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Hamiltonian systems with indefinite kinetic energy

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Abstract. Some interesting features of a class of two-dimensional Hamiltonians with indefinite kinetic energy are considered. It is shown that such Hamiltonians cannot be reduced, in general, to an equivalent dynamical Hamiltonian with positive definite kinetic energy quadratic in velocities. Complex nonlinear evolution equations like the K-dV, the MK-dV and the sine-Gordon equations possess such Hamiltonians. The case of complex K-dV equation has been considered in detail to demonstrate the generic features. The two-dimensional real systems obtained by analytic continuation to complex plane of one-dimensional dynamical systems are also discussed. The evolution equations for nonlinear, amplitude-modulated Langmuir waves as well as circularly polarized electromagnetic waves in plasmas, are considered as illustrative examples.

Keywords. Hamiltonians; indefinite kinetic energy; complex nonlinear evolution equations; complex K-dV equation; Langmuir waves; electromagnetic waves.

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1. Introduction

The question of integrability of dynamical systems has been one of the outstanding problems in classical mechanics. The pioneering work of Hénon and Heiles (1964) on the nature of the 'third integral of motion', for a star moving in the potential due to a galaxy, showed the irregular (or chaotic) behaviour of the system when the energy of the star exceeded some critical value. In other words, the motion of the star undergoes a transition from periodic, regular behaviour to non-periodic, irregular behaviour as the energy is increased. Since then, the general behaviour of nonlinear dynamical systems has been extensively studied with special reference to the question of integrability. Since the integrability of a given dynamical system is intimately connected with the number of 'isolating integrals' offered by the system, much attention has been focussed on the study of existence of such integrals for the dynamical systems (see, for instance, Whiteman 1977 and the references therein).

Two-dimensional dynamical systems constitute the simplest, but non-trivial, examples of the generic behaviour. Integrability of such a system is ensured if it possesses two independent integrals of motion which are in involution (by definition, these are the 'isolating integrals' mentioned above). Since for conservative potentials, the energy (or the Hamiltonian) is an integral of motion, the conditions on the potential leading to the existence of a second integral of motion, which commutes with the Hamiltonian, have been fairly extensively investigated in recent years. One of the successful attempts, in this direction, is due to Ablowitz *et al* (1980) who conjectured that the question of integrability is closely related to the analytic properties of the solutions of the equations of motion in the complex time plane. With the help of the so-called Painlevé conjecture, Ramani *et al* (1982) were able to obtain explicitly a class of homogeneous, polynomial, two-dimensional potentials possessing a second islolating integral of motion. There are, however, quite a few two-dimensional Hamiltonians which do not fall under this category (Hietarinta 1984). This has led to a modification of the original definition of the Painlevé property. On the other hand, following Whittaker's method, Hall (1983) analyzed the conditions which a two-dimensional potential must satisfy in order that the system admits a second isolating integral, polynomial in velocities.

It may be pointed out that most of the current investigations on two-dimensional Hamiltonians have been concerned with the classical dynamical systems, for which the kinetic energy is a positive definite quantity. However, quite often one encounters systems that admit an 'energy integral' for which the 'kinetic energy' is not positive definite. For example, the stationary states of modulationally unstable plasmas (Varma and Rao 1980; Rao and Varma 1982) have precisely this property.

The Hamiltonian, in the stationary frame, represents the total energy of the interacting waves. Such Hamiltonians are very similar to the usual two-dimensional, classical dynamical Hamiltonians except for an important difference that the 'kinetic energy' in the former cases is not positive definite. It is then very pertinent to ask whether such systems could be reduced, using the transformation theory, to a form which has positive definite kinetic energy. The advantages of such an analysis are quite obvious; one can then use the standard results of nonlinear dynamics to study the regular or chaotic behaviour of the wave amplitudes when the free parameters of the system are varied. The main aim of the present paper is to address to this problem.

In §2, we deal with a general two-dimensional potential and determine the conditions under which it is possible to reduce the associated energy integral to a form with positive definite kinetic energy quadratic in velocities. In §3, we consider some physical examples from the field of plasma physics. Specifically, we discuss the complex K-dV equation, as well as the coupled equations describing the propagation of localized amplitude-modulated Langmuir waves, and the circularly polarized electromagnetic waves. Section 4 deals with the question of extension of real one-dimensional systems to the complex plane.

2. Systems with indefinite kinetic energy

Consider a two-dimensional system whose dynamics is governed by the Hamiltonian,

$$H = \tfrac{1}{2}(\dot{x}^2 - \dot{y}^2) + V(x,y), \tag{1}$$

where x and y are the generalized coordinates, $V(x,y)$ is the potential and the dot denotes differentiation with respect to the time variable. The equations of motion associated with this system are,

$$\ddot{x} = -\partial V/\partial x, \quad \ddot{y} = +\partial V/\partial y. \tag{2}$$

Clearly such a system differs from the usual, two-dimensional dynamical systems in that the kinetic energy is not positive definite, leading to a sign change in the equation of motion for y. The precise origin of such systems in physical situations will be discussed with some examples in the next section. For the present, we are interested in finding out the class of two-dimensional potentials $V(x,y)$ which admit an integral of motion $G(x,y,\dot{x},\dot{y})$ which has positive definite kinetic energy. In particular, we shall consider the case when the kinetic energy is quadratic in velocities. The potential $\phi(x,y)$ contained in $G(x,y,\dot{x},\dot{y})$ would then correspond to a classical dynamical system which is equivalent to the given system.

We follow the method due to Bertrand outlined by Whittaker (1944) and Dorizzi *et al* (1983). The general form of such an integral quadratic in velocities is given by,

$$G(x,y,\dot{x},\dot{y}) = f\dot{x}^2 + g\dot{x}\dot{y} + h\dot{y}^2 + \phi(x,y), \tag{3}$$

where f, g and h are functions of x and y. To evaluate these functions, we differentiate (3) with respect to time; this gives,

$$\frac{\partial f}{\partial x}\dot{x}^3 + \frac{\partial h}{\partial y}\dot{y}^3 + \left(\frac{\partial f}{\partial y} + \frac{\partial g}{\partial x}\right)\dot{x}^2\dot{y} + \left(\frac{\partial h}{\partial x} + \frac{\partial g}{\partial y}\right)\dot{x}\dot{y}^2 + \left(2f\ddot{x} + g\ddot{y} + \frac{\partial \phi}{\partial x}\right)\dot{x} + \left(g\ddot{x} + 2h\ddot{y} + \frac{\partial \phi}{\partial y}\right)\dot{y} = 0. \tag{4}$$

Equating to zero each of the coefficients, we get from the cubic terms in $\dot{x}$ and $\dot{y}$,

$$\partial f/\partial x = 0, \quad \partial h/\partial y = 0;$$
$$(\partial f/\partial y) + (\partial g/\partial x) = 0, \quad (\partial h/\partial x) + (\partial g/\partial y) = 0. \tag{5}$$

This system of equations can be easily solved to get,

$$f = ay^2 + by + \lambda,$$
$$g = -2axy - bx - cy + \mu,$$
$$h = ax^2 + cx + \nu, \tag{6}$$

where a, b, c, λ, μ and ν are constants. On the other hand, the linear terms in $\dot{x}$ and $\dot{y}$ yield,

$$(\partial\phi/\partial x) + 2f\ddot{x} + g\ddot{y} = 0,$$
$$(\partial\phi/\partial y) + g\ddot{x} + 2h\ddot{y} = 0, \tag{7}$$

where $\ddot{x}$ and $\ddot{y}$ are to be substituted in terms of $\partial V/\partial x$ and $\partial V/\partial y$ from equation (2).

If one now demands that G be canonical and has a positive definite kinetic energy, then $g \equiv 0$ for all x and y. This implies that a, b, c and μ are identically zero. Without any loss of generality, we can take $\lambda = \nu = 1/2$; equations (7) then yield,

$$(\partial\phi/\partial x) = -\ddot{x} = +(\partial V/\partial x)$$

and

$$(\partial\phi/\partial y) = -\ddot{y} = -(\partial V/\partial y). \tag{8}$$

From (8) it follows that $\partial^2\phi/\partial x\partial y = 0$, implying that $\phi(x,y)$ must be a separable potential, i.e. $\phi(x,y) = \psi_1(x) + \psi_2(y)$. This requires that $V(x,y)$ must also be separable. Thus, except for separable potentials, it is not possible to find a classical dynamical system with positive definite kinetic energy, which is equivalent to the system described by equations (2). However, this does not preclude the possibility of such a reduction when the kinetic energy contains terms of order higher than quadratic in velocity. On the other hand, there are non-separable, two-dimensional potentials that are integrable but have Hamiltonians of the form given by equation (1). In the following, we illustrate these features with some examples.

3. Illustrative examples

3.1 *Complex K-dV equation*

The Korteweg-de Vries (K-dV) equation is one of the simplest nonlinear evolution equations having wide ranging applications in various branches of physics (see, for instance, Bullough and Caudrey 1980). For a one-dimensional real field $u(x, t)$, it is given by

$$\frac{\partial u}{\partial t}+\alpha u\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}=0, \tag{9}$$

where α and β are constants. In the stationary frame, $\xi=(x-Mt)$, equation (9) becomes

$$\beta(\mathrm{d}^2u/\mathrm{d}\xi^2)=Mu-\tfrac{1}{2}\alpha u^2, \tag{10}$$

where M is a free parameter characterizing the velocity of the stationary wave. Localized as well as periodic solutions of (10) are well-known. In particular, the real, one-soliton solution is given by

$$u(\xi)=(3M/\alpha)\,\mathrm{sech}^2\{\tfrac{1}{2}\delta(\xi-\xi_0)\}, \tag{11}$$

where $\delta=(M/\beta)^{1/2}$ and ξ_0 represents the initial phase of the soliton. Note that for localized solutions M and β must have the same sign.

It is interesting to note that, with proper identification, namely, u as the position coordinate, ξ as the time coordinate and β as the mass parameter, (10) is identical with the equation of motion for a one-dimensional dynamical system in a potential,

$$V(u)=-(M/2)u^2+(\alpha/6)u^3. \tag{12}$$

The associated Hamiltonian, $H=\tfrac{1}{2}\beta(\mathrm{d}u/\mathrm{d}\xi)^2+V$ is an integral of motion since V does not depend on ξ. This is in accordance with the well-known fact that the K-dV equation is indeed integrable. The above similarity is true also for other nonlinear evolution equations like the modified K-dV equation, the sine-Gordon equation, etc.

Recently, we have shown (Buti 1983; Buti *et al* 1984 (unpublished)) that the above mentioned nonlinear equations admit, in addition to the usual real, localized solutions, a new class of complex, localized solutions for $u(\xi)$ with ξ being real. These solutions arise due to the branch cut in the complex u-plane. We will elaborate on these solutions for the K-dV equation. Localized, one-soliton solution of (10) can be obtained by contour integration in the (u_1, u_2) plane (Buti *et al* 1984). They can be expressed in a simple form, namely,

$$u(\xi)=\left(\frac{3M}{\alpha}\right)\operatorname{sech}^2\left\{\frac{\delta}{2}(\xi-\xi_0)+i\eta\right\}, \tag{13}$$

where η is a free parameter. For different values of η, (13) represents a new class of complex solutions for u with one free parameter. Equation (13) has real regular as well as singular (Ablowitz and Cornille 1979; Jaworski 1984) solutions also imbedded in it.

Defining $u=u_1+iu_2$, a coupled set of stationary equations for u_1 and u_2 are obtained from (10); they are given by,

$$\beta\frac{d^2u_1}{d\xi^2}=Mu_1-\frac{\alpha}{2}(u_1^2-u_2^2)$$

and

$$\beta\frac{d^2u_2}{d\xi^2}=(M-\alpha u_1)u_2. \tag{14}$$

These equations can be treated as the equations of motion in a two-dimensional potential $V(u_1,u_2)$. The potential V now becomes $V=V_1+iV_2$, where

$$V_1=-\tfrac{1}{2}M(u_1^2-u_2^2)+\tfrac{1}{6}\alpha(u_1^3-3u_1u_2^2),$$
$$V_2=-Mu_1u_2-\tfrac{1}{6}\alpha(u_2^3-3u_1^2u_2), \tag{15}$$

with V_1 and V_2 satisfying (for well-behaved $V(u)$) the Cauchy-Riemann conditions with respect to u_1 and u_2. Similarly, the Hamiltonian H becomes $H=H_1+iH_2$ where

$$H_1=\frac{1}{2}\beta\left\{\left(\frac{du_1}{d\xi}\right)^2-\left(\frac{du_2}{d\xi}\right)^2\right\}+V_1(u_1,u_2),$$
$$H_2=\beta\frac{du_1}{d\xi}\frac{du_2}{d\xi}+V_2(u_1,u_2). \tag{16}$$

Equation (14) can then be written as,

$$\beta\frac{d^2u_1}{d\xi^2}=-\frac{\partial V_1}{\partial u_1};\quad \beta\frac{d^2u_2}{d\xi^2}=+\frac{\partial V_1}{\partial u_2}, \tag{17}$$

which are identical with (2). Similarly, the equations of motion in terms of V_2 can be readily obtained from (17) on using the Cauchy-Riemann conditions. Since V_1 is not separable, as shown in the previous section, it is not possible to find a G of the type mentioned earlier. Nevertheless, the system of equations (17) is indeed integrable since one can easily show that H_1 and H_2 are independent integrals of motion and they are in involution. For the K-dV equation an additional integral constraint can be shown to be:

$$\frac{\sin^2(2\eta)}{u_2^2}\left\{u_1^2+\frac{\alpha}{3M}(u_1^2+u_2^2)^{3/2}\cos(2\eta)\right.$$
$$\left.-\left(\frac{\alpha}{6M}\right)^2(u_1^2+u_2^2)^2\sin^2(2\eta)\right\}=1-\sin^2(2\eta), \tag{18}$$

where the solution (13) has been used. Equation (18) is valid for $u_2 \neq 0$ i.e., for $\eta \neq 0$ or $2n\pi$. Note that the constraint (18) can be considered as describing a trajectory in the phase space. The existence of such a trajectory ensures bounded (in the present case, localized) solutions for u_1 and u_2.

The above analysis is equally applicable to the complex modified K-dV as well as the complex sine-Gordon equation. The details are very similar to that of K-dV equation (Buti *et al* 1984) and hence are omitted.

3.2 *Amplitude-modulated nonlinear Langmuir waves in plasmas*

Physical examples where the above discussion bears some relevance are provided by the set of governing equations (in the stationary frame) for the propagation of nonlinear Langmuir waves as well as the circularly-polarized electromagnetic waves. First, we consider Langmuir waves. The relevant governing equations are given by (Varma and Rao 1980; Rao and Varma 1982):

$$3\frac{d^2E}{d\xi^2} = (\lambda - 1)E + E\exp(\Phi - \tfrac{1}{4}E^2),$$

$$\frac{d^2\Phi}{d\xi^2} = -\frac{M}{(M^2 - 2\Phi)^{1/2}} + \exp(\Phi - \tfrac{1}{4}E^2), \tag{19}$$

where λ is a free parameter related to the nonlinear shift in the Langmuir wave frequency, $\xi = x - Mt$ is the stationary variable, M is the Mach number, and all the variables are suitably normalized. The Langmuir field $E(\xi)$ and the ion field $\Phi(\xi)$ are real; while $E(\xi)$ can have both signs, $\Phi(\xi)$ is always negative. In order to bring them to the canonical form, we make a change of variable, $E \to \sqrt{3/2}\,E$. Equations (13) then take the form,

$$\frac{d^2E}{d\xi^2} = -\frac{\partial V}{\partial E}, \quad \frac{d^2\Phi}{d\xi^2} = +\frac{\partial V}{\partial \Phi}, \tag{20}$$

where

$$V(E, \Phi) = M(M^2 - 2\Phi)^{1/2} + \tfrac{1}{6}(1 - \lambda)E^2 + \exp(\Phi - \tfrac{1}{6}E^2), \tag{21}$$

is the effective 'potential' that can be associated with a system of two degrees of freedom with (E, Φ) as the generalized coordinates and ξ as the 'time' variable. The Lagrangian L, for (20), is given by

$$L = \tfrac{1}{2}\{(dE/d\xi)^2 - (d\Phi/d\xi)^2\} - V, \tag{22}$$

where the 'kinetic energy' is, obviously, indefinite. The two conjugate momenta are given by

$$\Pi_{(E)} \equiv \partial L/\partial(dE/d\xi) = dE/d\xi,$$

$$\Pi_{(\Phi)} \equiv \partial L/\partial(dE/d\Phi) = -d\Phi/d\xi. \tag{23}$$

The Hamiltonian is, therefore, obtained as

$$H = \tfrac{1}{2}\{\Pi_{(E)}^2 - \Pi_{(\Phi)}^2\} + V(E, \Phi). \tag{24}$$

This Hamiltonian can be identified with the Hamiltonian H_1 of the previous example. However, unlike V_1, the potential $V(E, \Phi)$ does not satisfy the Laplace equation. In this sense, equations (20) are more general than (17). Since the Lagrangian L is independent (explicitly) of ξ, the Hamiltonian H given by (24) is an integral of motion. In fact, for localized solutions, it is simply, $H = 1 + M^2$. Despite the intense search, it has not been possible to obtain explicitly a second integral of motion for the system of (19). Also, in the absence of the standard Hamiltonian G with positive definite kinetic energy, care must be exercised in treating (20) on par with dynamical systems and in applying the methods developed for the latter.

3.3 *Nonlinear electromagnetic waves in plasmas*

We shall now consider briefly the case of nonlinear propagation of circularly polarized electromagnetic waves with modulated amplitude. The relevant governing equations are (Rao *et al* 1983)

$$\beta \frac{d^2\Psi}{d\xi^2} = \lambda\Psi + \frac{\Psi}{(1+\Psi^2)^{1/2}} \exp[\Phi - \beta(1+\Psi^2)^{1/2} + \beta],$$

$$\frac{d^2\Phi}{d\xi^2} = -\frac{M}{(M^2-2\Phi)^{1/2}} + \exp[\Phi - \beta(1+\Psi^2)^{1/2} + \beta], \tag{25}$$

where Ψ is the normalized amplitude of the wave electric field and Φ is the associated low-frequency potential. λ, β and M are parameters which are suitably normalized. Equations (25) can be derived from the Lagrangian,

$$L = \tfrac{1}{2}\{\beta^2(d\Psi/d\xi)^2 - (d\Phi/d\xi)^2\} - V, \tag{26}$$

where the effective potential V is now given by

$$V = -\tfrac{1}{2}\lambda\beta\Psi^2 - M(M^2-2\Phi)^{1/2} + \exp[\Phi - \beta(1+\Psi^2)^{1/2} + \beta]. \tag{27}$$

The corresponding Hamiltonian, which is an integral of motion is

$$H = \tfrac{1}{2}\beta^2(d\Psi/d\xi)^2 - \tfrac{1}{2}(d\Phi/d\xi)^2 + V. \tag{28}$$

After a simple redefinition of Ψ, (25) can be made similar to (20). As in the Langmuir wave case, the Hamiltonian (28) cannot be reduced to a form with positive definite kinetic energy. In addition, it is not clear whether these equations admit a second integral of motion which is exact. On the other hand, one can obtain integrals of motion in the form of a series for E or Ψ in terms of Φ (Varma and Rao 1980; Rao and Varma 1982; Rao *et al* 1983). With suitable orderings, these can be used to obtain explicity the approximate localized solutions of (19) as well as (25).

4. Extension of one-dimensional real dynamical systems to complex plane

It is worth pointing out that the features discussed in §3 are not peculiar to the nonlinear evolution equations only, but rather are generic in nature. They are present in two-dimensional systems generated from one-dimensional dynamical systems by making the dependent variable complex. For example, consider an one-dimensional conservative dynamical system governed by the Lagrangian $L = \frac{1}{2}\dot{q}^2 - V(q)$ where $V(q)$ is a regular function of q. The Hamiltonian for this system is simply, $H = \frac{1}{2}p^2 + V(q)$, where $p = \dot{q}$. From this, one can generate a two-dimensional system by making q a complex variable, say $q = q_1 + i q_2$. Clearly, if $p = p_1 + ip_2$, then (q_1, p_1) and (q_2, p_2) constitute the canonical variables for a system with two degrees of freedom (q_1, q_2). The potential V then becomes $V = V_1 + iV_2$ where $V_1(q_1, q_2)$ and $V_2(q_1, q_2)$ satisfy Cauchy-Riemann conditions with respect to q_1 and q_2. It is easy to verify that the equations of motion for q_1 and q_2 in terms of the potential V_1 are identical with the equations (2).

In the process of analytic continuation to the complex plane, the Lagrangian L becomes $L_1 + iL_2$, where $L_1 = \frac{1}{2}(\dot{q}_1^2 - \dot{q}_2^2) - V_1$ and $L_2 = \dot{q}_1\dot{q}_2 - V_2$. Similarly, for the Hamiltonian, we have $H = H_1 + iH_2$ where, in terms of the conjugate variables (q_1, p_1) and (q_2, p_2), H_1 and H_2 are given by $H_1 = \frac{1}{2}(p_1^2 - p_2^2) + V_1$ and $H_2 = p_1 p_2 + V_2$. One can easily show that both L_1 and L_2 (and similarly, H_1 and H_2) generate identical equations of motion for q_1 and q_2. Since H is an integral of motion, both H_1 and H_2 are also integrals of motion.

If H_1 and H_2 are written in terms of the momenta (corresponding to q_1 and q_2) obtained from L_1, then the Poisson bracket of H_1 and H_2 with respect to these new canonical variables vanishes identically. Exactly the same result follows for the momenta obtained from L_2 also.

As shown in the previous section, it is not possible, in general, to obtain a two-dimensional Hamiltonian $G(q_1, q_2, p_1, p_2)$ which has positive definite kinetic energy quadratic in velocity and is equivalent to the system described by H_1 (or H_2). Thus, it appears that the usual results of nonlinear dynamics like the KAM theorem etc may not be directly applicable to the systems governed by H_1 or H_2.

5. Conclusions

We have investigated some features of a class of two-dimensional systems with indefinite kinetic energy. Specifically, we have looked for the existence of an equivalent Hamiltonian which is quadratic in velocities and has positive definite kinetic energy. It is shown that, in general, it is not possible to obtain the equivalent Hamiltonian except for the case of separable potentials. However, this does not preclude the existence of non-separable, indefinite Hamiltonians that are integrable. These features are illustrated by taking some examples. By considering the case of the K-dV equation, we have shown that the coupled set of equations for real and imaginary parts yields precisely the Hamiltonians as mentioned above, one of which has indefinite kinetic energy. Such Hamiltonians can also be obtained by 'complexification' of real, one-dimensional dynamical systems. Finally, we have considered some examples from the field of plasma physics. It is shown that the governing equations for nonlinear Langmuir waves as well as electromagnetic waves can be treated as dynamical equations for a system with two-degrees of freedom, but, with indefinite kinetic energy. The present discussion is, however, limited to systems with kinetic energy quadratic in velocity.

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