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Effective suppression of period-doubling in two diffusively coupled logistic maps

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Abstract. We consider a system of two delay diffusively coupled logistic maps. We find that for moderate values of diffusion coupling, the period-doubling sequence is effectively suppressed. Our study supports the existence of certain generic features for systems consisting of two coupled maps.

Keywords. Period-doubling; coupled logistic maps; flip-flop cycles.

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1. Introduction

A number of recent studies (Deissler 1984; Kaneko 1985; Kapral 1985; Waller and Kapral 1984) have focussed on diffusively (or otherwise) coupled nonlinear map lattices in an attempt to model the complex phenomena seen in dynamical systems with many degrees of freedom. These phenomena include spatio-temporal intermittency, pattern formation, kink-antikink production and propagation, etc (Bishop *et al* 1983; Kuramoto 1981; Nozaki and Bekki 1983). Because of the complicated nature of the systems considered, most of the work has been numerical in nature. A variety of interesting phenomena (of the type seen in partial differential equations) have been seen in these numerical simulations of coupled maps. However, only limited analytic progress has been made, mostly in the study of entrainment (Yamada and Fujisaka 1983).

In an attempt to isolate some of the generic features associated with coupled maps, some authors (Gu *et al* 1984; Hogg and Huberman 1984; Kaneko 1983; Lee and Tomita 1984) have considered the case with two coupled logistic maps. This system is described by the equations

$$x_{n+1} = c^{(1)} x_n (1 - x_n) + g^{(1)}(x_n, y_n),$$
$$y_{n+1} = c^{(2)} y_n (1 - y_n) + g^{(2)}(x_n, y_n), \tag{1}$$

where (x_n, y_n) denotes the phase space location of the system at the nth iterate; $c^{(1)}$ and $c^{(2)}$ are the driving parameters at sites 1 and 2, respectively; and $g^{(1)}, g^{(2)}$ are the coupling functions. In an ecological context, (1) may be interpreted as two neighbouring populations (numbering x_n and y_n in the nth generation respectively) exchanging individuals through a diffusion process (either linear or more complicated). Typical physical systems which (1) could describe are coupled Josephson junctions or small aspect ratio Benard cells.

In this paper we investigate the period-doubling sequence of symmetric logistic maps with a delayed diffusive coupling (to be explained later). The diffusion term can have the effect of strongly suppressing the period-doubling sequence seen in the single logistic map. We describe the mechanism whereby this suppression occurs and the parameter regions in which it takes place. Some of the features observed by us appear to be common to such systems (Hogg and Huberman 1984; Kaneko 1983; Lee and Tomita 1984). This supports the idea that generic properties can be attributed to coupled map systems (Hogg and Huberman 1984).

The commonly investigated (Hogg and Huberman 1984; Kaneko 1983; Lee and Tomita 1984) version of (1) is the symmetric case with linear diffusive coupling as under

$$\begin{aligned} x_{n+1} &= f(x_n) + D(y_n - x_n), \\ y_{n+1} &= f(y_n) + D(x_n - y_n), \end{aligned} \tag{2}$$

where D is the diffusion constant and we use the notation $f(x) = cx(1-x)$ for the logistic function. This system displays a complicated sequence of limit cycles, phase locking-unlocking sequences and quasiperiodic behaviour before a transition to chaos. It is claimed that such phenomena are generic to coupled maps (Hogg and Huberman 1984). However, as pointed out by Yamada and Fujisaka (1983), (2) (and its N-dimensional generalization) is unphysical. This is because, even for moderate values of D, only a few specially chosen initial conditions lead to a motion confined to the $[0, 1] \times [0, 1]$ square (or any other finite region). To remedy this defect, Yamada and Fujisaka (1983) suggested a delayed diffusion coupling system

$$\begin{aligned} x_{n+1} &= f(x_n) + D(f(y_n) - f(x_n)), \\ y_{n+1} &= f(y_n) + D(f(x_n) - f(y_n)). \end{aligned} \tag{3}$$

This system (and its N-dimensional generalization) is well-behaved (in the sense of confinement within the $[0, 1] \times [0, 1]$ square) for arbitrary initial conditions in $[0, 1] \times [0, 1]$, $c \in [0, 4]$ and $D \in [0, 1]$. For $D = 0$, the logistic maps decouple and x_n, y_n evolve independently. This serves as a comparison line (in c-D parameter space) as we increase the value of D from zero. In the ecological context, (3) is particularly meaningful as a model of alternating migration and reproduction (e.g. reproduction in winter and migration in summer). Consider two neighbouring populations, described by x_n and y_n respectively ($x_n, y_n \in [0, 1]$). In the first stage (say reproduction) the populations are updated as

$$x_n' = f(x_n); \quad y_n' = f(y_n). \tag{4}$$

In the second stage, migrations take place according to

$$x_{n+1} = x_n' + D(y_n' - x_n'); \quad y_{n+1} = y_n' + D(x_n' - y_n'). \tag{5}$$

Combining (4) and (5), we arrive at (3). We have conducted a numerical/analytical study of (3), focussing on the period-doubling bifurcations. The tools we used are evaluations

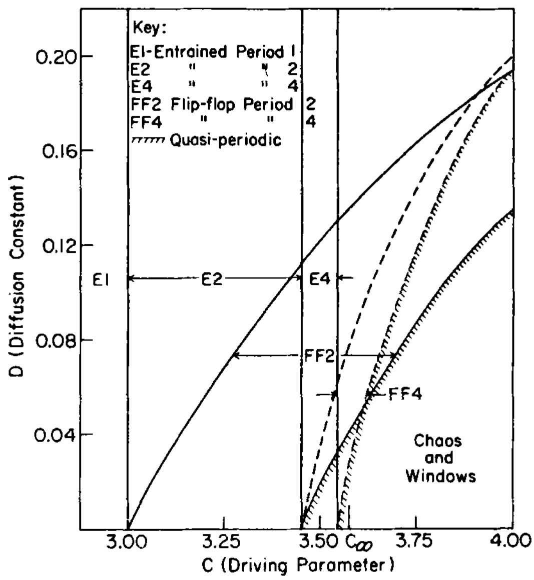

Figure 1. Parameter space diagram showing the stability regions of the lower period-2^n entrained and flip-flop cycles.

of the period-2^n limit cycles of (3); eigenvalues of the linearized map about these cycles; and the basins of attraction of these cycles. We present a qualitative description of our results below. Detailed numerical dependences are available from the parameter space diagram in figure 1, where the stability regions of the lower period limit cycles are shown.

2. Description of limit cycles

There are three types of period-2^n limit cycles for (3):

2.1 *Entrained limit cycles* ($x = y$)

In this case the period-2^n limit cycles are of the form $(x_i^*(c), x_i^*(c))$ where $x_i^*(c)$ denotes the ith point on the period-2^n limit cycle of the logistic map. Linearization about these limit cycles yields the eigenvalues $f^{(2^n)\prime}(x_i^*(c))$ and $(1-2D)^{2^n} f^{(2^n)\prime}(x_i^*(c))$ for the period-2^n limit cycle. These correspond to synchronized perturbations (eigenvector $\binom{1}{1}$) and antisynchronized perturbations (eigenvector $\binom{1}{-1}$) respectively. As $D \in [0, 1]$, $|1-2D|^{2^n} \leqslant 1$. Thus, the entrained limit cycles have the same stability region (in c) as the corresponding limit cycles for the single logistic map. (Notice that a cycle which is unstable for the single logistic map may have a coupled-map counterpart which is diffusively stabilized against antisynchronized perturbations. However, there would still be an unstable direction and numerically these orbits would not be observable). Entrainment can occur for $c > c_\infty$ also, where c_∞ denotes the c-value where chaos occurs for the single logistic map. Then an "entrained chaos" motion is seen.

2.2 *Flip-flop limit cycles (FFCs)*

For period-2^n these are of the form (x_i, y_i), where

$$x_{i+2^{n-1}} = y_i; \quad y_{i+2^{n-1}} = x_i. \tag{6}$$

Thus, there is a switch of the original values after 2^{n-1} iterations. Similar FFC have been found for (2) by Kaneko (1983) and Hogg and Huberman (1984). For period-2 we can analytically solve for the FFC. In (3) we substitute (Kaneko 1983)

$$\xi_n = \tfrac{1}{2}(x_n + y_n); \quad \eta_n = \tfrac{1}{2}(x_n - y_n). \tag{7}$$

The period-2 FFC is defined by $(\xi_0, \eta_0) \to (\xi_1, \eta_1) \equiv (\xi_0, -\eta_0) \to (\xi_0, \eta_0)$. We can solve for it as

$$\xi_0 = \frac{1}{2}\left(1 + \frac{1}{c(1-2D)}\right),$$

$$\eta_0 = \left\{\frac{1}{2c}\left[1 + \frac{1}{c(1-2D)}\right]\left[\frac{c}{2} - 1 - \frac{1}{2(1-2D)}\right]\right\}^{1/2}. \tag{8}$$

The following points are relevant vis-a-vis the period-2 FFC:

(a) It exists in the regions of (c, D) space where

$$\frac{c}{2} - 1 - \frac{1}{1(1-2D)} \geqslant 0,$$

or

$$D < D^* = \frac{c-3}{2(c-2)}. \tag{9}$$

In this paper, we consider only the region $D < 1/2$. This is because we wish to compare our results with those of Hogg and Huberman (1984) and Kaneko (1983) where moderate values of D are considered.

From (9), as $D^* \geqslant 0$, we see that $c \geqslant 3 = c^*_{1\to 2}$, where $c^*_{1\to 2}$ denotes the value of C at which the period $-1 \to 2$ bifurcation occurs for the single logistic map.

From figure 1, it is clear that there is an overlap between the stability region of the various FFC's and the entrained cycles. Thus, for $c^*_{2\to 4} \geqslant c > c^*_{1\to 2}$, there is a region of D-values in which the flip-flop and entrained period-2 cycles are stable. The competition between them is resolved by the relative sizes and locations of their basins of attraction. This point will be treated in greater detail later.

(b) At $D = 0$, the period-2 FFC is stable in the parameter region $c^*_{2\to 4} \geqslant c > c^*_{1\to 2}$ and unstable in the parameter region $c > c^*_{2\to 4}$. As D is increased, two bifurcation routes are possible, depending on the value of c. For $c^*_{2\to 4} \geqslant c > c^*_{1\to 2}$, the linearized eigenvalues of the FFC increase along the real line. At some value of D (depending on c, as shown in figure 1) one of the eigenvalues crosses the unit circle and a saddle point appears. At slightly higher D, the saddle point becomes a repellor. Finally, at $D = D^*$ (defined by (11)) the FFC has shrunken to zero size and coalesced with the period-1 entrained cycle, which is unstable in this region of c. For $c > c^*_{2\to 4}$, the FFC stays unstable (with complex eigenvalues, though) for small values of D. On increasing D, the

FFC becomes stable, with the eigenvalues still being complex. Further increase of D brings them down onto the real line. Subsequent bifurcations with increasing D are similar to those described for $c^*_{2\to 4} \geq c > c^*_{1\to 2}$.

For $c > c^*_{2\to 4}$, there is a small region of parameter space where a quasiperiodic attractor exists, before the period-2 cycle is stabilized. This is shown as the shaded region in figure 1. It is similar to that observed by Hogg and Huberman (1984) for (2). More details on this region will be presented elsewhere (Puri and Jackson 1986).

The same qualitative behaviour holds for the period-4 FFC. The stability region, however, is smaller than in the period-2 case as shown in figure 1. Shrinkage of the period-4 FFC results in a coalescence with the period-2 entrained cycle, which is unstable in this region.

We expect the same behaviour for higher period-2^n FFC, with correspondingly smaller stability regions. Our numerical results support this (Puri and Jackson 1986). These stability regions appear to have a nested structure in most regions of parameter space, as shown for periods-2 and -4 in figure 1. This would suggest that there is a bifurcation sequence in (c, D) space, with successively higher period-2^n FFCs becoming stable. This is not period-doubling in the conventional sense, because all the previous cycles coexist as attractors. However, there exist initial points in the unit square which go to higher period-2^n cycles as c and D are varied. Thus, in this restricted sense, period-doubling takes place on a set of initial conditions (Oono, Private Communication). Notice that we are not assured of the existence of initial points which undergo an infinite period-doubling sequence. However, if the nested structure holds for all period-2^n FFCs, there would be a region (of low, may be zero, measure) in parameter space where all period-2^n cycles coexist as attractors with chaos. We are presently investigating this possibility (Puri and Jackson 1986).

Thus, we find that diffusion can stabilize period-2^n FFCs. These stabilized cycles extend upto $c = 4$. Because of the rapidly shrinking regions in which higher period limit cycles are stable, two FFCs are dominant viz. period-2 and -4. At moderate values of diffusion, the period-2 and -4 FFCs have a larger basin of attraction than the entrained cycles or chaos. Most initial conditions end up on low period orbits. Thus, diffusion introduces temporal stability in the coupled system, in large regions of parameter space. The same seems to be true of the higher-dimensional generalization of (3) (Puri and Jackson 1986). This effect should be contrasted with studies of chemical reactions (Kuramoto 1981; Oono and Kohmoto 1985) where diffusion is essential to the appearance of chaos.

2.3 *Other limit cycles*

Our numerical results show a number of other period-2^n ($n \geq 1$) limit cycles. For period-2, the only stable limit cycle (other than entrained) is the FFC described earlier. For period-4, there is another (other than entrained and FFC) pair of stable limit cycles. To see how these arise, we consider the situation at $D = 0$, when the logistic maps are decoupled. The period-4 branch for the single logistic map is shown in figure 2. The period-4 FFC corresponds to the case where the oscillators are at a and b (or c and d). The other pair of limit cycles arises from the combinations a-c and a-d (which goes into c-a on iteration). For $c^*_{4\to 8} \geq c > c^*_{2\to 4}$, this pair is stable at $D = 0$. On increasing D, the eigenvalues (real) increases and the limit cycle becomes unstable. For $c > c^*_{4\to 8}$, this pair starts off unstable at $D = 0$. On increasing D, it becomes stable through the eigenvalues

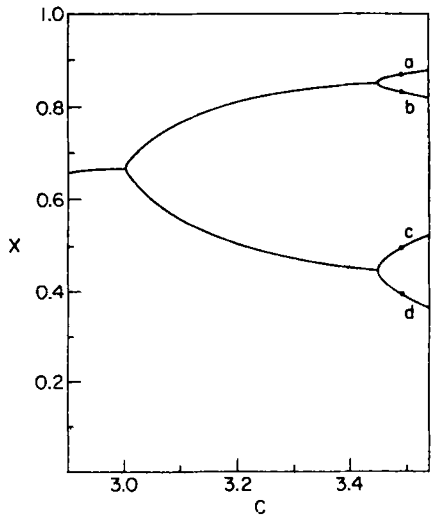

Figure 2. Lower period branches of the bifurcation diagram for the single logistic map.

(complex) crossing the unit circle. Subsequent bifurcations are the same as for $c^*_{4\to 8} \geqslant c > c^*_{2\to 4}$. The quasiperiodic attractor mentioned earlier is found after the period-4 non-FFCs have become unstable and before the period-2 FFC has become stable.

We have also investigated the basins of attraction of period-2 and -4 FFCs/entrained cycles for different parameter values. In figures 3(a)–(c) we show the shrinking basin of attraction for a period-2 FFC as D is increased, with $c = 3{\cdot}625$ ($> c_\infty$). Figure 3(d) shows the basin of attraction for a period-4 FFC for $c = 3{\cdot}625$ and $D = 0{\cdot}06$ (the same values as for figure 3(a)). Figures 3(a) and 3(d) account for 98 % of the points considered in the region $[0, 1] \times [0, 1]$. The basins are symmetric about the lines $x = 1/2$ and $y = 1/2$. This is because (x, y) $(1 - x, y)$ $(x, 1 - y)$ and $(1 - x, 1 - y)$ are all mapped into the same point by (3). An interesting feature of these basins is their self-similar nature along the diagonals.

3. Conclusions

To summarize, we have considered a system of two delay diffusively coupled logistic maps. We consider the period doubling sequence for this system and find that it is effectively suppressed by moderate values of the diffusion coupling. The mechanism of this suppression is described. Some of the features we find in our system are similar to those found by other authors with different systems (Gu *et al* 1984; Hogg and Huberman 1984; Kaneko 1983; Lee and Tomita 1984). This supports the contention that coupled map systems possess certain generic properties.

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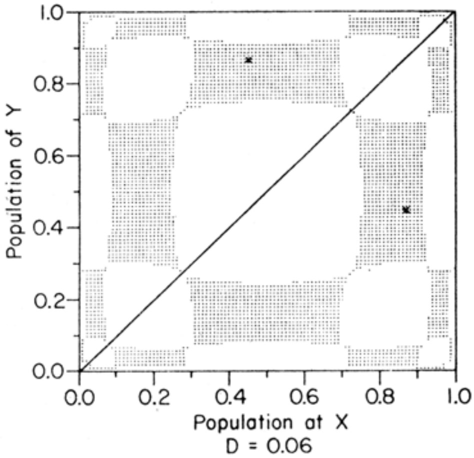

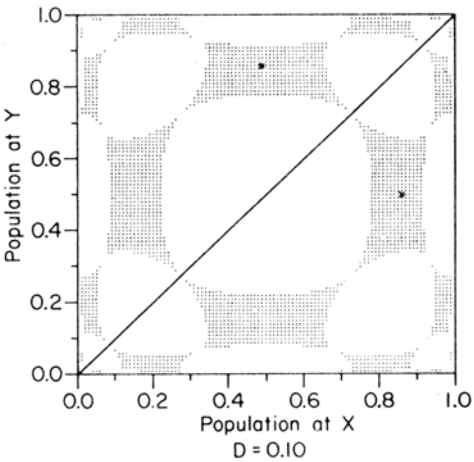

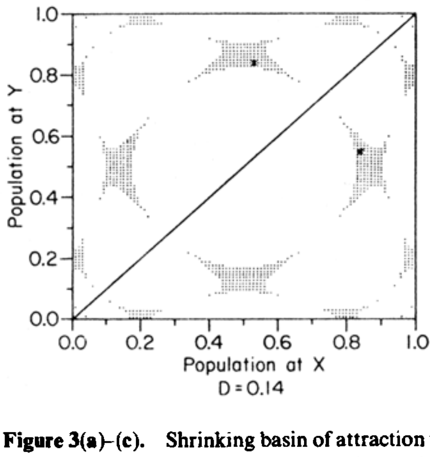

Figure 3(a)–(c). Shrinking basin of attraction for a period-2 flip-flop cycle as D is increased, with $c = 3{\cdot}625$. Points on the limit cycle are denoted by x.

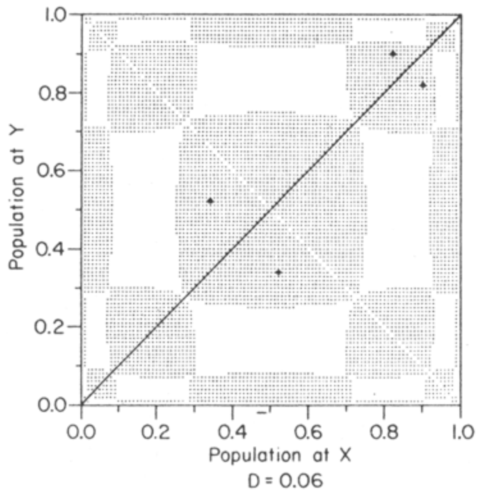

Figure 3(d). Basin of attraction for a period-4 flip-flop cycle for $c = 3{\cdot}625$ and $D = 0{\cdot}06$. Points on the limit cycle are denoted by 0.

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