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The complex sine-Gordon theory: soliton solutions through the virial approach

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Abstract. The one-soliton solutions found earlier through the inverse scattering method for the complex sine-Gordon theory by Lund ($m^2 < 0$) and by Vega and Maillet ($m^2 > 0$) are reobtained by using the virial theorem for solitons. An attempt is made to understand the physics of the virial approach.

Keywords. Complex sine-Gordon theory; solitons; virial theorem for solitons; trial orbits.

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1. Introduction

The phenomenology of strong interaction physics (Jacob 1975; Frampton 1974) is suggestive of an underlying string structure (Goddard *et al* 1973; Rebbi 1974) for hadrons. This makes it natural to try and obtain strings within the framework of a Lagrangian field theory. This problem has been tackled in two distinct ways: (i) The approach followed by Nielsen and Olesen (1973) and by Dashen *et al* (1975) is to show that such solutions exist even in conventional local field theories, and (ii) the geometrical approach followed by Kalb and Ramond (1974), Dragon (1974) and by Lund and Regge (1976 LR hereafter).

Following the geometrical approach, LR set up a classical Lagrangian which describes the interaction of strings through a massless scalar field of which they are in turn the source. This Lagrangian has interesting physical limits: the action integral is explicitly that of the Nambu (1970) string in a certain limit, while describing the motion of vortices in an incompressible, inviscid fluid in another limit (Rasetti and Regge 1975). Noting that the problem of the motion of the string is just the problem of finding the surface it describes in a Minkowski space-time, LR projected this surface onto a three-dimensional Euclidean space. The embedded manifold is constructed by way of linear differential equations obeyed by its normal and tangent vectors. These equations form an overdetermined set and in order that embeddability be possible, a set of two coupled nonlinear differential equations must be satisfied. Specifically, they were thus led to the following equations

$$\theta_{tt}-\theta_{xx} = -\sin\theta\cos\theta - \frac{\cos\theta}{\sin^3\theta}(\lambda_t^2-\lambda_x^2)$$

$$\cot^2\theta(\lambda_{tt}-\lambda_{xx}) = 2\frac{\cos\theta}{\sin^3\theta}(\lambda_t\theta_t-\lambda_x\theta_x), \tag{1}$$

which, as they noted, follow also from the Lagrangian

$$\mathscr{L}_{\mathrm{LR}} = \tfrac{1}{2}\{(\partial_\mu\theta)^2 - \sin^2\theta + \cot^2\theta(\partial_\mu\lambda)^2\}, \tag{2}$$

where μ runs from 0 to 1, and the metric is time-preferred.

We have briefly traced above the origin of the Lagrangian in (2) from a specific route. It is remarkable that: (a) Getmanov (1977) was led to consider essentially the same Lagrangian while looking for a classically invariant field theory which defines an exactly solvable, fully integrable Hamiltonian system admitting the existence of a nontrivial, momentum-dependent scattering matrix, and that (b) Pohlmeyer (1979) derived the same Lagrangian from the $O(4)$ nonlinear σ model by exploiting its conformal invariance and adopting a reduction procedure. Specifically the Lagrangians considered by Getmanov and Pohlmeyer, respectively, are:

$$\mathscr{L}_G = \frac{1}{2}\left[\frac{|\partial_\mu\psi|^2}{1-g^2|\psi|^2} - m^2|\psi|^2\right], \tag{3}$$

and

$$\mathscr{L}_p = \tfrac{1}{2}(\partial_\mu\alpha)^2 + \tfrac{1}{2}\tan^2\left(\frac{\alpha}{2}\right)(\partial_\mu\beta)^2 + \cos\alpha - 1. \tag{4}$$

The field ψ in (3) is a complex scalar field: let

$$\psi = \frac{1}{g}\sin\left(\frac{\alpha}{2}\right)\exp(i\beta/2), \tag{5}$$

$g=1$ and $m^2 = +1$: $\mathscr{L}_G$ is transformed into $\mathscr{L}_p$; hence the name complex sine-Gordon for $\mathscr{L}_G$. The choice

$$\psi = \cos\theta\exp(i\lambda), \tag{6}$$

$g=1$ and $m^2 = -1$ transforms $\mathscr{L}_G$ into $\mathscr{L}_{\mathrm{LR}}$.

The Lagrangian in (2) describes the dynamics of two scalar fields in 1 + 1 dimensions, one of them with a self-intraction of the sine-Gordon type and the other a massless field in a background geometry having its own dynamical evolution. This led Lund (1977; 1978) to conjecture that solving the set (1) may lead to an improvement in the understanding of the classical and quantum aspects of the gravitational field. Thus, even though Lund and Regge (1976) were able to give a particular solution of these equations in their initial paper, Lund (1977, 1978) returned to the problem of finding the solutions of these equations in a systematic manner by employing the inverse scattering method. Later, Vega and Maillet (1983) also employed the classical inverse method to obtain the solutions of the model in (3) [$m^2 > 0$], which they then went on to quantize semiclassically in the manner of Dashen *et al* (1975).

The purpose of this note is, in part, to show that one may profitably employ the virial approach (Malik *et al* 1985, MSJ hereafter) to obtain the one-soliton solutions of the system (3) $[m^2 \gtrless 0]$. Lack of systematic methods to deal with such coupled systems in the standard literature, general theoretical interest in the system and the difficulties intrinsic to the inverse scattering method make it worthwhile to approach the problem of its solutions from a different viewpoint. This aspect is dealt with in § 2. Having used the virial approach as a technique in MSJ—as also in the present study—an attempt to understand the physics of this approach provides another motivation for this note. This aspect and a few technical points are discussed in § 3.

2. Virial constraint and soliton solutions

We first show how the virial approach (Friedberg *et al* 1976, MSJ) leads to the solutions of set (1) in the static limit. We have

$$L_{\mathrm{LR}}^{(s)} = \int \mathrm{d}x \, \mathscr{L}_{\mathrm{LR}}^{(s)},$$

where $\mathscr{L}_{\mathrm{LR}}^{(s)}$ is given by (2) without the $\dot{\theta}$ and $\dot{\lambda}$ terms. Let $\theta(x) \to \theta(x')$ and $\lambda(x) \to \lambda(x')$ where $x' = \mu x$; the condition

$$\left.\frac{\delta L_{\mathrm{LR}}^{(s)\prime}}{\delta \mu}\right|_{\mu=1} = 0$$

now yields the following virial constraint

$$\theta_x^2 + \cot^2\theta\, \lambda_x^2 - \sin^2\theta = 0. \tag{8}$$

Differentiating (8) once with respect to x, we obtain the set of equations (1) without the time-derivative terms, i.e., the virial is consistent with the equations that we set out to solve (in general, consistency of this sort could imply an additional constraint between the fields). The fact that (8) is consistent with the equations of motion, but is a first-order equation simplifies the task of solving the original equations. The usual procedure now is to split (8) so as to have separate first-order equations for θ and λ, demand that these equations be consistent with the original second-order equations, etc (for details of the procedure, see MSJ). In the present case, however, this procedure is redundant; one can simply substitute (8) into the static limit of the first of equations (1) to obtain an equation involving only θ:

$$\theta_{xx} - \frac{1}{\sin\theta\cos\theta}\theta_x^2 + \frac{\sin^3\theta}{\cos\theta} = 0. \tag{9}$$

Equation (9) is easy to solve. Let

$$\sin\theta = \psi, \tag{10}$$

then we have

$$\psi\psi_{xx} - \psi_x^2 + \psi^4 = 0, \tag{11}$$

which may be reduced to

$$\psi_x^2 = c\psi^2 - \psi^4, \tag{12}$$

whence

$$\psi = \frac{c^{1/2}}{\cosh(c^{1/2}x)},$$

and from (10)

$$\theta = \sin^{-1}\left[\frac{c^{1/2}}{\cosh(c^{1/2}x)}\right]. \tag{13}$$

Substituting (13) into (8) we can easily obtain

$$= \tan^{-1}\left[\left(\frac{c}{1-c}\right)^{1/2}\tanh(c^{1/2}x)\right], \qquad (0 < c < 1) \tag{14}$$

(13) and (14) are the solutions obtained by Lund (1977; 1978).

Next, we deal with the system (3) $[m^2 > 0]$. Since the charge Q depends linearly on ψ_t, the classical solution for $Q \neq 0$ must be time-dependent; for the lowest energy state, $\psi \propto \exp(-iwt)$.

Let

$$\psi = \frac{1}{g}A(x)\exp(-iwt) \tag{15}$$

$$x \to mx,$$

and

$$v = w/m, \tag{16}$$

then the system (3) becomes

$$\mathscr{L}'_G = \frac{m^2}{2g^2}\left[\frac{v^2A^2 - A_x^2}{1-A^2} - A^2\right], \tag{17}$$

leading to the equation of motion

$$(1-A^2)A_{xx} + AA_x^2 + v^2A - A(1-A^2)^2 = 0. \tag{18}$$

Insofar as (18) is an ordinary nonlinear differential equation, one could try to solve it through the standard methods. For soliton-like solutions, however, the separatrix approach or the (equivalent) virial approach provides an unambiguous method. Thus, the virial constraint for the system (17) yields

$$\frac{1}{(1-A^2)}A_x^2 = A^2 - \frac{v^2A^2}{(1-A^2)}, \tag{19}$$

which is consistent with (18). Equation (19) is easy to solve; we obtain, after Lorentz-boosting,

$$\psi_{1s}\left\{\gamma(x-vt), \frac{r}{\tau}(t-vx), \tau\right\}$$

$$= \frac{1}{g}\frac{\left[1-\left(\frac{2\pi}{m\tau}\right)^2\right]^{1/2}\exp\left[i\frac{2\pi}{\tau}\gamma(t-vx)\right]}{\cosh\left[\left\{1-\left(\frac{2\pi}{m\tau}\right)^2\right\}^{1/2}m\gamma(x-vt)\right]}, \tag{20}$$

where

$$\gamma = 1/(1-v^2)^{1/2},$$

and $$w = 2\pi/\tau.$$

This is the one-soliton solution obtained by Vega and Maillet (1983).

3. Discussion

Given a system of coupled, nonlinear differential equations that admit of soliton-like solutions, one may tackle them through Rajaraman's (1979) method of trial orbits, or through the virial approach (MSJ). The two approaches were briefly compared in MSJ. We conclude this note with a few additional observations.

(a) For all 1-field, $1+1$ dimensional systems that we have studied, the one-soliton solutions can be obtained through the separatrix approach (Malik *et al* 1983). For such systems, the equation of the separatrix is equivalent to the virial theorem for solitons (MSJ). In the latter of these two references, soliton-like solutions are also obtained for coupled fields through the virial approach. We draw attention to the fact that the virial constraint used in all such cases in a *local* constraint, which is, indeed, a very stringent requirement. Recall that the usual virial constraint relates integrals of the kinetic and potential energy terms in the Lagrangian. Since a soliton solution has a rather special status, it may not seem to be very surprising that a special price has to be paid for its existence. Thus, amusingly enough, the requirement that a local virial constraint be satisfied may be looked upon as the cause of the existence of a soliton, somewhat akin to the requirement of a local gauge invariance being the cause of the existence of the photon! It is pertinent to mention here that recently Schiff (1982) has also studied classical field equations which admit of soliton-like solutions and are distinguished by a local virial constraint.

(b) In general, it cannot be expected that a mere substitution of the virial constraint itself into one of the equations to be solved will lead to decoupling. Thus, the complex sine-Gordon is a particularly simple system. Let us, however, pretend we failed to notice this simplicity. The usual procedure (MSJ) would then have been to split the virial constraint. A 'break-up' that works is:

$$\theta_x^2 = \gamma \tan^2\theta \tan^2\lambda, \tag{21}$$

$$\lambda_x^2 = \sin^2\theta \tan^2\theta - \gamma \tan^4\theta \tan^2\lambda. \tag{22}$$

Differentiating (21) w.r.t. x, using the static part of the first of the equations (1) and (22), we get

$$-\theta_x \sin\theta\cos\theta = \gamma \tan\lambda \sec^2\lambda\lambda_x, \tag{23}$$

whence squaring and using (21) and (22) leads to

$$p^2 - \gamma(x+1)^2 p + \gamma^2 x(x+1)^2 = 0, \tag{24}$$

where $p = \cos^2\theta$ and $x = \tan^2\lambda$.
One of the orbits that (24) leads to is

$$\tan^2\lambda = -1 + \cos^2\theta/\gamma \tag{25}$$

(the other orbit leads to the uninteresting situation where one of the fields is a constant).

We can now substitute (25) into the virial constraint and obtain a solution for θ; (25) itself then gives the solution for λ. We note that the orbit (25) does not depend uniquely on the split equations (21) and (22) of the virial constraint (8). Another break-up that leads to the same orbit and solutions is:

$$\theta_x^2 = \sin^2\theta - A^2\tan^2\theta, \tag{26}$$

$$\lambda_x^2 = A^2\tan^4\theta. \tag{27}$$

(c) Procedurally, it is simpler to combine Rajaraman's (1979) approach of trial orbits with the virial approach. One might guess an orbit and solve the virial equation, rather than try arbitrary break-ups of the latter. Even at the cost of redundancy, it should be noted that any two functions $\theta(x)$ and $\lambda(x)$ that solve (8) will not necessarily solve the time-independent part of (1), though the converse is true. Thus, since the orbit (25) is hyperbolic in the 'variables' $\cos\theta$ and $\tan\lambda$, we started with an elliptical orbit in these variables and obtained solutions that satisfied the orbit equation as well as the virial equation. The functions thus obtained did not, however, solve (1). The fact that we did not succeed in obtaining any new one-soliton solutions for the complex sine-Gordon theory, despite having tried many trial orbits, could possibly be due to the fact that the use of the virial relation (by itself) already decouples the original equations, leaving little scope for playing around with break-ups.

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