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On some operators in multichannel scattering

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Abstract. We investigate the strong limit of an operator valued sequence used in other form in the nonrelativistic theory of multichannel scattering, and also some of its consequences.

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1. Introduction

In the nonrelativistic theory of multichannel scattering (Sandhas 1972), Lippmann's Identity is used extensively, although it has not yet been derived rigorously in Hilbert space. Bencze and Chandler (1982) established a result related to it in weak limit. In § 2 of this paper we derive a few more related results in strong limit which may throw some light on the status of Lippmann's Identity and be of interest in the nonrelativistic theory of multichannel scattering.

2. Evaluation of a few strong limits

The Lippmann's Identity (Sandhas 1972) in the theory of multichannel scattering is usually written as

$$\lim_{\eta\to 0+} |A_\eta^\beta(\varepsilon_\alpha, \phi_\alpha(\varepsilon_\alpha))\rangle = \lim_{\eta\to 0+} i\eta\, G_\beta(\varepsilon_\alpha + i\eta)|\phi_\alpha(\varepsilon_\alpha)\rangle$$
$$= \delta_{\beta\alpha}|\phi_\alpha(\varepsilon_\alpha)\rangle, \tag{1}$$

where $|\phi_\alpha(\varepsilon_\alpha)\rangle$ is the eigenstate of the channel Hamiltonian H_α with eigenvalue ε_α: $H_\alpha = H_0 + V_\alpha$; $H_\alpha|\phi_\alpha(\varepsilon_\alpha)\rangle = \varepsilon_\alpha|\phi_\alpha(\varepsilon_\alpha)\rangle$. The total Hamiltonian of the system is $H = H_\alpha + \bar{V}_\alpha = H_\beta + \bar{V}_\beta = H_0 + V$; H_0 and V are operators for total kinetic energy (K.E.) and potential energy, respectively: $\bar{V}_\alpha$ = potential between bound clusters in α-channel: $G_\beta(z) = (z - H_\beta)^{-1}$ is the resolvent (Green's) operator in β-channel etc (Mukherjee 1981). Since H_α (through H_0) contains the K.E., operators for the centre-of-mass motion of bound clusters in α-channel, the eigenstate $|\phi_\alpha(\varepsilon_\alpha)\rangle$ is an improper state and not a member of Hilbert space $\mathscr{H}$, and hence (1) cannot be treated as an equation in

Hilbert space. Moreover, the sense in which the limit $\eta \to 0$ is taken in (1) is unspecified in the literature (Bencze and Chandler 1982).

The strong limit of $|A_\eta^\beta(\varepsilon_\beta, g)\rangle$ is easily evaluated by using the spectral representation of the Green's function $G_\beta(z)$. If $E(x)$ is the spectral projector corresponding to the resolution of identity of a self-adjoint operator A, then (Dunford and Schwartz 1963) for any two vectors $|f_1\rangle$ and $|f_2\rangle$ in Hilbert space, the resolvent operator $R(z) = (z-A)^{-1}$ may be written as

$$\langle f_1|R(x+i\eta)|f_2\rangle = \int_{-\infty}^{+\infty} \frac{\mathrm{d}(\langle f_1|E(y)|f_2\rangle)}{x+i\eta-y}$$

$$= -\int_{-\infty}^{+\infty} \frac{\langle f_1|E(y)|f_2\rangle}{(x+i\eta-y)^2}\mathrm{d}y \tag{2}$$

where as usual, we assume

$$A = \int_{-\infty}^{+\infty} \lambda \mathrm{d}(E(\lambda)); \quad \int_{-\infty}^{+\infty} \mathrm{d}\,\|E(y)\|^2 < \infty \text{ etc;}$$

$$E(a,b) = \underset{\eta\to 0+}{S\text{-lim}} \frac{1}{2\pi i}\int_a^b [R(x-i\eta)-R(x+i\eta)]\,\mathrm{d}x \tag{3}$$

$E(x) = E(-\infty, x)$ etc. It is easily established that, in the strong limit (Baümgartel and Wollenberg 1983)

$$\underset{\eta\to 0+}{S\text{-lim}} \langle f_1|i\eta R(x+i\eta)E(\{x\})|f_2\rangle = \langle f_1|E(\{x\}|f_2\rangle, \tag{4}$$

where

$$E(\{x\}) = \underset{\delta\to 0+}{S\text{-lim}} [E(x+\delta)-E(x-\delta)] \tag{5}$$

is the projection operator for the single point x. Equation (4) follows from the property (Prugovecki 1981; Dunford and Schwartz 1963) of the product of projectors $E(\Delta_1)E(\Delta_2) = E(\Delta_1 \cap \Delta_2)$ for any pair of Borel sets Δ_1, and Δ_2 giving

$$\begin{aligned} E(y)[E(x+\delta)-E(x-\delta)] &= 0, & -\infty < y \leqslant x-\delta \\ &= E(y)-E(x-\delta), & x-\delta < y \leqslant x+\delta \\ &= E(x+\delta)-E(x-\delta), & x+\delta < y < \infty \end{aligned} \tag{6}$$

so that the (finite) contribution to the left side of (4) comes from the third interval in (6) in its range of integration, in the limit of $\eta \to 0+$ and $\delta \to 0+$. Similarly by using (4) and (2), one establishes the following limits of norms

$$\lim_{\eta\to 0+} \|(i\eta R(x+i\eta)-E(\{x\}))|h\rangle\|^2 = \lim_{\eta\to 0+} \|i\eta R(x+i\eta)|h\rangle\|^2 = 0, \tag{7}$$

where $|h\rangle = (R'-\{x\})|f_2\rangle$; R' being interval $(-\infty, \infty)$ [see p. 627 of Prugovecki 1981). So from (7), by definition of strong limit

$$\underset{\eta\to 0+}{S\text{-lim}}\, i\eta R(x+i\eta) = E(\{x\}). \tag{8}$$

Using (8) we can write for the Green's function $G_\beta(x+i\eta)$ for any vector $|g\rangle$ in Hilbert space:

$$\underset{\eta\to 0+}{S\text{-lim}}\, |A_\eta^\beta(\varepsilon_\alpha, g)\rangle = \underset{\eta\to 0+}{S\text{-lim}}\, i\eta\, G_\beta(\varepsilon_\alpha+i\eta)|g\rangle = E_\beta(\{\varepsilon_\alpha\})|g\rangle, \tag{9}$$

where $E_\beta(x)$ is the spectral projector for the Hamiltonian H_β. Inverting (Baümgartel and Wollenberg 1983) (3) we get

$$p_\beta(\lambda) = \underset{\eta\to 0+}{S\text{-lim}}\, \frac{1}{2\pi i}[G_\beta(x-i\eta)-G_\beta(x+i\eta)]$$

$$= \underset{\varepsilon\to 0+}{S\text{-lim}} \int_{-\infty}^{+\infty} \delta_\varepsilon(x-\lambda)\,\mathrm{d}[E_\beta(x)] \tag{10}$$

where

$$\delta_\varepsilon(x) = \frac{1}{\pi}\cdot\frac{\varepsilon}{\varepsilon^2+x^2}$$

is a Dirac δ-sequence. It is readily checked that we have for all η

$$p_\beta(\lambda)\, i\eta\, G_\beta(\lambda+i\eta) = i\eta\, G_\beta(\lambda+i\eta)\cdot p_\beta(\lambda) = p_\beta(\lambda) \tag{11}$$

and

$$\underset{\eta\to 0+}{S\text{-lim}}\, p_\beta(\lambda)\cdot i\eta\, G_\beta(\lambda'+i\eta) = 0, \quad \lambda' \neq \lambda. \tag{12}$$

It is noted further that the vector

$$|\xi_\beta(\varepsilon, g)\rangle = p_\beta(\varepsilon)|g\rangle \tag{13}$$

for any arbitrary vector $|g\rangle$ in Hilbert space is an eigenstate of H_β and $i\eta\, G_\beta(\varepsilon_\beta+i\eta)$ with eigenvalues ε_β and unity, respectively:

$$i\eta\, G_\beta(\varepsilon_\beta+i\eta)|\xi_\beta(\varepsilon_\beta, g)\rangle = |\xi_\beta(\varepsilon_\beta, g)\rangle = \frac{1}{\varepsilon_\beta} H_\beta|\xi_\beta(\varepsilon_\beta, g)\rangle \tag{14}$$

$$\langle\xi_\beta(\varepsilon_\beta, g)|\, i\eta\, G_\beta(\varepsilon_\beta+i\eta) = \langle\xi_\beta(\varepsilon_\beta, g)| = \langle\xi_\beta(\varepsilon_\beta, g)|\, H_\beta\cdot\frac{1}{\varepsilon_\beta}, \tag{15}$$

in the same sense the plane wave is an eigenfunction of kinetic energy operator. Since $|g\rangle$ is an arbitrary vector of Hilbert space, the infinite number of vectors $|\xi_\beta(\varepsilon_\beta, g)\rangle$ span an infinite dimensioned linear manifold $M_\beta(\varepsilon_\beta)$ to which the Hilbert space is mapped onto by $p_\beta(\varepsilon_\beta)$ and that the projections of $|A_\eta^\beta(\varepsilon_\beta, g)\rangle$ in $M_\beta(\varepsilon_\beta)$, for all these vectors are non-zero. We may express $p_\beta(\varepsilon_\beta)$ in Stieltjes integral as

$$p_\beta(\varepsilon_\beta) = \delta(\varepsilon_\beta - H_\beta) = \int |\phi_\beta(\varepsilon'_\beta)\rangle\delta(\varepsilon_\beta-\varepsilon'_\beta)\mathrm{d}\varepsilon'_\beta\,\mathrm{d}^3\mathbf{K}'_1\,\mathrm{d}^3\mathbf{K}'_2 \ldots \langle\phi_\beta(\varepsilon'_\beta)|, \tag{16}$$

where the eigenstate $|\phi_\beta(\varepsilon'_\beta)\rangle$ of H_β describes a cluster of m bound groups of particles moving freely with total energy

$$\varepsilon'_\beta = \sum_j \left(-\varepsilon_j^{\beta'} + \frac{\hbar^2}{2\mu_j} k_j'^2 \right); \; -\varepsilon_j^{\beta'} \text{ and } \frac{\hbar^2}{2\mu_j} k_j'^2$$

are respectively the binding energy and the c.m. kinetic energy of the j-th bound group and the Stieltjes integral (16) refers to summation over all bound states and integration over all the continuum states which includes the continuum states of each of the groups forming bound states in β-channel with total energy $\varepsilon'_\beta = \varepsilon_\beta$. Writing

$$\hat{A}_\eta^\beta(\varepsilon'_\beta; \varepsilon_\beta, g) = \langle \phi_\beta(\mathbf{K}'_1, \mathbf{K}'_2, \ldots) | A_\eta^\beta(\varepsilon_\beta, g) \rangle$$

$$= \frac{i\eta}{\varepsilon_\beta + i\eta - \varepsilon'_\beta} \langle \phi_\beta(\varepsilon'_\beta) | g \rangle \xrightarrow[\eta \to 0+]{} \delta_{\varepsilon_\beta, \varepsilon'_\beta} \langle \phi_\beta(\varepsilon_\beta) | g \rangle \tag{17}$$

for the (generalized) Fourier component of $|A_\eta^\beta(\varepsilon_\beta, g)\rangle$ and

$$\lim_{\varepsilon \to 0+} \int_{a-\varepsilon}^{a+\varepsilon} \delta(x-b)\,\mathrm{d}x = \delta_{a,b} \tag{18}$$

for the Krönecker delta function $\delta_{a,b}$, in terms of Dirac-delta function, we have, from (9), (13) and (16), for any suitable test function $\hat{\omega}(\mathbf{K}'_1, \mathbf{K}'_2, \ldots) = \langle \omega | \phi_\beta(\mathbf{K}'_1, \mathbf{K}'_2, \ldots) \rangle$,

$$\lim_{\eta \to 0+} \langle \xi_\beta(\varepsilon_\beta, \omega) | A_\eta^\beta(\varepsilon_\beta, g) \rangle$$

$$= \lim_{\eta \to 0+} \int \langle \omega | \phi_\beta(\varepsilon'_\beta) \rangle \mathrm{d}^3\mathbf{K}'_1 \mathrm{d}^3\mathbf{K}'_2 \ldots \delta(\varepsilon_\beta - \varepsilon'_\beta) \hat{A}_\eta^\beta(\varepsilon'_\beta; \varepsilon_\beta, g) \tag{19}$$

$$= \lim_{\eta \to 0+} \int_{S_\beta(\varepsilon_\beta)} \langle \omega | \phi_\beta(\varepsilon'_\beta) \rangle \mathrm{d}^3\mathbf{K}'_1 \mathrm{d}^3\mathbf{K}'_2 \ldots \hat{A}_\eta^\beta(\varepsilon'_\beta; \varepsilon_\beta, g) \tag{20}$$

$$= \int_{S_\beta(\varepsilon_\beta)} \hat{\omega}(\mathbf{K}'_1, \mathbf{K}'_2, \ldots) \mathrm{d}^3\mathbf{K}'_1 d^3\mathbf{K}'_2 \ldots \langle \phi_\beta(\mathbf{K}'_1, \mathbf{K}'_2, \ldots) | g \rangle, \tag{21}$$

where in the Stieltjes integral (21), $S_\beta(\varepsilon_\beta)$ is the surface of a sphere $\sum_j(-\varepsilon_j^{\beta'} + \hbar^2 k'^2/2\mu_j) = \varepsilon_\beta$ as determined by Radon transform (Gelfand *et al* 1966) in (19). Now using the distribution theoretic definitions of (single layer) of surface-distribution concentrated on a surface Σ, for test functions ψ_1 in a suitable space, defined by (Kanwal 1983)

$$\langle \psi_1 | f_1 \delta(\Sigma) \rangle = \int_\Sigma f_1 \psi_1 \,\mathrm{d}S, \tag{22}$$

we note from (21) and (22) that, for every Hilbert space vector $|g\rangle$, $|A_\eta^\beta(\varepsilon_\beta, g)\rangle$ defines a sequence of distributions such that its Fourier components

$$\hat{A}_\eta^\beta(\varepsilon'_\beta; \varepsilon_\beta, g) = \langle \phi_\beta(\varepsilon'_\beta) | A_\eta^\beta(\varepsilon_\beta, g) \rangle; \quad \phi_\beta(\varepsilon_\beta) \in M_\beta(\varepsilon_\beta)$$

for $\eta \to 0+$ converges to a (single-layer) surface distribution, concentrated over the surface of a N-dimensional sphere of radius

$$r = \sqrt{\varepsilon_\beta} = \left(\sum_j \left(-\varepsilon_j^\beta + \frac{\hbar^2 k_j^2}{2\mu_j}\right)\right)^{1/2}$$

where the test functions $\omega(\mathbf{K}_1', \mathbf{K}_2', \ldots)$ belong to the C_0^∞ or Schwartzian space (Kanwal 1983). We thus conclude that the limit of the sequence

$$D\text{-}\lim_{\eta \to 0+} |\hat{A}_\eta^\beta(\varepsilon_\beta, g)\rangle = \delta(S_\beta(\varepsilon_\beta))|g\rangle,$$

where the symbol D-lim means that the limit is taken in distribution theoretical sense defined by (22) for which it is enough that the surface integral of (22) exists for every test function ψ_1 and that is satisfied in the present case as shown by (19)–(21). It may be noted that $E_\beta(\{\varepsilon_\beta\})|g\rangle$ expressed in terms of the eigenstates of H_β (Prugovecki 1981) may be written as

$$\begin{aligned} S\text{-}\lim_{\eta \to 0+} |A_\eta^\beta(\varepsilon_\beta, g)\rangle &= S\text{-}\lim_{\eta \to 0+} i\eta G_\beta(\varepsilon_\beta + i\eta)|g\rangle \\ &= S\text{-}\lim_{\eta \to 0+} \int_{\varepsilon_\beta - \eta}^{\varepsilon_\beta + \eta} \mathrm{d}\varepsilon_\beta' p_\beta(\varepsilon_\beta')|g\rangle = \mathscr{P}_\beta(\varepsilon_\beta)|g\rangle. \end{aligned} \tag{24}$$

$\mathscr{P}_\beta(\varepsilon_\beta)$ was used by Mukherjee (1981). Equations (23) and (24) have an important bearing on the problem of non-uniqueness of the solutions of Lippmann-Schwinger equation. It is known that the wavefunction defined by (Mukherjee 1981)

$$|\Psi_\alpha(\varepsilon_\alpha + i\eta)\rangle = i\eta\, G(\varepsilon_\alpha + i\eta)|\phi_\alpha(\varepsilon_\alpha)\rangle \tag{25}$$

$$= |\phi_\alpha(\varepsilon_\alpha)\rangle + G_\alpha(\varepsilon_\alpha + i\eta)\,\bar{V}_\alpha|\Psi_\alpha(\varepsilon_\alpha + i\eta)\rangle \tag{26}$$

$$= i\eta\, G_\beta(\varepsilon_\alpha + i\eta)|\phi_\alpha(\varepsilon_\alpha)\rangle + G_\beta(\varepsilon_\alpha + i\eta)\bar{V}_\beta|\Psi_\alpha(\varepsilon_\alpha + i\eta)\rangle \tag{27}$$

have meaning only in the distribution theoretic sense (see Faddeev 1961). But the Lippmann's identity in (1) is evaluated (Faddeev 1961) so that the left side of (1) is equal to the right side of (1) for every point in the co-ordinate space, which in classical analysis is also known as "pointwise convergence" of the limit $\eta \to 0+$. Since $|A_\eta^\beta(\varepsilon_\beta, g)\rangle$ and $|A_\eta^\beta(\varepsilon_\beta, \phi_\alpha(\varepsilon_\alpha))\rangle$ is a part of the wavefunctions which is itself valid in the distribution theoretic sense, it is only proper that the sequence $|A_\eta^\beta(\varepsilon_\beta, \phi_\alpha(\varepsilon_\alpha))\rangle$ is evaluated in the distribution theoretic sense, rather than in the ordinary function theoretic sense of pointwise convergence. In that case, the distributional limit $\eta \to 0+$ of (26) and (27) is

$$|\Psi_\alpha(\varepsilon_\alpha + i0)\rangle = |\phi_\alpha(\varepsilon_\alpha)\rangle + G_\alpha(\varepsilon_\alpha + i0)\bar{V}_\alpha|\Psi_\alpha(\varepsilon_\alpha + i0)\rangle, \tag{28}$$

$$|\Psi_\alpha(\varepsilon_\alpha + i0)\rangle = \mathscr{P}_\beta(\varepsilon_\alpha)|\phi_\alpha(\varepsilon_\alpha)\rangle + G_\beta(\varepsilon_\alpha + i0)\bar{V}_\beta|\Psi(\varepsilon_\alpha + i0)\rangle. \tag{29}$$

This result was obtained in a heauristic way earlier by Mukherjee (1981). It is clear that (21) remains valid for $|g\rangle = |\phi_\alpha(\varepsilon_\alpha)\rangle$ when a suitable test space for ω is chosen. Since $|\phi_\alpha(\varepsilon_\alpha)\rangle$ usually consists of a product of bound state wavefunction of bound-groups of particles in α-channel and the plane-wave motion of the centre of such groups, it appears enough to choose ω in C_0^∞ of compact support or Schwartzian space S (Kanwal 1983). We therefore regard (28) and (29) to be valid equations in the distribution-theoretic sense in the same spirit, where the limit $\eta \to 0$ of (26)–(27) is also evaluated distribution-theoretically which was not done in any calculation earlier and we end up with (28) and (29) which incidentally removes the non-uniqueness problems in the solutions of the Lippmann-Schwinger equation thereby giving a fresh justification of a similar conclusion arrived at earlier (Mukherjee 1981). This clearly happens because (Mukherjee 1981) $\mathscr{P}_f(\varepsilon_\alpha)$ does not give Krönecker delta in channel indices (although it gives rise to Krönecker delta in energy variable in momentum space, as in (17)), so that we do not have any more simultaneous existence of homogeneous and inhomogeneous equations for wavefunction $|\Psi_\alpha(\varepsilon_\alpha + i0)\rangle$ in each channel and hence no more the problem of nonuniqueness of the solution of Lippmann-Schwinger equation.

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