition of a two-valued operator commuting with space-time variables and denoting the 'particleness' of the constituents. This operator has the value $+1$ if the constituent is a particle and -1 if it is an antiparticle. The Hamiltonian can then be made charge conjugation invariant. A superselection rule is needed to ensure that physical states correspond to a definite value of this operator for each constituent except in the case of an identical particle-antiparticle system such as positronium, when the demands of generalized Pauli exclusion principle (and C invariance) require that any one constituent be a particle and the other an antiparticle. Two points may be made. Firstly, although the 'particleness' operator does not explicitly enter the free Hamiltonian in the spin-zero case, it is needed for a unified description of particles and antiparticles and for the construction of a C-odd electromagnetic current for the spin-zero case. Secondly, the distinction between particles and antiparticles is unimportant (and 'particleness' operator can be put equal to unity) if one is to consider particles only, as in nuclear physics.

The plan of the paper is as follows. In § 2 we consider spin-zero particles. In § 3 we consider spin-half particles. We consider in Appendix A the general form of CM constraint. In Appendix B we consider the modifications needed to describe interactions which depend, in effect, on total four-momentum squared.

2. Spin-zero case

We consider a system of N-structureless particles subject to direct interactions. The system will be described by introducing an $8N+8$ dimensional phase space subject to constraints that will be specified shortly. We introduce the total momentum four-vector P^μ of the system and its canonically conjugate variable Q^μ will be taken to label the world-line of the CM of the system with respect to the space-time axes of the observer. More precisely Q^μ labels possible rigid translations of the object and P^μ is their generator. Q^μ become the coordinates of the CM along the physical path by virtue of constraints satisfied. This point is important for no suitable definition of the CM exist along general paths. The coordinates and momenta of the constituent particles relative to the CM will be denoted by four-vectors ξ_a^μ, π_a^μ ($a = 1$ to N). These four-vectors turn out to be orthogonal to P^μ and, therefore, space-like for a physical system (we exclude from consideration light-like composite objects without excluding light-like constituents). We postulate the usual canonical Poisson bracket (PB) relations (with time-favoured metric):

$$[Q^\mu, P^\nu] = -g^{\mu\nu}, \tag{1}$$

$$[\xi_a^\mu, \pi_b^\nu] = -\delta_{ab}\, g^{\mu\nu} \tag{2}$$

with other PB's vanishing. For simplicity, we shall view the motion with respect to the time t recorded by a clock at rest in the CM even though the equations actually possess a 'chronometric invariance' in the sense of Dirac (1964) in that the equations of motion are invariant under the substitution of any invariant time whatsoever. If the canonical equation of motion

$$\dot{g} \equiv \frac{\mathrm{d}g}{\mathrm{d}t} = [g, H], \tag{3}$$

is taken over from NR mechanics, one is left with the task of constructing a Hamiltonian H which gives rise to a description with correct relativistic kinematics. The somewhat familiar choice, largely due to Rohrlich (1979a) and with a simple physical interpretation (see below), is

$$H = \sum_a (m_a^2 - \pi_a^2)^{1/2} + V - (P^2)^{1/2}. \tag{4}$$

Here V is a scalar function of the differences of ξ's and possibly of the momenta. The internal variables are not all independent and a number of constraints can be imposed in a form which is fully consistent with the equations of motion for arbitrary interaction. These take the form of the constraints

$$\xi_a \cdot P \approx 0, \quad \pi_a \cdot P \approx 0, \tag{5a, 5b}$$

which are second class because each of them has non-zero PB with at least one other constraint. Since π_a^μ are the momenta in the CM frame, they satisfy

$$\sum_a \pi_a^\mu \approx 0, \tag{6}$$

which is preserved because V is a function of the differences of ξ's. Similarly, since ξ's are the position variables relative to the CM they must satisfy a second class constraint of the general form

$$\sum_a (u_a \xi_a^\mu + v_a \pi_a^\mu) \approx 0, \tag{7}$$

where u_a and v_a are certain unknown scalar functions of the π's and the differences of ξ's chosen so as to preserve the constraint (Appendix A). The explicit form of the constraint is known for general interaction only in the two-particle equal mass case when $u_a = 1, v_a = 0$. It is also known for free particles, for piece wise constant central potentials and in the NR limit.

The dynamically possible trajectories are fixed by the following Lorentz invariant constraints which are first class (i.e. they have vanishing PB with all other constraints):

$$\chi_1 \equiv H \approx 0, \tag{8}$$

$$\chi_2 \equiv \left(\frac{P_0}{|P_0|} - 1\right) \approx 0. \tag{9}$$

These constraints have simple physical interpretations. Equation (8) is a realization of the general observation of Dirac (1964) that the Hamiltonian of a theory with chronometric invariance must vanish. With the particular choice (4) of the Hamiltonian, this has the physical significance that the mass of the composite system is the sum of the kinetic energy and the potential energy of the constituents in the CM frame. Equation (9) on the other hand expresses the positivity of the total energy and does not play any dynamical role.

With the constraints (5)–(8) and the circumstance that the initial value of Q^0 is irrelevant, one is left with $6N$ degrees of freedom. It may be desirable (it is in fact necessary for transition to quantum mechanics) to reformulate the theory so that the superfluous degrees of freedom associated with second class constraints could be eliminated from the dynamical theory. The method for doing this for general constraints is due to and described by Dirac (1964) and consists in introducing a new bracket, the Dirac bracket DB, defined in terms of the old PB and the constraints, such that the DB of the constraints with any dynamical variable vanishes. The constraints can then be considered to be identities or definitions of the superfluous variables. In the present case the DB, incorporating the constraints of (5) which can be written down simply by inspection and which agrees with the result of Dirac prescription, is

$$[\xi_a^\mu, \pi_b^\nu]^* = \left(\frac{P^\mu P^\nu}{P^2} - g^{\mu\nu}\right)\delta_{ab}, \tag{10}$$

where the single star superscript identifies the DB incorporating (5). As Dirac (1964) shows this change of bracket does not affect the equations of motion which are equally valid with the new bracket as with the old if (and only if) the constraints are conserved. The constraints (6)–(7) can also be handled similarly (see Appendix A) but the result is simple only in the two-particle equal mass case, in which case we have the final Dirac brackets with two star superscripts $(a, b = 1, 2)$:

$$[\xi_a^\mu, \pi_b^\nu]^{**} = \left(\frac{P^\mu P^\nu}{P^2} - g^{\mu\nu}\right)(\delta_{ab} - \tfrac{1}{2}). \tag{11}$$

In this case we can write

$$p_a^\mu = \tfrac{1}{2}P^\mu + \pi_a^\mu, \quad q_a^\mu = Q^\mu + \xi_a^\mu \tag{12}$$

and express everything in terms of constituent particle coordinates and momenta, but the PB of the constituent particle variables does not have a simple explanation.

The physical nature of gauge invariance can be illustrated by considering the canonical transformations generated by χ_1 and identifying it as a gauge change. Thus for any observable g, the canonical transformation

$$g \to g_1 = g + \varepsilon[\chi_1, g] = g + \varepsilon[H, g] \tag{13}$$

represents the evolution of the system corresponding to the replacement $t \to t + \varepsilon$. This does not change the state if the state is defined broadly to include all such systems (i.e. corresponding to all t). Thus we are considering an ensemble which is here a family of physical systems such that it contains all those systems which have or could have evolved one from the other (this point of view is capable of generalization, see Appendix B). We also note that the fundamental quantities P^μ and

$$M^{\mu\nu} = (Q \wedge P)^{\mu\nu} + \sum_a (\xi_a \wedge \pi_a)^{\mu\nu} \tag{14}$$

constitute the generators of the Poincaré group which is the basic symmetry group of physical states. They are separable when written in terms of constituent particle variables (see Appendix A). However, this does not imply that the theory is separable in the sense of Foldy (1961). The question of separability of constrained dynamical systems is a vexed one (Samuel 1982; Balachandran *et al* 1982b). It is reasonable to extend the definition of separability to include the decomposition of the constraints as well as the covariant Hamiltonian. Relativistic kinematics and transformation laws make this impossible in the present case (the mass of a composite system differs from the masses of the clusters by their relative kinetic energy). We should mention, however, that the physically *essential* requirement is that of *weak separability* which can be formulated as the requirement that the dynamics (i.e. the accelerations) in each of two separating clusters becomes independent of the conditions in the other cluster as the intercluster distance goes to infinity (Currie 1966). This condition is easily seen to be satisfied for the present case and, as one can check, a satisfactory scattering theory can be formulated in the present context.

Finally it is perhaps expected that because of the constraints (5), the Dirac brackets involving Q^μ are all altered, except the one with P^ν. We get for the changed brackets.

$$[Q^\mu, \xi_a^\nu]^* = \xi_a^\mu P^\nu / P^2, \tag{15}$$

$$[Q^\mu, \pi_a^\nu]^* = \pi_a^\mu P^\nu / P^2, \tag{16}$$

$$[Q^\mu, Q^\nu]^* = \sum_a \frac{(\xi_a \wedge \pi_a)^{\mu\nu}}{P^2}. \tag{17}$$

It can be shown that these brackets are not modified by the incorporation of the constraints (6)–(7) so (15)–(17) are valid equally well with two-star superscripts. An explicit representation of Q^μ will be obtained shortly in the quantum case which is what we consider from now on.

In making the transition to quantum theory we simply follow Dirac (1964). We reinterpret the DB incorporating the constraints (5)–(7) (the 'two-star' superscript bracket defined explicitly for the two particle equal mass case by (11)) as the quantum bracket i.e. $-i$ times the commutator of dynamical variables which will now be understood to be linear operators in some linear vector space $\mathscr{H}$. (The second class constraints are then to be understood as the definitions of the superfluous variables to be eliminated, if possible). The time development of these operators follows the Heisenberg form of equation of motion. This can be obtained from the quantum analogue of (3) with the bracket appearing there being interpreted as a DB. An explicit representation of $\mathscr{H}$ can be obtained as follows. We introduce the usual (non-covariant) internal variables $(0, \boldsymbol{\xi}'_a)$, $(0, \boldsymbol{\pi}'_a)$ which agree with (ξ_a^μ, π_a^μ) in the subspace of $\mathbf{P} = 0$ states. Then by Lorentz covariance

$$\xi_a^0 = \frac{\boldsymbol{\xi}'_a \cdot \mathbf{P}}{(P^2)^{1/2}}, \tag{18}$$

$$\boldsymbol{\xi}_a = \boldsymbol{\xi}'_a + \mathbf{P} \frac{\boldsymbol{\xi}'_a \cdot \mathbf{P}}{P_0 (P^2)^{1/2} + P^2}, \tag{19}$$

and a similar equation for π_a^μ. The primed variables are constrained only by (6) and (7) which must hold identically. In the special case of two equal mass particles they satisfy

$$[\xi'_{ai}, \pi'_{bj}] = i\delta_{ij}(\delta_{ab} - \tfrac{1}{2}) \tag{20}$$

with other commutators vanishing. In the general case, when (7) holds with unknown coefficients, the complete expression for the fundamental commutators is not explicitly known. It is nevertheless possible to show that the relative momenta commute and so do the *differences* of ξ's and their mutual commutator takes the form

$$[\xi'_{ai} - \xi'_{bi}, \pi'_{cj}] = i(\delta_{ac} - \delta_{bc})\delta_{ij}. \tag{21}$$

It is then clear that we can take the P^μ and either N-1 relative momenta or $N-1$ of the differences of ξ's e.g. $\boldsymbol{\xi}'_1 - \boldsymbol{\xi}'_2$, $\boldsymbol{\xi}'_1 - \boldsymbol{\xi}'_3$, ... $\boldsymbol{\xi}'_1 - \boldsymbol{\xi}'_N$ as constituting a complete set of commuting variables and obtain a representation of $\mathscr{H}$ in the form of $L^2(R^4 \times R^{3(N-1)})$. The physical states $|\phi\rangle$ in $\mathscr{H}$ satisfy the constraint.

$$\chi_1|\phi\rangle = 0, \quad \chi_2|\phi\rangle = 0, \tag{22a,b}$$

and would form a subspace Φ of $\mathscr{H}$ but for the divergence difficulties associated with gauge invariance: the L^2 norm of states satisfying (22a) is infinite. This difficulty is somewhat formal, for to verify (22a) one needs to make an accurate measurement of energy which is not possible in a limited time. It does, however, suggest the need for caution in formal manipulations. Divergence difficulties apart, Φ is isomorphic to the familiar Hilbert space of NR quantum mechanics, and this is very satisfying. A rigorous resolution of the divergence difficulty remains to be worked out but, in practical applications, one can simply ignore the CM part of the wavefunction.

The quantum version of (15)–(17) can then be solved to give in all cases

$$Q^0 = -i\frac{\partial}{\partial P_0}, \tag{23}$$

$$\mathbf{Q} = i\frac{\partial}{\partial \mathbf{P}} + \sum_a \frac{(\xi'_a \times \pi'_a) \times \mathbf{P}}{P_0\,(P^2)^{1/2} + P^2}. \tag{24}$$

The operators $(\mathbf{J}, \mathbf{K})$ defined in terms of the tensor $M^{\mu\nu}$ by $(M^{ij} = \varepsilon_{ijk} J^k, \quad K^k = M^{ko})$ have the familiar single-particle form when expressed in terms of these variables. They show that the effect of a general Lorentz transformation on the variables (ξ'_a, π'_a) is to rotate them and a rotationally invariant function of these variables is actually a world scalar.

We wish to add two more points. Lorentz invariance allows the potential energy form factors to depend on P^2 apart from any dependence on the relative momenta. Such dependence can arise even for ladder diagrams because of the contribution of antiparticles (which are included in the Feynman propagator and excluded here) or from higher order (crossed) diagrams or from annihilation or recombination type diagrams. Such dependence of the interaction on P^2 is very awkward for the present theory, since it implies a non-trivial dynamics for the CM which would not satisfy the kinematical constraint $\dot{Q}^2 = 1$ (thus making inaccessible the time t). A possible solution is discussed in Appendix B. Secondly, we might mention that if a subsidiary constraint $Q^0 = q^0$ were imposed, then one can show using path integral formalism (Faddeev 1981) that the present formulation is equivalent to one of the Bakamjian and Thomas form (Bakamjian *et al* 1953) with q^0 acting as the time in that form (i.e. one can eliminate P_0, Q_0 from the phase space). The constraint (8) ensures that $P_0 = (\mathbf{P}^2 + h^2)^{1/2}$ where $h = \sum_a (m_a^2 - \pi_a^2)^{1/2} + V$ and P_0 acts as the Hamiltonian in the Bakamjian form. However, manifest covariance is lost in this form, and the observables such as position do not have simple behaviour under Lorentz transformation.

3. Spin-half case

To accomodate the spin-half case, we follow the standard trick of linearizing the square root type energy operator. We can easily convince ourselves that apart from the sign ambiguities inherent in the gamma matrices, the unique invariant form for the kinetic energy (KE) operator is

$$h_a = s_a(-i\gamma_a \cdot \pi_a + \gamma_{5a} m_a), \tag{25}$$

where the factor s_a can be ± 1. (Our choice for the Dirac matrices agrees with Gasiorowicz (1967) and where necessary, we have found the so-called high energy or spinor representation as the most convenient). This form is non-Hermitean in general, being the square root of an expression which can become negative for time like π_a. It is sufficient for our purpose that it is relativistically invariant and Hermitean in the CM frame. There does not appear to be any deep physical significance in the choice of γ_{5a} multiplying the mass term but the manifest covariance will be lost if the KE is chosen to be of the form $\boldsymbol{\alpha}_a \cdot \boldsymbol{\pi}'_a + \beta_a m_a$ in the CM frame. (Correct behaviour under space inversion will be shown to hold nevertheless). With the help of a non-unitary Foldy-Wouthuysen (F.W.) transformation, (25) can be written as

$$h_a = A_a s_a \gamma_{5a} \varepsilon_a A_a^{-1}, \tag{26}$$

with $\quad \varepsilon_a = (m_a^2 - \pi_a^2)^{1/2}$

and $$A_a = \frac{i\gamma_{5a}\gamma_a \cdot \pi_a + m_a + \varepsilon_a}{[2\varepsilon_a(\varepsilon_a + m_a)]^{1/2}} = S_a(\mathbf{P}) U_a S_a^{-1}(\mathbf{P}), \tag{27}$$

where $S_a(\mathbf{P})$ is the matrix of the form exp $(\mathbf{b}.\boldsymbol{\alpha}_a/2)$ corresponding to a pure Lorentz transformation taking $((P^2)^{1/2}, \mathbf{0})$ to $(P_0, \mathbf{P})$, and U_a is the unitary FW transformation

$$U_a = \frac{-i\gamma_{5a}\gamma_a \cdot \boldsymbol{\pi}'_a + m_a + \varepsilon'_a}{[2\varepsilon'_a(\varepsilon'_a + m_a)]^{1/2}} \tag{28}$$

with $\varepsilon'_a = (m_a^2 + \boldsymbol{\pi}'^2_a)^{1/2} = \varepsilon_a$. We can then write the Hamiltonian of free particles as

$$H_0 = S(\mathbf{P})\, U\left[\sum_a s_a \gamma_{5a} \varepsilon'_a\right] U^\dagger S^{-1}(\mathbf{P}) - (P^2)^{1/2}, \tag{29}$$

where $S = \prod_a S_a$, $U = \prod_a U_a$. This form of the Hamiltonian is invariant under the n-parameter transformations

$$SU \exp\left(i \sum_a \theta_a \gamma_{5a}\right) U^\dagger S^{-1}. \tag{30}$$

We can therefore impose n mutually commuting first class constraints

$$SU s_a \gamma_{5a} U^\dagger S^{-1} \approx 1, \tag{31}$$

in the usual sense of being valid when acting on physical states. The constraints ensure that the KE of each particle is positive for physical states. They also serve another purpose. The dynamical variables $(\xi'_a, \boldsymbol{\pi}'_a)$ are not all independent. One relation is

$$\sum_a \boldsymbol{\pi}'_a \approx 0, \tag{32}$$

which is obviously conserved. The relation conjugate to (32) must involve the coordinates ξ'_a, be free of spinorial structure since a matrix constraint implies more than one constraint and should have a vanishing time derivative. This is impossible because of *zitterbewegung* effects and the best that one can achieve is to have a vanishing time derivative for the mean constraint

$$\bar{\chi} = SU\chi\, U^{\dagger}S^{-1}. \tag{33}$$

(The nomenclature and the definition of the mean position variables is similar to the one for the usual Dirac equation (Davydov 1966)). Since this type of Hamiltonian does not have a simple interpretation in terms of classical phase space, the usual method of transition to quantum mechanics is inapplicable here and we are free to invent a suitable procedure provided it is physically plausible. Thus we shall say that χ is a valid CM constraint if $[\bar{\chi}, H]$ (calculated with a set of fundamental brackets unconstrained by the CM constraints e.g. $[\xi'_{ai}, \pi'_{bj}] = i\delta_{ab}\delta_{ij}$) vanishes. These constraints (32)–(33) can then be incorporated in the fundamental brackets as before and the resulting equations of motion are to be accepted as the correct one. For free particles

$$\chi = \sum_a \{\varepsilon'_a, \xi'\} \approx 0 \tag{34}$$

and the derivative of $\bar{\chi}$ vanishes only because of the first class constraints (31) above. Since these (positivity of KE and CM constraint) are both necessary and desirable even when interactions are present, we are forced to retain the invariance (30) and introduce only those interactions which preserve it. However, before introducing the interaction, it is necessary to dispose of some questions.

As the Hamiltonian is non-Hermitean, it is necessary to use an indefinite metric. The metric operator is

$$\eta = (SS^{\dagger})^{-1}. \tag{35}$$

This ensures that the norm of physical states satisfying the constraint (31) automatically comes out positive if the norm is defined in the usual way as

$$\langle\phi|\phi\rangle = (\eta\phi, \phi). \tag{36}$$

(This is another reason for the necessity of the constraint (31)). Hermiticity in this metric means that the operator has the form

$$A = S\bar{A}S^{-1}, \tag{37}$$

where $\bar{A}$ is Hermitean in the usual sense. The Hamiltonian is thus Hermitean. Invariance under space inversion $\mathscr{I}$ is ensured by the condition

$$H_0(\mathbf{P}, \pi'_a) = PH_0(-\mathbf{P}, -\pi'_a)\, P^{-1}, \tag{38}$$

where

$$P = S(\mathbf{P})\left(\prod_a \gamma_{5a}\right) S^{-1}(-\mathbf{P}). \tag{39}$$

Time reversal invariance must be described by an antilinear operator to ensure that kinetic energies remain positive. It amounts to the relation (in momentum representation)

$$\mathscr{T} H_0 \mathscr{T}^{-1} \equiv T H_0^*(-\mathbf{P}, -\pi'_a)\, T^{-1} = H_0(\mathbf{P}, \pi'_a) \tag{40}$$

where, in the spinor representation,

$$T = S(\mathbf{P}) \left(\prod_a i\alpha_{2a} \right) S^{*-1}(-\mathbf{P}), \tag{41}$$

$\mathscr{I}$ and $\mathscr{T}$ are respectively unitary and antiunitary operators. Noting that $\mathscr{I}$ and $\mathscr{T}$ also carry the instruction of changing the sign of the momenta with them (and complex conjugation in the case of the latter) we see that $\mathscr{I}^2 = 1$ and $\mathscr{T}^2 = (-1)^N$ as expected.

We consider now charge conjugation invariance. If the antiparticles are to have a negative internal parity relative to the particles, then they must be described by a negative energy spinor. To keep the kinetic energy positive, we must take $s_a = -1$ for antiparticles and $s_a = +1$ for particles. Thus s_a is an operator (the particleness operator) whose eigenvalues are ± 1. This operator commutes with all the usual quantum mechanical variables but is odd under charge conjugation. For the charge conjugation operator we may take

$$C = S(\mathbf{P}) \left(\prod_a \beta_a \right) S^{-1}(\mathbf{P})\, C_0, \tag{42}$$

with $C_0 s_a C_0^{-1} = -s_a$ for all a, C_0 having no space or spinor structure.

The form of the centre of mass coordinates Q^μ and the six quantities $(\mathbf{J}, \mathbf{K})$ constituting the tensor $M^{\mu\nu}$ must both be modified to take account of the spin of the particles. A suitable form for $(\mathbf{J}, \mathbf{K})$ is

$$\mathbf{J} = S\left[i\frac{\partial}{\partial \mathbf{P}} \times \mathbf{P} + \sum_a (\xi'_a \times \pi'_a + \tfrac{1}{2}\sigma_a) \right] S^{-1}, \tag{43}$$

$$\mathbf{K} = S\left[iP_0 \frac{\partial}{\partial \mathbf{P}} + i\mathbf{P}\frac{\partial}{\partial P_0} + \sum_a \frac{\mathbf{P} \times (\xi'_a \times \pi'_a + \frac{1}{2}\sigma_a)}{P_0 + (P^2)^{1/2}} \right] S^{-1}. \tag{44}$$

Correspondingly we can define a set of pseudo-Hermitean and covariant centre of mass variables Q^μ with

$$Q^0 = -i\, S \frac{\partial}{\partial P_0} S^{-1}, \tag{45}$$

$$\mathbf{Q} = S\left[i\frac{\partial}{\partial \mathbf{P}} + \sum_a \frac{(\xi'_a \times \pi'_a + \frac{1}{2}\sigma_a) \times \mathbf{P}}{P_0 (P^2)^{1/2} + P^2} \right] S^{-1}. \tag{46}$$

These imply that

$$M^{\mu\nu} = (Q \wedge P)^{\mu\nu} + \sum_a (\xi_a \wedge \pi_a + \tfrac{1}{2}\sigma_a^{\perp})^{\mu\nu}, \tag{47}$$

where

$$\sigma_a^{\perp\mu\nu} = \sigma_a^{\mu\nu} - \frac{\sigma_a^{\mu\rho} P_\rho P^\nu + \sigma_a^{\rho\nu} P_\rho P^\mu}{P^2}, \tag{48}$$

is the transverse-transverse part of the spin tensor. The commutation relations (15)–(16) remain unchanged but equation (17) is modified in the suggestive manner $\xi_a \wedge \pi_a \to \xi_a \wedge \pi_a + \frac{1}{2}\sigma_a^{\perp}$.

We now discuss the form of interaction. If we write the Hamiltonian in the form

$$H = \sum_a (-i\gamma_a \cdot \pi_a + \gamma_{5a} m_a) s_a + \bar{V} - (P^2)^{1/2}, \tag{49}$$

then to preserve the constraints (31) we must choose

$$\bar{V} = SUVU^{\dagger}S^{-1}, \tag{50}$$

where V must commute with γ_{5a} (for all a). This restricts us to the Dirac matrices 1, σ_a, γ_{5a}, $\gamma_{5a}\sigma_a$. If we use the constraint (31) (i.e. if we do not distinguish between terms whose matrix elements between physical states is the same even though they can, for example, give rise to differing electromagnetic effects), we are left with a two-body potential of the form

$$V = V_{ss} + \sigma_1 \cdot \mathbf{V}_{ts} + \mathbf{V}_{st} \cdot \sigma_2 + \sigma_1 \cdot \mathbf{V} \cdot \sigma_2, \tag{51}$$

where the form factors V_{ss} etc are of the form

$$V_1 + V_2 s_1 s_2, \tag{52}$$

and each of the form factors must be constructed from the internal variables such as the π', the differences of ξ' etc (and may also be a function of P^2, see Appendix B). The particular form given above is automatically C-invariant. Parity and time reversal invariance has to be checked separately as in non-relativistic mechanics. It should be noted that the rotational invariance here ensures Lorentz invariance.

The explicit form of the Hamiltonian equation (49) shows that the corresponding wave functions are exactly analogous to those found recently by an approximate solution of the Bethe-Salpeter equation except for a relabelling of the Dirac algebra $i\gamma \to \alpha, \gamma_5 \to \beta$ (Thakur 1979, 1983). The Hamiltonian of (49) is, however, different from the Breit Hamiltonian in essentially two respects. First, the use of CM variables automatically ensures invariant masses, thus precluding a difficulty commonly present when constituent particle variables are used (Artru 1984). Our construction is the result of our insistence that the CM move in a straight line even for a composite system of spinning particles. Secondly, our use of mean position variables in constructing the interaction, equation (50), has the consequence that spin-orbit coupling is not a consequence of relativity, rather it is arbitrary and has to be determined from other considerations. More specifically, if the interaction (51) is chosen to be spin independent, then the dynamics remains spin-independent in contrast to, say, Dirac equation. This is the price one has to pay for complete discard of wrong sign spinors (i.e. negative energy spinors for particles or positive energy spinors for antiparticles) even when interaction is present.

Lastly we wish to point out an important difference between the present theory and other approaches. Our construction based as it is on finite dimensional non-unitary representations of the Lorentz group involves indefinite metric and has a manifestly covariant form. If a solution exists within a vector space with a positive definite norm, it is unlikely to be manifestly covariant, and what is more important, would be different from Dirac equation even for free particles. Such a theory can only describe fermions with electromagnetic properties rather different from that of Dirac fermions.

Appendix A. CM constraint

The constraint of equation (7) is conserved if

$$[H, \sum_a (u_a \xi_a^\mu + v_a \pi_a^\mu)] \approx 0, \tag{A.1}$$

where the zero on the right is an arbitrary combination of the expressions (6) and (7) with coefficients μ and λ respectively. Evaluating the commutator, we have

$$[H, u_a] - 2\sum_b \frac{\partial V}{\partial \xi_{ab}^2}(v_a - v_b) = \lambda u_a, \tag{A.2}$$

$$[H, v_a] - \frac{u_a}{(m_a^2 - \pi_a^2)^{1/2}} = \lambda v_a + \mu. \tag{A.3}$$

We may impose the constraint $\sum_a v_a = 0$ in view of the constraint (6). Thus

$$\mu = -\frac{1}{N}\sum_a \frac{u_a}{(m_a^2 - \pi_a^2)^{1/2}}. \tag{A.4}$$

Also

$$\left[H, \sum_a u_a\right] = \lambda \sum_a u_a. \tag{A.5}$$

λ represents the freedom of multiplying (7) by an arbitrary function. If we therefore set $\sum_a u_a = 1$, we get $\lambda = 0$. Simple solutions exist only in the cases already mentioned.

If the constituent particle coordinates and momenta are q_a, p_a, we can write

$$q_a^\mu = Q^\mu + \xi_a^\mu + \frac{v_a}{u_a}\pi_a^\mu, \quad p_a^\mu = u_a P^\mu + \pi_a^\mu. \tag{A.6}$$

These ensure that

$$M_{\mu\nu} = \sum_a (q_a \wedge p_a)_{\mu\nu}, \quad P_\mu = \sum_a p_{a\mu} \tag{A.7}$$

which may be considered a justification for the actual choice (16). Equation (A.7) ensures that $M^{\mu\nu}$ and P^μ are separable in the sense of Foldy.

Finally we shall write down the explicit form of the Dirac bracket incorporating the constraints (6)–(7). Let

$$\chi_\mu = \sum_a \pi_{a\mu}, \tag{A.8}$$

$$\psi_\mu = \sum_a (u_a \xi_{a\mu} + v_a \pi_{a\mu}), \tag{A.9}$$

where u_a and v_a are to be regarded as functions of *differences* of ξ's and of π's. Then

$$[\chi_\mu, \chi_\nu]^* = 0, \tag{A.10}$$

$$[\chi_\mu, \psi_\nu]^* = \left(g_{\mu\nu} - \frac{P_\mu P_\nu}{P^2}\right), \tag{A.11}$$

and

$$[\psi_\mu, \psi_\nu]^* = T_{\mu\nu} = -T_{\nu\mu}, \tag{A.12}$$

is an unknown expression. Following the general procedure of Dirac (1964) we find

$$\begin{aligned}[f, g]^{**} = [f, g]^* - [f, \chi_\mu]^* \, T^{\mu\nu}[\chi_\nu, g]^* \\ + [f, \chi_\mu]^* \left(g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2}\right)[\psi_\nu, g]^* \\ - [f, \psi_\mu]^* \left(g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2}\right)[\chi_\nu, g]^*. \end{aligned} \tag{A.13}$$

To verify that this bracket vanishes identically if one of the entries is χ_μ or ψ_μ, it is necessary to recall that the constraints (5) are already identities with the single star bracket. Equation (A.13) has the consequence that the unknown quantity $T_{\mu\nu}$ does not enter the explicit expression of a bracket if either of its entries has a vanishing (single star) bracket with χ_μ. This is useful for transition to quantum mechanics.

Appendix B. Potentials depending on total mass

As pointed out in the text, potential energy form factors chosen to reproduce effects whose origin lies in the domain of field theory are, unlike in the Galilean case, expected to depend on P^2 also. Such interactions destroy the kinematical nature of the centre of mass (i.e. CM does not move uniformly). Consequently, we cannot use a clock moving with the CM to set up equations of motion and the formalism devised here becomes inapplicable. A possible solution of the difficulty is the following: we enlarge the phase space by introducing two scalars α and M commuting with all other variables and having

$$[\alpha, M] = 1. \tag{B.1}$$

We define the Hamiltonian to be

$$H = \left[\frac{M^2 - P^2}{2M}\right] + \left[\sum_a (m_a^2 - \pi_a^2)^{1/2} + V - M\right] \tag{B.2}$$

with the first class constraints

$$P^2 - M^2 \approx 0, \tag{B.3}$$

$$\sum_a (m_a^2 - \pi_a^2)^{1/2} + V - M \approx 0, \tag{B.4}$$

other constraints remaining as before. (One has to demand in addition $P_0 \geqslant M > 0$, this could also be written as a constraint). We now use the extra degree of freedom available to demand that V be independent of P^2 but may depend on M which we identify with the mass of the composite system. In this manner we can also consistently discuss energy-dependent potentials such as the optical potentials of nuclear physics in a Lorentz invariant manner, while retaining the kinematical nature of the centre of mass. The least satisfactory aspect of this extension is the physical interpretation of α. This depends on the interpretation of the constraints (B.3) and (B.4) as the generators of local gauge invariance (cf. equation (13)). The first constraint (B.3) is kinematical while the second is dynamical. We shall assume that the local gauge invariance is actually integrable so that one can define an equivalence class of physical systems such that two systems are in this class if their dynamical variables can be connected to each other by gauge transformations generated by (B.3) and (B.4). (These ensure that every system remains gauge-equivalent to every other system into which it can evolve). We shall take the point of view that such a theory describes the ensemble of gauge-equivalent physical systems (i.e. the whole equivalence class rather than one particular element). It then appears that α is a parameter labelling the elements of this class. This labelling is in general time-dependent ($\dot{\alpha} = \partial H/\partial M \neq 0$) unless the interaction happens to be independent of M. Then α may be identified with coordinatized proper time, for a change in α can be exactly compensated by a change in t for arbitrary dynamical variables. This interpretation may be expected to be approximately valid if the interaction depends on M only weakly. Interactions which depend on M arise only when complex relativistic systems (which can temporarily lose their identity) are considered and the description of such systems in relativistic particle dynamics cannot entirely parallel non-relativistic mechanics. We should add that the above ideas are easily taken over to quantum mechanics.

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