Pramāṇa – J. Phys., Vol. 27, No. 5, November 1986, pp. 637–646. © Printed in India.

Spherically symmetric free fall collapse

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MS received 13 January 1986; revised 25 August 1986

Abstract. The general dynamical equations for spherical gravitational collapse are derived by introducing the eigenvalue of the conformal Weyl tensor in the 2-2 component of the Einstein tensor and assuming the material content of the models to be a perfect fluid. Since this eigenvalue is coupled always with the material energy density, it has been interpreted as the *energy density of the free gravitational field* whose presence is related with anisotropy and inhomogeneity. As a particular case, the collapse of a spherically symmetric dust (zero pressure) with vanishing radial acceleration (free fall collapse) is discussed. It is shown that the model is inhomogeneous with non-vanishing shear of the congruence of world lines of the dust particles. The model contains gravitational radiation by Szekere's criterion since both shear invariant and the spatial gradient of density are non-vanishing. This is in contrast to the Oppenheimer-Synder model for which both the above mentioned characteristics are absent. A particular solution which is anisotropic and inhomogeneous has been given to prove the emission of gravitational radiation by the freely falling dust and in this case the energy density of the free gravitational field contains a type N term superposed on the coulombian field.

Keywords. Gravitational collapse; gravitational radiation; free fall collapse; energy momentum tensor.

PACS Nos 04·20; 04·30

1. Introduction

Gravitational collapse occurs in the Newtonian framework as well as in general relativity. By the Newtonian force law $F = -M/r^2$, one can see that for a fixed mass, as the radius of the star decreases, the gravitational force increases very rapidly. By choosing an appropriate equation of state, one can construct pressure gradients to oppose the collapse. That is, in both Newtonian and relativistic theories collapse can be avoided and stable equilibrium achieved for any given mass M by choosing an equation of state which gives sharply rising pressures under compression at sufficiently low densities. The pressure gradient that prevents a fluid element from falling appears in Einstein's theory as the sources of an acceleration. This acceleration, by keeping the fluid element at a fixed r value, causes it to depart from geodesic motion. That is, in relativistic theory we have the equation describing the balance between gravitational force and pressure gradient for a spherically symmetric static star as (see Tolman 1958):

$$p' = -(\rho + p)\nu'/2. \tag{1}$$

Throughout this paper a prime and an overhead dot denote respectively differentiation with respect to r and t. In the Newtonian case ($\rho \gg p$) it reduces to

$$p' = -\rho v'/2. \tag{2}$$

Thus, we can infer that general relativity predicts stronger gravitational forces in a stationary body than does Newtonian theory, because pressure also contributes to attraction. These forces, among their other important effects, can pull certain white-dwarf stars and super massive stars into gravitational collapse under circumstances where Newtonian theory would have predicted stable hydrostatic equilibrium. The characteristic difference between Newtonian theory and general relativity can be seen if we consider the motion of freely falling particles. Let us consider a particle dropped into a Newtonian potential field $\psi = -M/r$. Unless it is aimed precisely towards $r = 0$, it always scatters and rebounds passing the scattering centre $r = 0$ at some distance of closest approach, $r_{\min}$ with $0 < r_{\min} < b$ where b is the impact parameter. A study of the geodesics in Schwarzschild metric shows that particles whose velocities correspond to impact parameters $b < (27)^{1/2} M$ do not scatter, but fall on to $r = 0$, following a spiral path to conserve angular momentum. This can be interpreted as when relativistic effects become important, gravitational forces may overwhelm centrifugal forces, which is not the case in the Newtonian theory. Suppose, we consider a ball of dust of uniform density. All the dust particles will attract one another, so the sphere will tend to contract. And we have the acceleration of a dust particle at the surface as $-M/r^2$ where M and r are the mass and radius of the sphere respectively. As the volume of the sphere tends to zero, the contraction rate rapidly rises giving the proper time of collapse as

$$\tau = \pi(R_i^3/8M)^{1/2}, \tag{3}$$

where R_i is the initial radius of the sphere (see Thorne 1967). That is, for example if the internal pressures of the sun were suddenly removed then it will shrink to a point in just over half an hour.

When matter is in free fall (zero acceleration) with increasing density, and remains in this state, such a collapse is called catastrophic, a characteristic of which being emission of gravitational radiation. According to Misner (1969) a mass moving near its own Schwarzschild radius at free fall velocities (which are nearer to the velocity of light) will inevitably emit large amounts of gravitational radiation.

The first spherically symmetric free fall collapse was discussed by Oppenheimer and Snyder (1939). They have used the well-known Friedman cosmological metric to describe the interior geometry of a collapsing sphere filled with pressure free fluid (dust particles). Since the geometry is conformally flat, the free gravitational field vanishes identically in this case and there is no emission of gravitational radiation. According to Szekeres (1966) the propagation of gravitational radiation in a perfect fluid distribution is characterized by the presence of shear and rotation of neighbouring fluid particles and the spatial gradient of the matter density. In the Oppenheimer-Snyder model the density is uniform, and both shear and rotation are identically zero. Hence, we would like to construct an inhomogeneous spherically symmetric model with free fall collapse and show that for this model both the shear of the neighbouring particles and spatial gradient of material density are non-vanishing. Thus, even though the collapsing dust emits gravitational radiation it cannot escape from the outer surface of the star due to Birkhoff's theorem. However, the shearing forces can distort the sphere into an ellipsoid which has the r-direction as the principal axis and is degenerate in the θ, ϕ-plane.

The energy momentum tensor and the metric are introduced in §2. The field equations are introduced in §3 and the general dynamical equations for spherical gravitational collapse are derived. In §4, the problem of a spherically symmetric gravitating dust with free fall is discussed and in §5 it was shown with the help of a particular solution that the dust emits gravitational radiation even though it gets trapped in the surface layers of the star and cannot escape through the outer surface. In §6, the interior solution is shown to match with the exterior Schwarzschild solution over a surface r_b = constant (or $R(r_b, t) = R_b$) and the paper ends with our concluding remarks in §7.

2. Energy momentum tensor and the metric form

For a perfect fluid distribution of matter given by the energy momentum tensor

$$T_a^b = (\rho + p)u_a u^b - p g_a^b, \; u_a u^a = 1, \tag{4}$$

the matter conservation law, equation of continuity, equations of motion and the thermodynamic relationship are given respectively in the form (see Misner 1969):

$$(n u^a)_{;a} = 0, \tag{5}$$

$$\rho_{,a} u^a + (\rho + p) u^a_{;a} = 0, \tag{6}$$

$$(\rho + p) u^a_{;b} u^b = (g^{ab} - u^a u^b) p_{,b}, \tag{7}$$

$$e_{,a} u^a + p(1/n)_{,a} u^a = 0, \tag{8}$$

where $\rho = n(1+e)$. Here n and e are respectively baryon number density and specific internal energy. The geometry for such an energy momentum distribution is chosen as

$$ds^2 = -(\exp \lambda) dr^2 - R^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) + (\exp \nu) dt^2, \tag{9}$$

where λ, R and ν are functions of r and t only. Here r and R are respectively the comoving (Lagrangian) and position (Eulerian) coordinates; i.e. $R(r, 0) = r$. Defining the directional derivatives

$$D_r \equiv \exp(-\lambda/2)(\partial/\partial r), \quad D_t \equiv \exp(-\nu/2)(\partial/\partial t), \tag{10}$$

the velocity of the fluid is given by

$$U = u^a (\partial R/\partial x^a) = D_t R, \tag{11}$$

where $u^a = (0, 0, 0, \exp(-\nu/2))$. Similarly, we define

$$\Gamma = D_r R, \tag{12}$$

where $(\Gamma^2 - 1)$ reduces in the special relativistic limit to the kinetic energy per unit mass of the fluid. The baryon conservation equation (5) may be expressed as

$$\exp(-\lambda/2) = 4\pi R^2 n, \tag{13}$$

and hence (12) takes the form,

$$\Gamma = \exp(-\lambda/2)R' = 4\pi n R^2 R'. \tag{14}$$

3. Field equations

The geometric part of the field equations for the metric (9) is written by introducing the eigenvalue of the Weyl conformal tensor in the 2-2 component of the Einstein tensor (see Krishna Rao 1971, 1972, 1973). Thus, Einstein's field equations take the form

$$8\pi T_1^1 = -8\pi p = -[1/R\exp(\lambda)]\{(R'^2/R) + R'\nu'\} + [1/R\exp(\nu)]\{2\ddot{R} + (\dot{R}^2/R) - \dot{R}\dot{\nu}\} + (1/R^2), \tag{15}$$

$$8\pi T_2^2 = 8\pi T_3^3 = -8\pi p = 8\pi\varepsilon - [1/R\exp(\lambda)]\{2R'' - (R'^2/R) - R'\lambda' + R'\nu'\} + [1/R\exp(\nu)]\{2\ddot{R} - (\dot{R}^2/R) + \dot{R}\dot{\lambda} - \dot{R}\dot{\nu}\} - (1/R^2), \tag{16}$$

$$8\pi T_4^4 = 8\pi\rho = -[1/R\exp(\lambda)]\{2R'' + (R'^2/R) - R'\lambda'\} + [1/R\exp(\nu)]\{(\dot{R}^2/R) + \dot{R}\dot{\lambda}\} + (1/R^2), \tag{17}$$

$$8\pi T_4^1 = 0 = [1/R\exp(\lambda)](2\dot{R}' - R'\dot{\lambda} - \dot{R}\nu'). \tag{18}$$

where

$$8\pi\varepsilon = [\exp(-\lambda)][(R''/R) + (R'/R)^2 - \frac{\nu''}{2} - \frac{\nu'^2}{4} - (R'\lambda'/2R) + (R'\nu'/2R) + (\lambda'\nu'/4)] + [\exp(-\nu)][(\ddot{\lambda}/2) + (\dot{\lambda}^2/4) - (\ddot{R}/R) + (\dot{R}/R)^2 - (\dot{R}\dot{\lambda}/2R) + (\dot{R}\dot{\nu}/2R) - (\dot{\lambda}\dot{\nu}/4)] + (1/R^2). \tag{19}$$

is the eigenvalue of the Weyl conformal tensor in Petrov's classification (Krishna Rao 1966).

Now, the expression for $\exp(-\lambda)$ is computed from the combination $\{(15) + (17) - (16)\}$ as

$$\exp(-\lambda) = [1 + U^2 - \{8\pi(\rho+\varepsilon)R^2/3\}](R')^{-2}, \tag{20}$$

or

$$\Gamma^2 = 1 + U^2 - \{8\pi(\rho+\varepsilon)R^2/3\}. \tag{21}$$

By comparing the expression (20) with the one obtained by Misner and Sharp (1964), we get

$$M(r,t) = 4\pi(\rho+\varepsilon)R^3/3, \tag{22}$$

where the mass function $M(r, t)$ is defined as

$$M(r,t) = \int_0^R \rho(4\pi R^2 \, dR). \tag{23}$$

The expression for $M(r, t)$ given by (22) shows that $(\rho + \varepsilon)$ plays the role of mean density of matter within a sphere of radius R. As pointed out earlier (Krishna Rao 1971, 1972, 1973) the coupling of ε with the material energy density suggests that the former may be interpreted as the energy density of the free gravitational field. The force distribution for this free gravitational field has the effect of distorting a sphere into an ellipsoid which has the r-direction as the principal axis and is degenerate in the θ, ϕ-plane (Szekeres 1965). The explicit relationship between the energy density of the free gravitational field and the shear invariant has been obtained in § 5.

From (18) it is easy to note that

$$D_t \lambda = 2(\partial U/\partial R), \tag{24}$$

where we have written $\partial/\partial R = (R')^{-1}(\partial/\partial r)$.

Now the matter conservation equation (5) may be written after eliminating λ with the help of (24), and a slight rearrangement of terms, as

$$D_t(nR^2) = -nR^2(\partial U/\partial R). \tag{25}$$

Again from (15), (18) and (20) we get

$$D_t U = (1/2)\Gamma^2(\partial \nu/\partial R) - (4\pi/3)(\rho + \varepsilon + 3p)R. \tag{26}$$

The first term on the right hand side of (26) represents the mechanical forces through the radial component of the equations of motion (7), (or (27) below) and the second term gives the combined gravitational forces due to the fluid and the field. Equations (8), (11), (25) and (26) are termed as 'dynamical equations' (Misner 1969).

The radial component of the equations of motion (7) may be written as

$$(\rho + p)D_r \nu = -2D_r p. \tag{27}$$

(Compare this expression with (1) given for the static case).

The expression for ν is obtained by integrating (27) on each t-constant hypersurface. For bounded distributions it is convenient to take $\nu = 0$ at the outer surface of the body so that the coordinate time t becomes the proper time there.

Another useful expression obtained by applying the operator D_t to (20) and then using (15) and (18) is

$$D_t\{4\pi(\rho + \varepsilon)R^3/3\} = -p(4\pi R^2 U) = -pD_t(4\pi R^3/3), \tag{28}$$

which may be interpreted as the rate of gain of combined energy of the field and the material enclosed in a volume $(4\pi R^3/3)$ equal to the work done in unit time by the pressure during the contraction $(U < 0)$ of the volume. For conformally flat models $(\varepsilon = 0)$, which include the Robertson-Walker models, (28) reduces to the familiar form.

Similarly, by applying the operator D_r to (20) and using (17) and (18) we get

$$D_r\{4\pi(\rho + \varepsilon)R^3/3\} = 4\pi R^2 \Gamma \rho. \tag{29}$$

Also from (21), (26) and (28) we get

$$D_t\Gamma = UD_r\nu. \tag{30}$$

4. Free fall collapse

In this section we obtain a solution of the field equations given in § 3. In our attempt to obtain a generalization of the Oppenheimer-Snyder (1939) model we are guided mainly by the fact that gravitational radiation is present in the model. Hence, we look for those properties of the time-like congruence u^a and the physical variables associated with the material distribution which characterize the presence of gravitational radiation. As mentioned earlier, it was shown by Szekeres (1966) that rotation, shear and the spatial gradient of material density constitute the gravitationally active part of the fluid, the part that can be found by observing the free gravitational field. It may be mentioned here that the space-time of the Oppenheimer-Snyder model is conformally flat and hence all the three physical characteristics mentioned above are absent.

Our choice of spherical symmetry limits the number of non-vanishing invariants associated with the time-like congruence u^a to just three given below:

$$a = \text{acceleration} = \{\exp(-\lambda/2)\}(\nu'/2), \tag{31}$$

$$\theta = \text{expansion} = \{\exp(-\nu/2)\}\{(\dot{\lambda}/2) + (2\dot{R}/R)\}, \tag{32}$$

$$\sigma = \text{shear invariant} = \{\exp(-\nu/2)\}\{(\dot{\lambda}/2) - (\dot{R}/R)\}. \tag{33}$$

From (31) for vanishing radial acceleration of the material particles (free fall), we have $\nu' = 0$. Thus, without loss of generality we can assume that $\nu = 0$. Thus, our metric simplifies to

$$ds^2 = dt^2 - \exp(-\lambda)dr^2 - R^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{34}$$

With this simplification equation (30) gives

$$\Gamma(r,t) = \Gamma(r,t_0) = \Gamma(r). \tag{35}$$

That is, Γ is a constant of motion for each shell. Also putting $\nu = 0$ into the field equations the mathematical structure simplifies considerably. From (18)

$$\dot{\lambda}/2 = \dot{R}'/R', \tag{36}$$

which on integration with respect to time gives

$$\exp(\lambda/2) = R'/\Gamma(r) = R'/[1 + U^2 - \{8\pi(\rho+\varepsilon)R^2/3\}]^{1/2}. \tag{37}$$

It may be noted that while arriving at (37) we have used (14) and (30).

From (21) we get

$$\alpha \equiv \dot{R}/R = \{8\pi(\rho+\varepsilon)/3 + (\Gamma^2 - 1)/R^2\}^{1/2}, \tag{38}$$

and with these simplifications the expressions for shear σ contraction θ, the material energy density ρ and the energy density of the free gravitational field ε are given by

$$\sigma = (R/R')\alpha', \tag{39}$$

$$\theta = \sigma + 3\alpha, \tag{40}$$

$$8\pi\rho = (2R/R')\alpha\alpha' + 3\alpha^2 - 2\Gamma\Gamma'/RR' - (\Gamma^2 - 1)/R^2, \tag{41}$$

$$8\pi\varepsilon = (R/R')(\dot{\alpha}' + \alpha\alpha') + \Gamma\Gamma'/RR' - (\Gamma^2 - 1)/R^2. \tag{42}$$

Since the pressure gradients can be neglected in the free fall collapse, if we take $p = 0$ with the help of (22) and (28) we can write (38) as

$$\dot{R}^2 - 2M(r)/R = \Gamma^2 - 1. \tag{43}$$

The physical meaning of this equation and its relation with the geodesic equation for the radial motion of a test particle in the Schwarzschild exterior space-time were discussed by Misner (1969). Also, as he pointed out that 'shell crossing' in this case would imply infinite material density. Further, it may be mentioned that in analogy with the Newtonian case $\Gamma^2 > 1$, $\Gamma^2 = 1$ and $\Gamma^2 < 1$ correspond respectively to hyperbolic, parabolic and elliptic motion of the dust particles. Also, (26) simplifies to

$$\ddot{R} = -M(r)/R^2, \tag{44}$$

and from (29) we get

$$\partial M/\partial R = 4\pi R^2\Gamma\rho. \tag{45}$$

In our case α satisfies the differential equation

$$\dot{\alpha}' + 3\alpha\alpha' - (\Gamma\Gamma'/R^2) + R'(\Gamma^2 - 1)/R^3 = 0, \tag{46}$$

whose first integral is given by (41). The relations between θ, σ, ε and ρ are given by

$$(\theta - \sigma)^2 = 24\pi(\rho + \varepsilon) + 9(\Gamma^2 - 1)/R^2, \tag{47}$$

$$8\pi\varepsilon = \dot{\sigma} + (\sigma/3)(2\sigma + \theta) + (\Gamma\Gamma'/RR') - (\Gamma^2 - 1)/R^2. \tag{48}$$

It is to be stressed again that the free gravitational field affects the motion of matter by inducing shear in the world lines, which also enters the contraction equation (40) tending to cause the world lines to converge. Thus, a spherical object gets distorted into an ellipsoid which has the r-direction as the principal axis and is degenerate in the θ, ϕ-plane.

The inhomogeneity and anisotropy of the 3-spaces t = constant are best understood by the presence of shear terms in the expressions for the Ricci tensor $R^*_{ab}(a, b = 1, 2, 3)$, its spur R^* and the Gaussian curvature $K(\bar{x}, e^a)$ of the surface formed by all geodesics through the point $\bar{x}$ orthogonal to the direction e^a where $e^a e_a = -1$, $e_a u^a = 0$ (see Ellis 1971). In the present case it is easy to verify from (34) and (36) that the rates of contraction per unit distance along the radial and transverse directions are given respectively by $(\dot{R}'/R')$ and $(\dot{R}/R)$ showing anisotropy of contraction velocities of dust particles. If $(\dot{R}/R)$ is a function of t only then α' will vanish implying $\sigma = 0$ which is also the condition for isotropy. Consequently from (46) we get

$$R'(\Gamma^2-1)-R\Gamma\Gamma'=0, \tag{49}$$

and therefore $\varepsilon = 0$. In contrast to the uniform density models for which $\rho = \rho(t)$ and $\sigma = \varepsilon = 0$, the model presented here contains gravitational radiation by Szekers' (1966) criterion since σ and $\partial\rho/\partial r$ are non-vanishing which are also characteristics respectively for anisotropy and inhomogeneity. In the next section we present a particular solution of (43) which clearly demonstrates the presence of gravitational radiation in the interior surface layers of the star.

5. A particular solution

It was stated earlier that gravitational radiation emitted during free fall collapse gets trapped inside the surface layers of the star and the exterior geometry is described by the Schwarzschild static solution. This fact can be demonstrated by considering a particular solution of the general problem discussed in the preceding section. Of the three types of possible motion for the freely falling dust particles, we consider the simplest case $\Gamma^2 = 1$. Thus, (43) on integration gives (see Papapetrou 1974)

$$R(r,t)=\left[\frac{9M(r)}{2}\right]^{1/3}(r-t)^{2/3}, \tag{50}$$

and

$$\exp(\lambda)=R'^2=\frac{[2M(r)+(r-t)M'(r)]^2}{[36(r-t)^2M^4(r)]^{1/3}}. \tag{51}$$

Now, since the metric is completely determined, the expressions for σ, θ, ρ and ε are written explicitly with the help of (39)–(42), (50) and (51):

$$\sigma=\frac{2M(r)}{(r-t)[2M(r)+(r-t)M'(r)]}, \tag{52}$$

$$\theta=\frac{2}{(r-t)}\left[\frac{M(r)}{2M(r)+(r-t)M'(r)}-1\right], \tag{53}$$

$$8\pi\rho=\left[\frac{4M'(r)}{3(r-t)}\right]\left[\frac{1}{2M(r)+(r-t)M'(r)}\right], \tag{54}$$

$$8\pi\varepsilon=\left[\frac{8M(r)}{3(r-t)^2}\right]\left[\frac{1}{2M(r)+(r-t)M'(r)}\right]. \tag{55}$$

The term $[2M(r)+(r-t)M'(r)]$ appearing in the denominator of the above expressions is essentially the effect of gravitational radiation on shear, expansion, material energy density and field energy density. This fact may be explained by noting that the term $(r-t)^{-2}$ appearing in (55) is of the order R^{-3} (through equation (50)) which represents the typical spherical fall off for a type D field. Thus the second term denotes the superposition of a gravitational radiation field (type N) over the Coulombian type D field. Also, on a fixed surface $r = r_b$, $M(r_b) = $ constant and hence for $r > r_b$ we observe only the type D field.

Finally, the dust particles moving radially reach the centre of symmetry $R = 0$ at the moment $t = r$. Thus, at some finite time τ all particles have reached the centre of symmetry and gravitational collapse has been completed resulting in the formation of a black hole.

6. Boundary conditions

The exterior spacetime for the gravitational field of a spherically symmetric collapsing dust is given by the Schwarzschild metric

$$(\mathrm{d}s^2)_o = \left(1 - \frac{2M}{R}\right)\mathrm{d}T^2 - \left(1 - \frac{2M}{R}\right)^{-1} \mathrm{d}R^2 - R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \quad (56)$$

where M is the mass enclosed within a sphere of coordinate radius r_b (= constant) and the suffix o denotes outside the boundary r_b. Since $R = R(r, t)$, we can write (56) in the form

$$(\mathrm{d}s^2)_o = \left(1 - \frac{2M}{R}\right)\mathrm{d}T^2 - \left(1 - \frac{2M}{R}\right)^{-1} (R'\,\mathrm{d}r + \dot{R}\,\mathrm{d}t)^2$$
$$- R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \quad (57)$$

To obtain a relation between the differentials $\mathrm{d}T$, $\mathrm{d}t$, $\mathrm{d}r$ we write the exterior solution in (t, r, θ, ϕ) system by putting $\rho = 0$ in the interior solution. Thus, we get

$$(\mathrm{d}s^2)_o = \mathrm{d}t^2 - [\exp(\lambda)]\,\mathrm{d}r^2 - R^2(\mathrm{d}\phi^2 + \sin^2\theta\,\mathrm{d}\phi^2), \quad (58)$$

where

$$[\exp(-\lambda)] = \left(1 + U_0^2 - \frac{8\pi}{3}\varepsilon_0 R^2\right)(R')^{-2}. \quad (59)$$

Comparing (57) and (58), we get

$$\mathrm{d}T = \left(1 - \frac{2M}{R}\right)^{-1} \{\Gamma_0\,\mathrm{d}t + U_0 \exp(\lambda_0/2)\,\mathrm{d}r\}, \quad (60)$$

with $M = (4\pi/3)\varepsilon_0 R^3$. Now, on the boundary $r = r_b =$ constant, respectively the exterior and interior solutions take the form

$$(\mathrm{d}s^2)_b = \left(1 - \frac{2M}{R_b}\right)\mathrm{d}T^2 - \left(1 - \frac{2M}{R_b}\right)^{-1} \dot{R}_b^2\,\mathrm{d}t^2 - R_b^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \quad (61)$$

and

$$(\mathrm{d}s^2)_b = \mathrm{d}t^2 - R_b^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \quad (62)$$

where $R_b = R(r_b, t)$. Comparing (61) and (62) we get

$$\mathrm{d}T = \left(1 - \frac{2M}{R_b}\right)^{-1} \Gamma_b\,\mathrm{d}t. \quad (63)$$

The continuity of the first derivatives of the metric potentials is equivalent in the present case to the continuity of T_1^1 and T_4^1 which is trivial since these expressions vanish on both sides of the boundary.

7. Concluding remarks

The eigenvalue of the Weyl conformal tensor which is introduced in the transverse (2-2) component of the Einstein tensor and interpreted as the energy density of the free gravitational field plays an important role in identifying the presence of gravitational radiation in the interior of a freely falling spherically symmetric dust. For the particular solution given in § 5, the superposition of the type N field on the Coulomb field clearly demonstrates that the collapsing dust emits gravitational radiation which however is trapped in the inner surface layers of the star. The shearing forces produced by the free gravitational field may explain for the existence of elliptic galaxies since even in certain static distributions ε is twice as large as ρ (see Krișhna Rao and Annapurna 1985).

The presence of the free gravitational field (represented by ε) has also a bearing on the embedding problem. It was shown earlier that for a spherically symmetric model filled with a perfect fluid to be of class one either $\varepsilon = 0$ or $\varepsilon + \rho - 3p = 0$ (see Krishna Rao 1971). Since neither of these conditions holds for our present model (with $p = 0$) and a general spherically symmetric space-time is atmost of class two (see Karmarkar 1948) we conclude that the space-time described by the metric (34), (50) and (51) is also of class two. This is in contrast to the Oppenheimer-Snyder model which is of class one.

Acknowledgement

The authors' thanks are due to the referees for their comments on the original manuscript.

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