Pramāṇa – J. Phys., Vol. 27, No. 4, October 1986, pp. 523–536. © Printed in India.

Field theoretic model of composite particles: a variational approach

S P MISRA and S PANDA
Institute of Physics, Bhubaneswar 751 005, India

MS received 12 May 1986

Abstract. We consider here a variational method for describing composite particles. We use equal time anticommutators/commutators, and regard the field operators as arbitrary except for the constraint from the above. The expectation value of the Hamiltonian is extremized both with respect to the field operators and with respect to the wave functions, the former extremization being the new feature here. Comparison with other calculations has been made, and correspondence with them has also been established. We further show here how an "effective mass" can arise from a potential.

Keywords. Field theory; Hamiltonian method; bound states.

PACS No. 11·10

1. Introduction

We had earlier considered a field theoretic model of composite hadrons (Misra 1978). The basic concept introduced there was that of constituent field operators describing the composite object at rest, from which hadrons in motion were considered by Lorentz boosting. The model was used for a wide class of physical problems (Misra 1980; Misra and Panda 1980). However, it had been applied with an ansatz for the constituent field operators which was ad hoc. We now suggest a method for the determination of these constituent field operators. The method introduces some modifications in the earlier model. It also gives an alternative description for bound states as such. We observe e.g. that the "constituent" field operators can depend quite significantly on the specific composite object of which they are constituents compared to what had been proposed earlier.

The paper is organized as follows. In § 2, we define the notations and discuss the specific extremum principle for fermionic composites. In § 3 we discuss the same for bosonic constituents, where some differences in principle are noticed. In § 4, we illustrate the method with some simple examples, where contact is made with conventional spectroscopy. In § 5 we discuss how the notion of an "effective mass" can arise.

2. Fermionic bound states

We shall illustrate the method with the *notations* for the electron-proton system, interacting through quantum electrodynamics. This helps to fix our ideas, and,

generalization to any other fermion-(anti)fermion bound state with an arbitrary potential and interaction will be straightforward. We shall here discuss the bound state with zero total momentum, while the constituent particles are relativistic. We start with the obvious remark that the electron-proton system has a solution *in field theory* which will describe the hydrogen atom at rest, and we are attempting here e.g. to find a good approximation to this solution.

With $t = 0$, we first write the four-component electron field operator as

$$\psi_e(\mathbf{x}, 0) = e(\mathbf{x}, 0) + \tilde{e}(\mathbf{x}, 0), \tag{1}$$

where we write

$$e(\mathbf{x}, 0) = (2\pi)^{-3/2} \int u_1(\mathbf{k}) e(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x}) \, d\mathbf{k} \tag{2}$$

and

$$\tilde{e}(\mathbf{x}, 0) = (2\pi)^{-3/2} \int v_1(\mathbf{k}) \tilde{e}(\mathbf{k}) \exp(-i\mathbf{k} \cdot \mathbf{x}) \, d\mathbf{k}. \tag{3}$$

In the above, the two-component operator $e(\mathbf{k}) = e_{lr}(\mathbf{k}) u_{lr}$ with $e_{lr}(\mathbf{k})$ annihilating an electron of spin r, and with $u_{l\frac{1}{2}} = \binom{1}{0}$ and $u_{l-\frac{1}{2}} = \binom{0}{1}$. Here and henceforward summation over repeated indices is understood. We take the form of $u_1(\mathbf{k})$ as,

$$u_1(\mathbf{k}) = \begin{pmatrix} f_1(K) \\ g_1(K) \sigma \cdot \mathbf{k} \end{pmatrix}. \tag{4}$$

Similarly in (3), the two-component positron creation operator is given as $\tilde{e}(\mathbf{k}) = \tilde{e}_{ls}(\mathbf{k}) v_{ls}$, where $v_{l\frac{1}{2}} = \binom{0}{-i}$ and $v_{l-\frac{1}{2}} = \binom{i}{0}$, and,

$$v_1(\mathbf{k}) = \begin{pmatrix} g_1(K) \sigma \cdot \mathbf{k} \\ f_1(K) \end{pmatrix}. \tag{5}$$

It was noted in Misra (1978) that the condition that $\psi_e(\mathbf{x}, 0)$ satisfies the usual equal time anticommutator relation requires that

$$f^2 + K^2 g^2 = 1. \tag{6}$$

We can use (6) more conveniently through the substitution (Yaouanc *et al* 1985) $f_1(K) = \cos \frac{1}{2}(\phi_1(K))$ and $K g_1(K) = \sin \frac{1}{2}(\phi_1(K))$, which yields

$$u_1(\mathbf{k}) = \begin{pmatrix} \cos \frac{1}{2}\phi_1 \\ \sin \frac{1}{2}\phi_1 \sigma \cdot \hat{k} \end{pmatrix}. \tag{7}$$

For a constituent electron in the hydrogen atom, we shall determine $\phi_1(K)$ through a variational principle to be stated.

Parallel to (1), we now take for the proton,

$$\psi_p^c(\mathbf{x}, 0) = p(\mathbf{x}, 0) + \tilde{p}(\mathbf{x}, 0) \tag{8}$$

with

$$\tilde{p}(\mathbf{x}, 0) = (2\pi)^{-3/2} \int v_2(\mathbf{k}) \tilde{p}(\mathbf{k}) \exp(-i\mathbf{k} \cdot \mathbf{x}) d\mathbf{k}, \tag{9}$$

where,

$$v_2(\mathbf{k}) = \begin{pmatrix} \sin \frac{1}{2}\phi_2 \sigma \cdot \hat{k} \\ \cos \frac{1}{2}\phi_2 \end{pmatrix}. \tag{10}$$

$\tilde{p}(\mathbf{k}) = \tilde{p}_{lr}(\mathbf{k})v_{lr}$ is the two-component creation operator of the proton. For later convenience, we have expanded the charge conjugate of the proton field operator. We shall also determine $\phi_2(K)$ from the same variational principle.

The effective Hamiltonian for the electron-proton system as is relevant for the hydrogen atom is

$$H \equiv \int \mathscr{H}(\mathbf{x}, 0)\,\mathrm{d}\mathbf{x}$$
$$= \int [\mathscr{H}_1(\mathbf{x}, 0) + \mathscr{H}_2(\mathbf{x}, 0) + v(\mathbf{x}, 0)]\,\mathrm{d}\mathbf{x}, \tag{11}$$

where,

$$\mathscr{H}_1(\mathbf{x}, 0) = e^+(\mathbf{x}, 0)(-i\boldsymbol{\alpha}\cdot\nabla + \beta m_1)e(\mathbf{x}, 0), \tag{12}$$

$$\mathscr{H}_2(\mathbf{x}, 0) = :\tilde{p}^+(\mathbf{x}, 0)(-i\boldsymbol{\alpha}\cdot\nabla + \beta m_2)\tilde{p}(\mathbf{x}, 0):, \tag{13}$$

and

$$v(\mathbf{x}, 0) = \int \mathrm{d}\mathbf{y}: \bar{e}(\mathbf{x}, 0)\gamma^\mu e(\mathbf{x}, 0)\bar{\tilde{p}}(\mathbf{y}, 0)\gamma^\nu \tilde{p}(\mathbf{y}, 0): v_{\mu\nu}(\mathbf{x} - \mathbf{y}). \tag{14}$$

To illustrate the present method, we shall choose the Coulomb gauge, such that we have,

$$v_{00}(\mathbf{x} - \mathbf{y}) = (2\pi)^{-3}\int \frac{4\pi\alpha}{|\mathbf{q}|^2}\exp[i\mathbf{q}\cdot(\mathbf{x} - \mathbf{y})]\,\mathrm{d}\mathbf{q}, \tag{15a}$$

and with, $i, j = 1, 2, 3$,

$$v_{ij}(\mathbf{x} - \mathbf{y}) = (2\pi)^{-3}\int \frac{4\pi\alpha}{|\mathbf{q}|^2}\left(-\delta_{ij} + \frac{q_i q_j}{|\mathbf{q}|^2}\right)\exp[i\mathbf{q}\cdot(\mathbf{x} - \mathbf{y})]\,\mathrm{d}\mathbf{q}. \tag{15b}$$

The negative sign of the potential will emerge from a reordering in (14).

We shall now first obtain the known result that the variation of the state vector is equivalent to the eigenvalue equation, and *then proceed to modify the principle.* For a bound-state $|B\rangle$, we define the energy "functional" as

$$E(|B\rangle, \lambda) = \langle B|H|B\rangle - \lambda(\langle B|B\rangle - 1). \tag{16}$$

Extremum of E with respect to the variations of $|B\rangle$ and λ then yields the equations

$$H|B\rangle = \lambda|B\rangle \tag{17}$$

and

$$\langle B|B\rangle = 1. \tag{18}$$

This shows, as is known, that *extremizing* E with condition (18) yields (Stanley and Robson 1980) the eigenvalues of H.

In the present case, $H = H(\phi_1, \phi_2)$ is also a functional of ϕ_1 and ϕ_2 of (7) and (10) which are so far undetermined. We shall here take that they are *also* determined by the same extremization principle. Thus we define

$$E(|B\rangle, \phi_1, \phi_2, \lambda) = \langle B|H(\phi_1, \phi_2)|B\rangle - \lambda(\langle B|B\rangle - 1). \tag{19}$$

Now, in addition to (17) and (18), we also obtain the equations†

$$\left\langle B\left|\frac{\delta H(\phi_1, \phi_2)}{\delta\phi_1(\mathbf{k})}\right|B\right\rangle = 0 \tag{20}$$

† Yaouanc *et al* (1985) also do a variation in $\phi(K)$, but in the context of chiral symmetry breaking.

and

$$\left\langle B \left| \frac{\delta H(\phi_1, \phi_2)}{\delta \phi_2(\mathbf{k})} \right| B \right\rangle = 0, \tag{21}$$

where we have taken $\phi_A(\mathbf{k}) = \phi_A(K), A = 1, 2$. Equations (17), (20) and (21) with the condition (18) determine in a self-consistent manner, the structure of the field operators and the state vector $|B\rangle$ corresponding to the eigenvalue λ. Obviously λ is the expectation value of H when the above three equations are satisfied. Variation with respect to ϕ_1 and ϕ_2 as in (20) and (21) is the new feature of the present model. This yields the bound state eigenvalues as well as the structure of the constituent operators through ϕ_1 and ϕ_2.

Let us now consider two states $|B_1(\mathbf{p})\rangle$ and $|B_2(\mathbf{p})\rangle$ which are eigenstates of total momentum and are properly normalized. We then consider the equations

$$\langle B_1(\mathbf{p})|H|B_2(\mathbf{0})\rangle = \delta(\mathbf{P}) M_{B_1 B_2} = (2\pi)^3 \delta(\mathbf{p}) \langle B_1(\mathbf{0})|\mathscr{H}(0)|B_2(\mathbf{0})\rangle. \tag{22}$$

The last relationship arises from space translational invariance, with $\mathscr{H}(0)$ as the Hamiltonian density at space time origin. This leads to the definition of the *mass* matrix through

$$M_{B_1 B_2} \equiv \langle B_1|M|B_2\rangle = (2\pi)^3 \langle B_1(\mathbf{0})|\mathscr{H}(0)|B_2(\mathbf{0})\rangle, \tag{23}$$

which will be our basic equation.

We now take in particular a spin singlet state of the electron-proton system given as

$$|B(\mathbf{0})\rangle = \frac{1}{\sqrt{2}} \int \mathrm{d}\mathbf{k}\, \tilde{h}(\mathbf{k}) e(\mathbf{k})^+ \tilde{p}(-\mathbf{k})|\mathrm{vac}\rangle, \tag{24}$$

where $\tilde{h}(\mathbf{k})$ is the *wave function of the bound system* in momentum space. We ignore any possible mixing between $S = 0$ and $S = 1$ states, and revert to (19). We also choose the normalization corresponding to (18) as $\langle B(\mathbf{P})|B(\mathbf{0})\rangle = \delta(\mathbf{P})$. This yields the familiar condition for the wave function $\tilde{h}(\mathbf{k})$ as

$$\int |\tilde{h}(\mathbf{k})|^2 \,\mathrm{d}\mathbf{k} = 1. \tag{25}$$

Equation (19) now becomes,

$$E(\tilde{h}, \phi_1, \phi_2, \lambda) = t_1(\tilde{h}, \phi_1) + t_2(\tilde{h}, \phi_2) + v(\tilde{h}, \phi_1, \phi_2) - \lambda\left(\int |\tilde{h}|^2 \,\mathrm{d}\mathbf{k} - 1\right). \tag{26}$$

In the above, from (23) and (12) we have, on simplification

$$t_1(\tilde{h}, \phi_1) \equiv (2\pi)^3 \langle B(\mathbf{0})|\mathscr{H}_1(0)|B(\mathbf{0})\rangle = \int \tilde{h}(\mathbf{k})^* [m_1 \cos\phi_1(\mathbf{k}) + K \sin\phi_1(\mathbf{k})] \tilde{h}(\mathbf{k})\,\mathrm{d}\mathbf{k}. \tag{27}$$

Similarly, from (13) we have

$$t_2(\tilde{h}, \phi_2) \equiv (2\pi)^3 \langle B(\mathbf{0})|\mathscr{H}_2(0)|B(\mathbf{0})\rangle = \int \tilde{h}(\mathbf{k})^* [m_2 \cos\phi_2(\mathbf{k}) + K \sin\phi_2(\mathbf{k})] \tilde{h}(\mathbf{k})\mathrm{d}\mathbf{k}. \tag{28}$$

Evaluation of $v(\tilde{h}, \phi_1, \phi_2)$ is a little more complicated. Here we have, with (14), (15a) and (15b),

$$v(\tilde{h}, \phi_1, \phi_2) = (2\pi)^3 \int \tilde{h}(\mathbf{k}_1')^* F(\mathbf{k}_1', \mathbf{k}_1, \phi_1, \phi_2) \frac{-4\pi\alpha}{(\mathbf{k}_1 - \mathbf{k}_1')^2} \tilde{h}(\mathbf{k}_1) d\mathbf{k}_1 d\mathbf{k}_1' \tag{29}$$

where

$$\begin{aligned} F(\mathbf{k}_1', \mathbf{k}_1, \phi_1, \phi_2) = {} & \cos(\phi_1/2)\cos(\phi_1'/2)\cos(\phi_2/2)\cos(\phi_2'/2) \\ & + \sin(\phi_1/2)\sin(\phi_1'/2)\sin(\phi_2/2)\sin(\phi_2'/2) \\ & + 2[\sin(\phi_1'/2)\sin(\phi_2'/2)\cos(\phi_1/2)\cos(\phi_2/2) \\ & + \cos(\phi_1'/2)\cos(\phi_2'/2)\sin(\phi_1/2)\sin(\phi_2/2) \\ & - (\hat{k}_1' \cdot \hat{q})(\hat{k}_1 \cdot \hat{q})\{\cos(\phi_1'/2)\cos(\phi_2/2)\sin(\phi_1/2)\sin(\phi_2'/2) \\ & + \cos(\phi_1/2)\cos(\phi_2'/2)\sin(\phi_1'/2)\sin(\phi_2/2)\}] \\ & + (\hat{k}_1' \cdot \hat{k}_1)\{\cos(\phi_1/2)\cos(\phi_1'/2)\sin(\phi_2/2)\sin(\phi_2'/2) \\ & + \cos(\phi_2'/2)\cos(\phi_2/2)\sin(\phi_1/2)\sin(\phi_1'/2)\}. \end{aligned} \tag{30}$$

In the above, we have substituted $\phi_A = \phi_A(K)$ and $\phi_A(K') = \phi_A'$, $A = 1, 2$. We are now in a position to consider the equations parallel to (17), (20) and (21). Those are, respectively,

$$\frac{\delta t_1}{\delta \tilde{h}(\mathbf{k})^*} + \frac{\delta t_2}{\delta \tilde{h}(\mathbf{k})^*} + \frac{\delta v}{\delta \tilde{h}(\mathbf{k})^*} = \lambda \tilde{h}(\mathbf{k}), \tag{31}$$

$$\frac{\delta t_1}{\delta \phi_1(\mathbf{k})} + \frac{\delta v}{\delta \phi_1(\mathbf{k})} = 0, \tag{32}$$

and

$$\frac{\delta t_2}{\delta \phi_2(\mathbf{k})} + \frac{\delta v}{\delta \phi_2(\mathbf{k})} = 0, \tag{33}$$

which determine the three functions $\tilde{h}(\mathbf{k})$, $\phi_1(\mathbf{k})$ and $\phi_2(\mathbf{k})$. The eigenvalue λ is merely the expectation value of $t_1 + t_2 + v$ for the above set of solutions. We make the obvious remark that here, as expected ϕ_1, ϕ_2 and $\tilde{h}$ are inter-related and further that the equations in ϕ_1 and ϕ_2 are highly nonlinear. In the next section we shall see that these equations yield e.g. the conventional relativistic wave equation for the hydrogen atom. However, we shall discuss equations (31) to (33) a little bit more here.

Equation (33) can be explicitly written as, with (28) and (29),

$$\begin{aligned} & (-m_2 \sin\phi_2(\mathbf{k}) + K\cos\phi_2(\mathbf{k}))|\tilde{h}(\mathbf{k})|^2 \\ & + (2\pi)^{-3} \int d\mathbf{k}_1' d\mathbf{k}_1 \tilde{h}(\mathbf{k}_1')^* \frac{\delta F(\mathbf{k}_1', \mathbf{k}_1, \phi_1, \phi_2)}{\delta \phi_2(\mathbf{k})} \frac{-4\pi\alpha}{|\mathbf{k}_1 - \mathbf{k}_1'|^2} \tilde{h}(\mathbf{k}_1) = 0. \end{aligned} \tag{34}$$

When we neglect the interaction term in (34), we obtain that

$$\sin\phi_2/K = \cos\phi_2/m_2 = 1/(m_2^2/K^2)^{\frac{1}{2}}. \tag{35}$$

We next assume the proton mass to be large, which yields that $\phi_2 = 0$. In this case (30) becomes considerably simpler, and we have,

$$F(\mathbf{k}_1', \mathbf{k}_1, \phi_1, 0) = \cos(\phi_1/2)\cos(\phi_1'/2) + (\hat{k}_1', \hat{k}_1)\sin(\phi_1/2)\sin(\phi_1'/2). \quad (36)$$

We then consider (32), drop the interaction term, and obtain as in (35)

$$\sin\phi_1/K = \cos\phi_1/m_1 = 1/(m_1^2 + K^2)^{\frac{1}{2}}. \quad (37)$$

Retaining terms upto order k^2, (31) then yields the familiar nonrelativistic equation

$$(m_1 + \frac{K^2}{2m_1} + m_2)\tilde{h}(\mathbf{k}) + (2\pi)^{-3}\int \frac{-4\pi\alpha}{|\mathbf{k}-\mathbf{k}'|^2}\tilde{h}(\mathbf{k}')d\mathbf{k}' = \lambda\tilde{h}(\mathbf{k}). \quad (38)$$

We have also examined the effect of terms we have neglected, and have found that it is of order α^6, whereas the renormalization effects in (11) give a larger contribution.

A few remarks here may be in order. The variation of the wave function is such that the extremum in energy is a minimum. However, it may be noted that the extremum for the variation of ϕ_1 and ϕ_2 here is a *maximum*. This is rather an unexpected feature, but is obvious when one looks at t_1. It appears that the variation of the fermion field operator is such that it *pumps in* as much energy as it can *to* the localized system, and the localized system through the wave function readjusts itself to make it a minimum. The final solution becomes a compromise between these two effects. A less esoteric explanation is that the field operators minimize the action, and hence the Lagrangian density, whereas the wave function minimizes the energy (Misra and Patnaik 1985).

It is obvious that the above method is applicable to any system of two Dirac particles, when the effective Hamiltonian is known. It can also be generalized to three particle bound states, e.g., for the baryons. The constituent field operators taken as an ansatz in Misra (1978) for the hadrons here get determined from the calculations of the mass levels. Further, the dynamics of the bound state is closely linked with those constituent field operators, and the simple ansatz we had taken earlier (Misra 1978) about the *universality* of constituent operators is likely to have only limited validity. This "handicap" however is compensated by the fact that the present method enables us to systematically calculate these and thus link the dynamics of spectroscopy with that of even high energy collisions (Misra 1978; Misra 1980; Misra and Panda 1980).

3. Bosonic bound states

We shall consider here the bosonic bound state of a *scalar* particle and antiparticle bound by a phenomenological and known potential. The Hamiltonian density parallel to (11) is taken as

$$\mathscr{H}(x) = \mathscr{H}_0(x) + v(x), \quad (39)$$

where

$$\mathscr{H}_0(x) = :\{(\delta_t\Phi)^*(\partial_t\Phi) + (\nabla\Phi)^*(\nabla\Phi) + m^2\Phi^*\Phi\}: \quad (40)$$

and as an illustration,

$$v(\mathbf{x}, 0) = \tfrac{1}{2}\int \Phi^*(\mathbf{x}, 0)\Phi(\mathbf{x}, 0)v(\mathbf{x}-\mathbf{y})\Phi^*(\mathbf{y}, 0)\Phi(\mathbf{y}, 0)d\mathbf{y}. \quad (41)$$

With *only the constraint of equal time commutators*, we write the constituent field operator $\Phi(x)$ in the rest frame of the composite particle as

$$\Phi(x) = (2\pi)^{-3/2} \int \frac{\mathrm{d}\mathbf{k}}{[2\theta(k)]^{\frac{1}{2}}} \exp[-i\theta(K)t + i\mathbf{k}\cdot\mathbf{x}] + \exp[i\theta(K)t - i\mathbf{k}\cdot\mathbf{x}\, b(\mathbf{k})^+\}, \tag{42}$$

where $\theta(K)$ is an *arbitrary* function of $K = |\mathbf{k}|$, and $a(\mathbf{k})^+$ and $b(\mathbf{k})^+$ are usual particle and antiparticle creation operators. As earlier we shall determine the $\theta(K)$'s by the variational method.

We take the state $|B(0)\rangle$ as

$$|B(0)\rangle = \int \tilde{h}(\mathbf{k})\mathrm{d}\mathbf{k}\, a(\mathbf{k})^+ b(-\mathbf{k})^+ |\mathrm{vac}\rangle. \tag{43}$$

As before the energy functional is

$$E(\tilde{h}, \theta, \lambda) = t(\tilde{h}, \theta) + v(\tilde{h}, \theta) - \lambda(\int |\tilde{h}|^2 \mathrm{d}\mathbf{k} - 1) \tag{44}$$

where,

$$t(\tilde{h}, \theta) \equiv (2\pi)^3 \langle B(0)|\mathscr{H}_0(0)|B(0)\rangle = \int \tilde{h}^*(\mathbf{k})\left[\theta(K) + \frac{K^2 + m^2}{\theta(K)}\right]\tilde{h}(\mathbf{k})\mathrm{d}\mathbf{k} \tag{45}$$

and,

$$v(\tilde{h}, \theta) = (2\pi)^{-3} \int \frac{\mathrm{d}\mathbf{k}_1' \mathrm{d}\mathbf{k}_1}{4\theta(K_1')\theta(K_1)} \tilde{h}(\mathbf{k}_1')^* \tilde{v}(\mathbf{k}_1' - \mathbf{k}_1)\tilde{h}(\mathbf{k}_1). \tag{46}$$

We note that, the θ variation yields

$$\left(1 - \frac{K^2 + m^2}{\theta^2}\right)\tilde{h}(\mathbf{k})^* \tilde{h}(\mathbf{k}) + \frac{\partial v(\tilde{h}, \theta)}{\partial \theta(K)} = 0 \tag{47}$$

such that, if we neglect the potential above, we get $\theta = (K^2 + m^2)^{\frac{1}{2}} \equiv \omega(K)$, which is the usual solution. Variation of $\tilde{h}$ gives as before the eigenvalue equation, with λ as the eigenvalue.

We however note a difference between earlier analysis for fermions and the present one for the bosons. Earlier, as we had noted, the extremum for ϕ variation for a fixed $\tilde{h}$ yielded a *maximum* of the energy. However, we may see that for a "small" potential the field operator variation in (47) through θ results now in a *minimum* of energy. Thus, the bosonic field operator also tries to minimize energy of the localized solution along with the wave function, in contrast wtih the fermionic field operator.

We may further note that in both the places for fermions and bosons, we have *not* used the equations of motion, but have replaced it by a variational principle. This includes naturally the effect due to the potential, which is in contrast to the method of first quantising the system with the *free* field Hamiltonian, and, *then*, including interactions. The obvious technique here is to fully utilize the equal time commutator/anticommutator algebra to construct the bound state. The advantage of this variational method for *nonperturbative* calculations with an arbitrary basis is obvious.

We next illustrate the method with a few simple examples.

4. Illustrative examples

4.1 *Relativistic hydrogen atom*

When the proton is taken as heavy, so that by (35), $\phi_2 = 0$. We take the hydrogen like state as

$$|B(\mathbf{0}),s\rangle = \int e(\mathbf{k})^+ \tilde{h}(\mathbf{k})\tilde{p}_s(-\mathbf{k})\mathrm{d}\mathbf{k}|\mathrm{vac}\rangle. \tag{48}$$

In the above $\tilde{h}(\mathbf{k})$ is the *two-spin component doublet wave function* normalized as

$$\int \tilde{h}(\mathbf{k})^+ \tilde{h}(\mathbf{k})\mathrm{d}\mathbf{k} = 1. \tag{49}$$

Equation (28) yields that $t_2 = m_2$, and (27) becomes

$$t_1 = \int \tilde{h}(\mathbf{k})^+ [m_1 \cos\phi_1(K) + K\sin\phi_1(K)]\tilde{h}(\mathbf{k})\mathrm{d}\mathbf{k}. \tag{50}$$

We now *define a four-component wavefunction* $\tilde{\psi}_1(\mathbf{k})$ as

$$\tilde{\psi}_1(\mathbf{k}) = \begin{pmatrix} \cos\dfrac{\phi_1}{2}\tilde{h}(\mathbf{k}) \\ \sin\dfrac{\phi_1}{2}(\sigma\cdot\hat{k})\tilde{h}(\mathbf{k}) \end{pmatrix}. \tag{51}$$

Then we easily see that

$$t_1 = \int \tilde{\psi}_1(\mathbf{k})^+ (\boldsymbol{\alpha}\cdot\mathbf{k} + \beta m_1)\tilde{\psi}_1(\mathbf{k})\mathrm{d}\mathbf{k}. \tag{52}$$

Also, with $\phi_2 = 0$ and the above four-component $\tilde{\psi}_1(\mathbf{k})$, (29) now simplifies to

$$v = (2\pi)^{-3}\int \tilde{\psi}_1(\mathbf{k}_1')^+ \frac{-4\pi\alpha}{|\mathbf{k}_1 - \mathbf{k}_1'|^2}\tilde{\psi}_1(\mathbf{k}_1)\mathrm{d}\mathbf{k}_1'\mathrm{d}\mathbf{k}_1. \tag{53}$$

Thus t_1 and v both become functionals of the *four-component* wave function $\tilde{\psi}_1(\mathbf{k})$, and through (51), variation of the wave function $\tilde{h}(\mathbf{k})$ and the field operator with $\phi_1(K)$ get absorbed into the variation of $\tilde{\psi}_1(\mathbf{k})$. The normalization (49) is now equivalent to

$$\int \tilde{\psi}_1(\mathbf{k})^+ \tilde{\psi}_1(\mathbf{k})\mathrm{d}\mathbf{k} = 1. \tag{54}$$

Hence the energy functional becomes

$$E(\tilde{\psi}_1, \lambda) = t_1(\tilde{\psi}_1) + v(\tilde{\psi}_1) - \lambda\left(\int \tilde{\psi}_1(\mathbf{k}_1)^+ \tilde{\psi}_1(\mathbf{k}_1)\mathrm{d}\mathbf{k}_1 - 1\right), \tag{55}$$

with the variational equation (31) as

$$(\boldsymbol{\alpha}\cdot\mathbf{k}_1 + m_1)\tilde{\psi}_1(\mathbf{k}_1) + m_2\tilde{\psi}_1(\mathbf{k}_1) + (2\pi)^{-3}\int \frac{-4\pi\alpha}{|\mathbf{k}_1 - \mathbf{k}_1'|^2} \times \tilde{\psi}_1(\mathbf{k}_1')\mathrm{d}\mathbf{k}_1' = \lambda\tilde{\psi}_1(\mathbf{k}_1). \tag{56}$$

In coordinate space, we obviously have here,

$$t_1 = \int \psi_1(\mathbf{r})^+ (-i\boldsymbol{\alpha}\cdot\nabla + \beta m_1)\psi_1(\mathbf{r})\mathrm{d}\mathbf{r} \tag{57}$$

and

$$v = \int \psi_1(\mathbf{r})^+ \left(-\frac{\alpha}{r}\right)\psi_1(\mathbf{r})\,\mathrm{d}\mathbf{r}. \tag{58}$$

In the above problem, the only "relativistic" particle is the electron, and hence, as illustrated, the four component wave function (51) absorbs the variation of the wave function along with the variation of the field operator, with the normalization condition (54). This feature, however, is absent when we want to vary the field operators of two or more relativistic particles, where the principle stated in the last section can be applied. The above analysis shows that the method suggested in the last section agrees with the accurate calculations of quantum electrodynamics. The effect of the recoil of the proton or its size can be easily incorporated in a parallel manner.

4.2 *Salpeter equation*

We shall now consider two quark flavours with a simple potential. Equations (12), (13) and (14) become

$$\mathscr{H}_1(\mathbf{x}) = q_1(\mathbf{x})^+(-i\boldsymbol{\alpha}\cdot\nabla + \beta m_1)q_1(\mathbf{x}), \tag{59}$$

$$\mathscr{H}_2(\mathbf{x}) = :\tilde{q}_2(\mathbf{x})^+(-i\boldsymbol{\alpha}\cdot\nabla + \beta m_2)\tilde{q}_2(\mathbf{x}): \tag{60}$$

and

$$v(\mathbf{x}) = \int \mathrm{d}\mathbf{y}: q_1(\mathbf{x})^+(\lambda_a/2)q_1(\mathbf{x})\hat{q}_2(\mathbf{y})^+(\lambda_a/2)\hat{q}_2(\mathbf{y}): v(\mathbf{x}-\mathbf{y}). \tag{61}$$

Our objective here is to make contact with Yaouanc *et al* (1985), and recognize the differences. We are here effectively taking the nonrelativistic version of (14) as in Yaouanc *et al* (1985). We first replace (24) by

$$|B(\mathbf{0})\rangle = \frac{1}{\sqrt{3}}\int \chi_{rs}(\mathbf{k})q^i_{1r}(\mathbf{k})^+\tilde{q}^i_{2s}(-\mathbf{k})\,\mathrm{d}\mathbf{k}|\mathrm{vac}\rangle. \tag{62}$$

The colour index "i" as well as the spin indices above are summed. As before $q_1(\mathbf{k})^+$ and $\tilde{q}_2(-\mathbf{k})$ are *two component* creation operators for the quark q_1 and the antiquark $\tilde{q}_2$. The normalization parallel to (49) is

$$\int \chi^*_{rs}(\mathbf{k})\chi_{rs}(\mathbf{k})\,\mathrm{d}\mathbf{k} = 1. \tag{63}$$

We next define the 4×4 matrix given as

$$\chi_{\alpha\beta}(\mathbf{k}) = u^{(1)}_{r\alpha}(\mathbf{k})\chi_{rs}(\mathbf{k})v^{(2)}_{s\beta}(-\mathbf{k})^*, \tag{64}$$

where the Greek indices α, β are indices for the four-component spinors, with the constituent spinors given by the earlier equations (4) and (5). Clearly, the inverse of (64) is given as

$$\chi_{rs}(\mathbf{k}) = u^{(1)}_{r\alpha}(\mathbf{k})^*\chi_{\alpha\beta}(\mathbf{k})v^{(2)}_{s\beta}(-\mathbf{k}). \tag{65}$$

We, however, note that $\chi_{\alpha\beta}(\mathbf{k})$ *arises* from the 2×2 matrix $\chi_{rs}(\mathbf{k})$, and thus is really a highly constrained 4×4 matrix, which makes it as inequivalent to Yaouanc *et al* (1985). We shall however continue to look into the otherwise complete similarity of the two.

For this purpose we now define the projection operators

$$\Lambda^{(1)}_{(+)}(\mathbf{k})_{\alpha\beta} = u^{(1)}_{r\alpha}(\mathbf{k})u^{(1)}_{r\beta}(\mathbf{k})^* \tag{66}$$

$$\Lambda^{(2)}_{(-)}(\mathbf{k})_{\alpha\beta} = v^{(2)}_{s\alpha}(-\mathbf{k})v^{(2)}_{s\beta}(-\mathbf{k})^*, \tag{67}$$

which were the basic objects in Yaouanc *et al* (1985). In the present subsection, when we write $\chi(\mathbf{k})$, we shall understand it as the 4×4 matrix of (64). We may note from (64) the identities†

$$\chi(\mathbf{k}) = \Lambda^{(1)}_{(+)}(\mathbf{k})\chi(\mathbf{k}) = \chi(\mathbf{k})\Lambda^{(2)}_{(-)}(-\mathbf{k}) = \Lambda^{(1)}_{(+)}(\mathbf{k})\chi(\mathbf{k})\Lambda^{(2)}_{(-)}(-\mathbf{k}), \tag{68}$$

which we shall frequently use. Now with $A = 1, 2$; $\Lambda^{A}_{(\pm)}(\mathbf{k})$ are hermitian, and, $\Lambda^{A}_{(+)}(\mathbf{k}) + \Lambda^{A}_{(-)}(-\mathbf{k}) = I$ and $\Lambda^{A}_{(+)}(\mathbf{k})\Lambda^{A}_{(-)}(-\mathbf{k}) = 0$, with A *not* summed.

We now try to simplify, $t_1 = (2\pi)^3 \langle B(\mathbf{0})|\mathscr{H}^{(0)}_1|B(\mathbf{0})\rangle$. With a little algebra, we then obtain that

$$t_1 = \mathrm{tr}[\chi(\mathbf{k})^+(\boldsymbol{\alpha}\cdot\mathbf{k} + \beta m_1)\chi(\mathbf{k})], \tag{69}$$

where in particular (65) and (68) have been used. We also further obtain, with (59) and (68),

$$\begin{aligned} t_2 &\equiv (2\pi)^3 \langle B(\mathbf{0})|\mathscr{H}^{(0)}_2|B_2(\mathbf{0})\rangle \\ &= -\mathrm{tr}[\chi(\mathbf{k})^+\chi(\mathbf{k})(\boldsymbol{\alpha}\cdot\mathbf{k} + \beta m_2)]. \end{aligned} \tag{70}$$

The *negative* sign above along with a change of order may be noted.

We next consider (61), and obtain using (65) along with some simplifications, that

$$\begin{aligned} v &\equiv (2\pi)^3 \langle B(\mathbf{0})|v(0)|B(\mathbf{0})\rangle \\ &= (2\pi)^{-3} \int \mathrm{tr}[\chi(\mathbf{k})^+\chi(\mathbf{k}')]\,(-\tfrac{4}{3})\,\tilde{V}(\mathbf{k}-\mathbf{k}')\,\mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{k}' \qquad (71) \\ &\equiv (2\pi)^{-3} \int \mathrm{tr}[\chi(\mathbf{k})^+(\chi(\mathbf{k}')\Lambda^{(2)}_{-}(-\mathbf{k}) - \Lambda^{(1)}_{-}(-\mathbf{k})\chi(\mathbf{k}'))] \\ &\quad \times (-\tfrac{4}{3})\,\tilde{V}(\mathbf{k}-\mathbf{k}')\,\mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{k}'. \qquad (72) \end{aligned}$$

In (72) we recognize that with (68) $\chi(\mathbf{k})$ is a *constrained* 4×4 matrix. We further note that the normalization here from (63) is

$$\int \mathrm{tr}[\chi(\mathbf{k})^+\chi(\mathbf{k})]\,\mathrm{d}\mathbf{k} = 1. \tag{73}$$

Hence, taking the functional parallel to (26) and using (69), (70), (72) and (73) we get here the eigenvalue equation,

$$\begin{aligned} &(\boldsymbol{\alpha}\cdot\mathbf{k} + \beta m_1)\chi(\mathbf{k}) - \chi(\mathbf{k})(\boldsymbol{\alpha}\cdot\mathbf{k} + \beta m_2) \\ &+ (2\pi)^{-3} \int (\chi(\mathbf{k}')\Lambda^{(2)}_{-}(-\mathbf{k}) - \Lambda^{(1)}_{-}(-\mathbf{k})\chi(\mathbf{k}'))\,(-\tfrac{4}{3})\,\tilde{V}(\mathbf{k}-\mathbf{k}')\,\mathrm{d}\mathbf{k}' \\ &= \lambda\chi(\mathbf{k}), \end{aligned} \tag{74}$$

which is the Salpeter equation (Yaouanc *et al* 1985). As an example of interlinking

† We note that $\Lambda_-(-\mathbf{k})$ of Yaouanc *et al* (1985) = Λ (**k**) here.

the above with the results of §2, let us note that for a *spin singlet state* as (24), $\chi_{rs}(\mathbf{k}) = \frac{1}{\sqrt{2}} u^{+}_{Ir} v_{Is} \tilde{h}(\mathbf{k})$, such that from (64)

$$\chi_{\alpha\beta}(\mathbf{k}) = \frac{1}{\sqrt{2}} \tilde{h}(\mathbf{k}) \begin{pmatrix} -\cos\frac{\phi_1}{2}\sin\frac{\phi_2}{2}\,\sigma\cdot\hat{k}, & \cos\frac{\phi_1}{2}\cos\frac{\phi_2}{2} \\ -\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}, & \sin\frac{\phi_1}{2}\cos\frac{\phi_2}{2}\,\sigma\cdot\hat{k} \end{pmatrix}. \tag{75}$$

With the above, we can easily verify that (69) and (80) yield in fact *the same* results as (27) and (28). Further, with v as in (61), we get from (71) and (75) that

$$\begin{aligned} v = (2\pi)^{-3} \int \tilde{h}(\mathbf{k}')^* \Bigg[(\hat{k}\cdot\hat{k}') \Big(&\cos\frac{\phi_1}{2}\cos\frac{\phi_1'}{2}\sin\frac{\phi_2}{2}\sin\frac{\phi_2'}{2} \\ &+ \sin\frac{\phi_1}{2}\sin\frac{\phi_1'}{2}\cos\frac{\phi_2}{2}\cos\frac{\phi_2'}{2}\Big) + \Big(\sin\frac{\phi_1}{2}\sin\frac{\phi_1'}{2}\sin\frac{\phi_2}{2}\sin\frac{\phi_2'}{2} \\ &+ \cos\frac{\phi_1}{2}\cos\frac{\phi_1'}{2}\cos\frac{\phi_2}{2}\cos\frac{\phi_2'}{2}\Big)\Bigg] (-\tfrac{4}{3})\,\tilde{V}(\mathbf{k}-\mathbf{k}')\tilde{h}(\mathbf{k})\,\mathrm{d}\mathbf{k}'\,\mathrm{d}\mathbf{k} \end{aligned} \tag{76}$$

$$\begin{aligned} &= (2\pi)^{-3} \int \tilde{h}(\mathbf{k}')^* \tfrac{1}{2} [1 + \cos\phi\cos\phi' + \sin\phi\sin\phi'(\hat{k}\cdot\hat{k}')] \\ &\quad \times (-\tfrac{4}{3})\,\tilde{V}(\mathbf{k}'-\mathbf{k})\tilde{h}(\mathbf{k})\,\mathrm{d}\mathbf{k}'\,\mathrm{d}\mathbf{k}. \end{aligned} \tag{77}$$

In (77), we have taken the masses to be the same, so that we will have from symmetry $\phi_1 = \phi_2 = \phi$ and $\phi_1' = \phi_2' = \phi'$.

In particular, we may take e.g. in (61) the harmonic oscillator potential as

$$-\tfrac{4}{3} V(\mathbf{x}-\mathbf{y}) = V_0^3 |\mathbf{x}-\mathbf{y}|^2. \tag{78}$$

We note that the Fourier transform of (78) does not really exist, and therefore gives rise to ill-defined quantities when we substitute it in (77) and evaluate. We circumvent this difficulty by regularizing the potential as below. We evaluate

$$\begin{aligned} v = \lim_{a\to 0} V_0^3 (2\pi)^{-3}\, 2\pi(\pi/a)^{3/2} \times \frac{1}{2} \int \tilde{h}(K_1')^* \Bigg[&1 + \cos\phi_1\cos\phi_1' \\ &+ \frac{\sin\phi_1\sin\phi_1'}{K_1 K_1'} \left(K^2 - \frac{\phi^2}{4} \right) \Bigg] \tilde{h}(K_1) \left(\frac{3}{2a} - \frac{\phi^2}{4a^2} \right) \\ &\times \exp\left(-\frac{\phi^2}{4a} \right) K^2\,\mathrm{d}K\,\phi^2\,\mathrm{d}\phi\,(\cos\theta) \end{aligned} \tag{79}$$

where we have changed the variables as $\phi = |\mathbf{q}| = |\mathbf{k}_1' - \mathbf{k}_1|$, $\mathbf{K} = \frac{1}{2}(\mathbf{k}_1' + \mathbf{k}_1)$. The above corresponds to the regularization procedure $V(\mathbf{r}) = V_0^3 r^2 \exp(-ar^2)$, with $a \to 0$, which in momentum space becomes with $\phi = |\mathbf{q}|$,

$$-\frac{4}{3}\tilde{V}(\mathbf{q}) = \lim_{a\to 0} V_0^3 \left(\frac{\pi}{a} \right)^{3/2} \left(\frac{3}{2a} - \frac{\phi^2}{4a^2} \right) \exp\left(-\frac{\phi^2}{4a} \right). \tag{80}$$

We have mentioned the above regularization process since here basically *expectation* values and hence integrals are involved. The equations for spectroscopy here are non-perturbative, but become highly nonlinear and cannot be easily solved.

We now note a few differences in the context of Yaouanc *et al* (1985). We note that here ϕ is *not* determined through chiral symmetry breaking of the vacuum but enters nontrivially in the equations of the bound state as a whole, and, determines the constituent field operators. The interlinking of the form of four-component constituent operators with spectroscopy is the new feature of the present variational model, which can be exploited for many applications of quark model elsewhere.

5. Effective mass

In the last section we had considered the quark-antiquark system. We shall here show in this framework a possible origin of an "effective mass" in Schrödinger's equation in a trivial approximation with a harmonic oscillator potential.

As observed in Yaouanc *et al* (1985), a simple non-relativistic approximation for $\phi(K)$ becomes

$$\phi(K) = CK. \tag{81}$$

We shall now work with this approximation, and thus replace ϕ variation by the variation of the parameter C. Further, we shall take

$$\begin{aligned} \tilde{h}(\mathbf{k}) &= \left(\frac{R^2}{\pi}\right)^{3/4} \exp\left(-\frac{R^2K^2}{2}\right) \\ &\equiv \sqrt{4\pi}\,\tilde{u}_{00}(K), \end{aligned} \tag{82}$$

and, replace $\tilde{h}(\mathbf{k})$ variation by the variation with respect to R. Our objective here is to test how the effective mass can arise.

From (69) and (70), we now obtain, with the approximations (81) and (82) that

$$\begin{aligned} t &= \int 2[m\cos\phi + K\sin\phi]\,|\tilde{u}_{00}(K)|^2\,K^2\,\mathrm{d}K \\ &\simeq \int [2m + K^2(2C - mC^2)]\,|\tilde{u}_{00}(K)|^2\,K^2\,\mathrm{d}K. \end{aligned} \tag{83}$$

We next consider the regularized potential (79). We note that this arose from (16) which was the zeroth component only corresponding to a vector exchange. We generalize this to the relativistic expression corresponding to (14). The calculation is a little lengthy, and the final result is being written here. We thus obtain, with (81) and (82) the expression for v as,

$$\begin{aligned} v = V_0^3 \int\Bigg[& R^4K^2\left(\frac{7}{5} - \frac{2}{5}\cos^2 CK\right) - C^2\left(\frac{1}{10} + \frac{2}{5}\sin^2 CK\right) \\ & - \frac{2}{5}R^2CK\sin 2\,CK - \frac{2}{5K^2}\sin^2 CK - \frac{C}{5k}\sin 2\,CK \\ & + \frac{2R^2}{5}\sin^2 CK\Bigg]\,|\tilde{u}_{00}(K)|^2\,K^2\,\mathrm{d}k \\ \simeq V_0^3 \int\Bigg[& R^4K^2 - \frac{2}{5}R^2C^2K^2 - \frac{9}{10}C^2\Bigg]\,|\tilde{u}_{00}(K)|^2\,K^2\,\mathrm{d}K. \end{aligned} \tag{84}$$

In (83) and (84) we have retained only terms upto K^2. The first term on the right hand side of (84) corresponds to the nonrelativistic potential, and, *second* term contributes to an "effective mass" *for the kinetic term*, which we now define as

$$\frac{1}{m_{\text{eff}}} = (2C - mC^2) - \frac{2}{5} R^2 C^2 V_0^3. \tag{85}$$

On the right hand side of (85), we have to substitute the values of R and C *after* minimization of $t + v$ with respect to R and C. Further, we may interpret $-\frac{9}{10} V_0^3 C^2$ as a correction to the rest mass $2m$. The above statements are derived from the identification that (83) and (84) correspond to the Schrödinger equation,

$$\left(2m - \frac{9}{10} V_0^3 C^2 + \frac{\mathbf{k}^2}{m_{\text{eff}}}\right) \tilde{u}_{00}(\mathbf{k}) + (2\pi)^{-3} \int \mathrm{d}\mathbf{k}' \times$$
$$\tilde{V}(\mathbf{k} - \mathbf{k}') \tilde{u}_{00}(\mathbf{k}') = \lambda \tilde{u}_{00}(\mathbf{k}), \tag{86}$$

where $\tilde{V}(\mathbf{k} - \mathbf{k}')$ is evaluated with (80), and, m_{eff} is given by (85). We may further note that the corrections to mass for harmonic oscillator potential become positive and are also generally dependent on the state. Such statements are obviously meaningful only if the system is nonrelativistic. For a relativistic system, in any case an "effective mass" is not well defined. Also, for potentials other than harmonic oscillator potential, we may not generally have a clean dependence on K^2 as in (85) and thus the correction through effective mass may only be approximate.

6. Discussion

We note that in the present analysis we have taken equal time anti-commutators/commutators of the interacting field operators as our starting point. Such anticommutators/commutators impose a constraint on the field operators, which are taken as otherwise arbitrary. We next define a bound state with some (unknown) wave function and evaluate the expectation value of an effective Hamiltonian. Then we determine the field operators as well as of the wave function by a variational principle. Thus we *do not* use the equations of motion of the field operators; this is replaced by a variation of the same to get an extremum for energy. We may contrast this with the *usual* approach of first considering the quantized *free fields* and then introducing the interaction term. Here the fields are never free and the solutions for these fields become closely interlinked with both the interaction as well as the specific bound state under consideration.

We also note that the present method also generates the conventional results such as the relativistic equation for the electron in a central potential or, the Salpeter equation for quark antiquark systems, with the advantage that it is basically nonperturbative. Further work to utilize this aspect is in progress.

Acknowledgement

The authors are indebted to H Mishra, S Naik and A R Panda for discussions.

References

Corson E M 1953 *Introduction to tensors, spinors and relativistic wave equations* (London & Glasgow: Blackie and Son)
Finger M *et al* 1979 *Phys. Rev.* **D20** 3253
Misra S P 1978 *Phys. Rev.* **D18** 1661, 1673
Misra S P 1980 *Phys. Rev.* **D21** 1231
Misra S P 1986 *Proceedings of Second Asia Pacific Physics Conference, Bangalore* (to appear)
Misra S P and Panda A R 1980 *Phys. Rev.* **D21** 3094
Misra S P and Panda S 1986 *Pramana (J. Phys.)* (to appear)
Misra S P and Patnaik T 1985 *Pramana (J. Phys.)* **24** 595
Schwinger J 1951 *Phys. Rev.* **82** 914
Stanley D P and Robson D 1980 *Phys. Rev.* **D21** 3180
Yaouanc A Le *et al* 1985 *Phys. Rev. Lett.* **54** 506