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Supersymmetric classical mechanics*

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Abstract. The purpose of the paper is to construct a supersymmetric Lagrangian within the framework of classical mechanics which would be regarded as a candidate for passage to supersymmetric quantum mechanics.

Keywords. Supersymmetry; classical mechanics; generalized Poisson bracket; equations of motion; canonical transformation; classical Lagrangian.

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1. Introduction

Supersymmetric quantum mechanics was originally proposed by Witten 1981 to study the breakdown of supersymmetry by non-perturbative effects. Cooper and Freedman (1982; see also Salmonson and Van Holten 1981) explored non-perturbative dynamical breaking of supersymmetry in various supersymmetric quantum-mechanical models and in addition they also considered the formulation of a supersymmetric Lagrangian at the classical level. To derive this they started from a supersymmetric field theory based on the superfield formalism of Salam and Strathdee (1975) by letting the dimension to become 1. The corresponding Hamiltonian for the supersymmetric quantum theory takes the form

$$\mathscr{H} = \tfrac{1}{2}p^2 + \tfrac{1}{2}W^2(q) - \tfrac{1}{2}[\psi^\dagger, \psi]\frac{\mathrm{d}W}{\mathrm{d}q}, \tag{1}$$

where $[q, p] = i\hbar$ and $\{\psi^\dagger, \psi\} = 1$. Here q, p are the conventional bosonic operators while $\psi^\dagger$ and ψ are the fermionic operators. The combinations $(p - iW)\psi$ and $(p + iW)\psi^\dagger$ are called supersymmetric charges designated by Q and $Q^\dagger$ respectively. The Q and $Q^\dagger$ satisfy the properties, namely $\{Q^\dagger, Q\} = 2H$ and furthermore Q and $Q^\dagger$ commute with H.

The purpose of the paper is to construct a supersymmetric classical Lagrangian which in the quantum limit leads to the same Hamiltonian as given in (1). Our method differs from that adopted by Cooper and Freedman (1982) and is based on the canonical method through the formulation of Poisson brackets, etc.

In §2 we define a suitably generalized Poisson bracket within the framework of classical mechanics involving bosonic and fermionic variables (Biswas 1985) which can

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The authors felicitate Prof. D S Kothari on his eightieth birthday and dedicate this paper to him on this occasion.

be regarded as a proper candidate for passage from classical to quantum supersymmetric mechanics.

In § 3 we start with Lagrange's equations of motion and arrive at the generalized Hamilton's equations through a generalized definition of Legendre transformation and in §4 we introduce canonical transformation in the context of both bosonic and fermionic coordinates and momenta and prove the invariance of the Poisson bracket relations under this transformation. In the last section we construct the classical Lagrangian which in the quantum limit renders the Hamiltonian given by (1).

2. Generalized Poisson bracket relations and their algebraic properties

In this section we shall define the generalized Poisson bracket by examining the classical limit of the generalized Dirac bracket of two operators.

We consider here a dynamical system whose states can be classified into even and odd eigenstates of the permutation operator $\mathscr{P}$ which commutes with the system Hamiltonian $\mathscr{H}$, i.e.

$$\mathscr{P}^{-1}\mathscr{H}\mathscr{P}=\mathscr{H}. \tag{2}$$

By definition $\mathscr{P}$ effects the interchange of particle coordinates for any pair of particles. Consequently the quantum-mechanical operators $\hat{\Omega}$ are also even or odd under $\mathscr{P}$:

$$\mathscr{P}^{-1}\hat{\Omega}\,\mathscr{P}=(-1)^{\pi(\hat{\Omega})}\hat{\Omega}. \tag{3}$$

Here $\pi(\hat{\Omega})=0$ for an even operator $\hat{\Omega}$ which takes even (odd) states into even (odd) states, and $\pi(\hat{\Omega})=1$ for an odd operator which takes odd (even) states into even (odd) states. Clearly

$$\pi(\hat{\Omega}_1\hat{\Omega}_2)=\pi(\hat{\Omega}_1)+\pi(\hat{\Omega}_2). \tag{4}$$

Thus $\hat{\Omega}_1\hat{\Omega}_2$ is odd only when one of them is odd.

We denote by

$$[\hat{A},\hat{B}]$$

the generalized Dirac bracket of two arbitrary operators $\hat{A}$ and $\hat{B}$. This bracket is defined so that it satisfies the following algebraic properties relating to (anti-) symmetry under interchange, generalized chain rule, linearity and Jacobi identity:

$$[\hat{A},\hat{B}]=-(-1)^{ab}[\hat{B},\hat{A}], \tag{5a}$$

$$[\hat{A},\hat{B}\hat{C}]=[\hat{A},\hat{B}]\hat{C}+(-1)^{ab}\hat{B}[\hat{A},\hat{C}], \tag{5b}$$

$$[\hat{A},\hat{B}+\hat{C}]=[\hat{A},\hat{B}]+[\hat{A},\hat{C}], \tag{5c}$$

$$[\hat{A}[\hat{B}\hat{C}]]+(-1)^{a(b+c)}[\hat{B}[\hat{C}\hat{A}]]+(-1)^{c(a+b)}[\hat{C}[\hat{A}\hat{B}]=0, \tag{5d}$$

where

$$\pi(\hat{A})=a,\quad \pi(\hat{B})=b,\quad \pi(\hat{C})=c$$

for arbitrary operators $\hat{A}$, $\hat{B}$ and $\hat{C}$. The above algebraic properties reduce to the wellknown properties of the ordinary Dirac bracket when $a = b = c = 0$. We now establish that these generalized algebraic properties lead us to the identification

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - (-1)^{ab}\hat{B}\hat{A}, \tag{6}$$

thus showing that $[\hat{A}, \hat{B}]$ becomes the anti-commutator of $\hat{A}$ and $\hat{B}$ when they are both odd operators, though otherwise it remains equal to their commutator.

To see this, let us expand $[\hat{A}\hat{B}, \hat{C}\hat{D}]$. This may be done in two equivalent ways. In the first instance the application of the chain rule on $\hat{C}\hat{D}$ is followed by its application to $\hat{A}\hat{B}$. The result is

$$[\hat{A}\hat{B}, \hat{C}\hat{D}] = \hat{A}[\hat{B}, \hat{C}]\hat{D} + (-1)^{bc}[\hat{A}, \hat{C}]\hat{B}\hat{D} + (-1)^{c(a+b)}\hat{C}\hat{A}[\hat{B}\hat{D}] + (-1)^{ca+cb+bd}\hat{C}[\hat{A}, \hat{D}]\hat{B}.$$

Next, the application of the chain rule in the reverse order yields

$$[\hat{A}\hat{B}, \hat{C}\hat{D}] = \hat{A}[\hat{B}, \hat{C}]\hat{D} + (-1)^{bc}\hat{A}\hat{C}[\hat{B}, \hat{D}] + (-1)^{b(c+d)}[\hat{A}, \hat{C}]\hat{D}\hat{B} + (-1)^{ca+bc+bd}\hat{C}[\hat{A}, \hat{D}]\hat{B}.$$

On equating the two equivalent expressions we get

$$[\hat{A}, \hat{C}](\hat{B}\hat{D} - (-1)^{bd}\hat{D}\hat{B}) = (\hat{A}\hat{C} - (-1)^{ac}\hat{C}\hat{A})[\hat{B}, \hat{D}]. \tag{7}$$

Since this result is true for arbitrary operators, it follows that $[\hat{\Omega}_1, \hat{\Omega}_2]$ is proportional to

$$\hat{\Omega}_1\hat{\Omega}_2 - (-1)^{\pi(\hat{\Omega}_1)\pi(\hat{\Omega}_2)}\hat{\Omega}_2\hat{\Omega}_1.$$

The proportionality constant has to be unity if the generalized bracket must reduce to the ordinary bracket when $\hat{\Omega}_1$ and $\hat{\Omega}_2$ are both even operators, thus leading us to (6). It is easy to verify that the bracket expression as given by (6) is also consistent with (5c) and (5d).

In order to be able to evaluate generalized Dirac brackets using their algebraic properties given by (5), we postulate the following relations for the fundamental brackets which are a generalization of the corresponding expressions in ordinary quantum mechanics:

$$[\hat{\alpha}_i, \hat{\alpha}_j] = [\hat{\beta}_i, \hat{\beta}_j] = 0, \tag{8a}$$

$$[\hat{\alpha}_i, \hat{\beta}_j] = i\hbar\delta_{ij}, \quad i, j = 1, 2 \ldots, N. \tag{8b}$$

Here $\hat{\alpha}$ denote the generalized coordinate operators and $\hat{\beta}$ the corresponding canonical momenta. They can be separated into even and odd operators:

$$\hat{\alpha}_i \equiv \hat{q}_i, \quad \hat{\beta}_i \equiv \hat{p}_i, \quad i = 1, 2 \ldots n, \tag{9a}$$

and

$$\hat{\alpha}_{n+s} \equiv \hat{\theta}_s, \quad \hat{\beta}_{n+s} \equiv \hat{\pi}_s, \quad s = 1, 2 \ldots N - n. \tag{9b}$$

Thus the first n coordinates and the corresponding momenta are even in nature and the remaining are odd:

$$\pi(\hat{q}_i) = \pi(\hat{p}_i) = 0, \quad i = 1, 2 \ldots n, \tag{10a}$$

$$\pi(\hat{\theta}_s) = \pi(\hat{\pi}_s) = 1, \quad s = 1, 2 \ldots N-n. \tag{10b}$$

Having thus generalized the structure and fundamental bracket relations of ordinary quantum mechanics, we now proceed to take the classical limit

$$\hbar \to 0.$$

As we now show this yields the generalization of the correspondence principle. In the classical limit the quantum-mechanical operator $\hat{\Omega}$ is replaced by its classical counterpart Ω (denoted here by the same symbol but without the hat) and the generalized Dirac bracket $[\hat{\Omega}_1, \hat{\Omega}_2]$ of any two operators $\hat{\Omega}_1$ and $\hat{\Omega}_2$ reduces in some sense (made precise below) to the generalized Poisson bracket of Ω_1 and Ω_2 denoted by $\{\Omega_1, \Omega_2\}$. To be precise, let us examine (7) which in the classical limit must read for arbitrary operators as

$$\frac{\hat{B}\hat{D} - (-1)^{bd}\hat{D}\hat{B}}{\{B, D\}} = \frac{\hat{A}\hat{C} - (-1)^{ac}\hat{C}\hat{A}}{\{A, C\}}.$$

The result of this division, which is independent of the choice of the pair of operators, is readily seen to be $i\hbar$ by choosing them to be even. It is concluded that in the classical limit the generalized Dirac bracket $[\hat{\Omega}_1, \hat{\Omega}_2]$ approaches zero in such a manner that its ratio to $i\hbar$ is finite and equal to the generalized Poisson bracket $\{\Omega_1, \Omega_2\}$. Hence we have the generalized correspondence principle:

$$\mathop{\mathscr{L}t}_{\hbar \to 0} \frac{[\hat{\Omega}_1, \hat{\Omega}_2]}{i\hbar} = \{\Omega_1, \Omega_2\}. \tag{11}$$

By virtue of this important result, it follows that the generalized Poisson bracket also enjoys identically the same algebraic properties as the generalized Dirac bracket, i.e.,

$$\{A, B\} = -(-1)^{ab}\{B, A\}, \tag{12a}$$

$$\{A, BC\} = \{A, B\}C + (-1)^{ab} B\{A, C\}, \tag{12b}$$

$$\{A, B+C\} = \{A, B\} + \{A, C\}, \tag{12c}$$

$$\{A\{BC\}\} + (-1)^{a(b+c)}\{B\{CA\}\} + (-1)^{c(a+b)}\{C\{A, B\}\} = 0. \tag{12d}$$

Here $\quad a = \pi(A) = \pi(\hat{A})$ etc.

Furthermore the fundamental generalized Poisson bracket relations, which are derived from (8a) and (8b), are required to be

$$\{\alpha_i, \alpha_j\} = \{\beta_i, \beta_j\} = 0, \tag{13a}$$

$$\{\alpha_i, \beta_j\} = \delta_{ij}. \tag{13b}$$

If we let $\hbar \to 0$ in the same equations (8a) and (8b), and utilize the connection between the generalized Dirac bracket and the (anti-) commutator, we observe that the odd coordinates θ and odd momenta π anti-commute among themselves and commute with the even coordinates q and even momenta p (which commute among themselves), i.e.

$$\alpha_i\alpha_j - (-1)^{\pi(\alpha_i)\pi(\alpha_j)}\alpha_j\alpha_i = 0 = \beta_i\beta_j - (-1)^{\pi(\beta_i)\pi(\beta_j)}\beta_j\beta_i, \tag{14a}$$

$$\alpha_i\beta_j - (-1)^{\pi(\alpha_i)\pi(\beta_j)}\beta_j\alpha_i = 0. \tag{14b}$$

Finally let us motivate the construction of a suitable expression for the generalized Poisson bracket of a given pair of classical dynamical variables, which involves taking their first order derivatives with respect to such (anti-) commuting coordinates α and momenta β and which is consistent with the properties (12a)–(12d) and the relations (13a) and (13b).

Since α and β are not all commuting we need to distinguish between left and right differentiation. To explain our notation, consider the differential $d\Omega$ of a dynamical variable Ω which is a function of α, β and time t:

$$d\Omega(\alpha, \beta, t) = dt\partial_t\Omega + \sum_{i=1}^{N} \Omega_{,\alpha_i}d\alpha_i + d\beta_i\,\partial_{\beta_i}\Omega. \tag{15}$$

Here $\Omega_{,\alpha}(\partial_\beta\Omega)$ denotes a right (left) partial derivative of Ω with respect to α (β).

It immediately follows that

$$\partial_\alpha\Omega = (-1)^{\pi(\alpha)(\pi(\alpha)+\pi(\Omega))}\Omega_{,\alpha}, \tag{16}$$

since $\quad \Omega_{,\alpha}d\alpha = d\alpha\,\partial_\alpha\Omega.$

The chain rule for left differentiation is:

$$\partial_\alpha(\Omega_1\Omega_2) = (\partial_\alpha\Omega_1)\Omega_2 + (-1)^{\pi(\alpha)\pi(\Omega_1)}\Omega_1\,\partial_\alpha\Omega_2, \tag{17}$$

and that for right differentiation is:

$$(\Omega_1\Omega_2)_{,\alpha} = \Omega_1\Omega_{2,\alpha} + (-1)^{\pi(\Omega_2)\pi(\alpha)}\Omega_{1,\alpha}\Omega_2. \tag{18}$$

These chain rules are required to ensure (12b). In ordinary classical mechanics, the Poisson bracket of any pair of dynamical variables $E_1(q, p, t)$ and $E_2(q, p, t)$ is defined as

$$\{E_1, E_2\} = \sum_{i=1}^{n}\left(\frac{\partial E_1}{\partial q_i}\frac{\partial E_2}{\partial p_i} - \frac{\partial E_2}{\partial q_i}\frac{\partial E_1}{\partial p_i}\right). \tag{19}$$

Here E_1, E_2 and q, p being all even in nature, the right and left derivatives are identical. The correct generalization of (19) to the generalized Poisson bracket of $\Omega_1(\alpha, \beta, t)$ and $\Omega_2(\alpha, \beta, t)$ which is consistent with (12a) and (12b) is given by either of the following two expressions. Either

$$\{\Omega_1, \Omega_2\} = \sum_{i=1}^{N}(\Omega_{1,\alpha_i}\partial_{\beta_i}\Omega_2 - (-1)^{\pi(\Omega_1)\pi(\Omega_2)}\Omega_{2,\alpha_i}\partial_{\beta_i}\Omega_1), \tag{20a}$$

or,

$$\{\Omega_1, \Omega_2\} = \sum_{i=1}^{N}(-1)^{\pi(\alpha_i)}(\Omega_{1,\alpha_i}\partial_{\beta_i}\Omega_2 - (-1)^{\pi(\Omega_1)\pi(\Omega_2)}\Omega_{2,\alpha_i}\partial_{\beta_i}\Omega_1) \tag{20b}$$

Note that $\pi(\alpha_i) = \pi(\beta_i) = \pi(\alpha_i)\pi(\alpha_i) = \pi(\alpha_i)\pi(\beta_i)$. The linearity property (12c) goes through in either case. It is also straightforward though tedious to verify that in both cases the generalized Jacobi identity (12d) follows. However (20b) implies

$$\{\alpha_i, \beta_j\} = (-1)^{\pi(\alpha_i)}\delta_{ij},$$

thus resulting in inconsistency with the relation imposed by (13b). Thus we are left with (20a) as our definition of the generalized Poisson bracket which is consistent with all the algebraic properties given by (12a)–(12d) and the relations given by (13a)–(13b). This can be regarded as a suitable candidate for transition from generalized classical mechanics to generalized quantum mechanics via the generalized correspondence principle given by (11).

3. Equations of motion

In this section we obtain the equations of motion in the generalized Poisson bracket form.

Let us start with Hamilton's principle which requires taking the variation of the definite integral of the Lagrangian $\mathscr{L}(\alpha, \dot{\alpha}, t)$ which depends on the (anti-) commuting coordinates α_i and the corresponding velocities $\dot{\alpha}_i$ besides perhaps explicitly depending on time t.

$$\delta\mathscr{L} = \sum_{i=1}^{N} \mathscr{L}_{,\alpha_i}\delta_{\alpha_i} + \mathscr{L}_{,\dot{\alpha}_i}\delta_{\dot{\alpha}_i}\,. \tag{21}$$

Here δ denotes the infinitesimal variation from the actual path to co-terminus neighbouring path. From (21) and Hamilton's principle follow Lagrange's equations of motion, namely,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathscr{L}_{,\dot{\alpha}_i}) = \mathscr{L}_{,\alpha_i}. \tag{22a}$$

Note that equivalently we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\partial_{\dot{\alpha}_i}\mathscr{L}) = \partial_{\alpha_i}\mathscr{L}. \tag{22b}$$

The Hamiltonian $H(\alpha, \beta, t)$ is related to the Lagrangian $\mathscr{L}(\alpha, \dot{\alpha}, t)$ through a Legendre transformation which may be effected in either of the following two ways. Either

$$\mathscr{H}(\alpha, \beta, t) = -\mathscr{L}(\alpha, \dot{\alpha}, t) + \sum_{i=1}^{N} \beta_i\dot{\alpha}_i, \tag{23a}$$

or

$$\mathscr{H}(\alpha, \beta, t) = -\mathscr{L}(\alpha, \dot{\alpha}, t) + \sum_{i=1}^{N} \dot{\alpha}_i\beta_i. \tag{23b}$$

Correspondingly this enforces either of the following two identifications for momenta in terms of velocities when we take the variation of (23a) and (23b) and demand the vanishing of the coefficient of $\delta\dot{\alpha}_i$. Either

$$\beta_i = \mathscr{L}_{,\dot{\alpha}_i}, \tag{24a}$$

or

$$\beta_i = \partial_{\dot{\alpha}_i}\mathscr{L}. \tag{24b}$$

Correspondingly the resulting Hamilton's equations of motion assume either of the following two forms. Either

$$\dot{\alpha}_i = \partial_{\beta_i}\mathscr{H}, \quad \dot{\beta}_i = -\mathscr{H}_{,\alpha_i}, \tag{25a}$$

or

$$\dot{\alpha}_i = \mathscr{H}_{,\beta_i}, \quad \dot{\beta}_i = -\partial_{\alpha_i}\mathscr{H}. \tag{25b}$$

Consequently the equation of motion for an arbitrary dynamical variable $\Omega(\alpha, \beta, t)$

$$\dot{\Omega} = \partial_t\Omega + \sum_{i=1}^{N} \Omega_{,\alpha_i}\dot{\alpha}_i + \dot{\beta}_i\,\partial_{\beta_i}\Omega$$

assumes the Poisson bracket form given by

$$\dot{\Omega} = \partial_t\Omega + \{\Omega, \mathscr{H}\}$$

with the corresponding Poisson bracket expression defined by either (20a) or (20b). Recall that $\pi(\mathscr{H}) = 0$.

Since we have already argued in the last section that the proper definition of Poisson bracket is given by (20a), it follows that the appropriate Legendre transformation is given by (23a) and not (23b). Hence the proper definition of momenta in terms of velocities is given by (24a). It is understood that these equations can be solved for velocities in terms of momenta in carrying out Legendre transformation (23a). The correct generalization of Hamilton's equations of motion is given by (25a).

4. Canonical transformations and invariance of generalized Poisson bracket

In this section we shall establish the invariance of Poisson bracket defined by (20a) under a canonical transformation. This transformation is defined as a reparametrization of the (anti-) commuting coordinate momenta

$$\gamma = \gamma(\alpha, \beta, t), \tag{26a}$$

$$\delta = \delta(\alpha, \beta, t), \tag{26b}$$

such that the equations of motion expressed in terms of the transformed variables

$$\partial_{\delta_i}\mathscr{H}' = \dot{\gamma}_i, \tag{27a}$$

$$\mathscr{H}'_{,\gamma_i} = -\dot{\delta}_i, \tag{27b}$$

are identical in their content with the equations of motion (25a) expressed in terms of the old variables (α, β). The transformed Hamiltonian $H'(\gamma, \delta, t)$ differs from the old Hamiltonian $H(\alpha, \beta, t)$ by the partial time derivative of the generating function of the canonical transformation which (besides depending explicitly on time) can be chosen to be a function of old coordinates α and new coordinates γ, or of old momenta β and new coordinates γ, or of old coordinates α and new momenta δ, or of old momenta β and new momenta δ. These varieties of the generating function, denoted by $F_1(\alpha, \gamma, t)$, $F_2(\beta, \gamma, t)$, $F_3(\alpha, \delta, t)$ and $F_4(\beta, \delta, t)$ respectively, are interconnected by Legendre transformations:

$$F_2(\beta,\gamma,t) = F_1(\alpha,\gamma,t) - \sum_{i=1}^{N} \beta_i\alpha_i, \tag{28a}$$

$$F_3(\alpha,\delta,t) = F_1(\alpha,\gamma,t) + \sum_{i=1}^{N} \delta_i\gamma_i, \tag{28b}$$

$$F_4(\beta,\delta,t) = F_1(\alpha,\gamma,t) - \sum_{i=1}^{N} \beta_i\alpha_i + \sum_{i=1}^{N} \delta_i\gamma_i. \tag{28c}$$

In these equations it is understood that the elimination of $\alpha(\beta)$ in favour of $\gamma(\delta)$ is done by the following set of simultaneous equations

$$F_{1,\alpha_i} = \beta_i, \quad F_{1,\gamma_i} = -\delta_i, \quad i = 1, 2 \ldots N. \tag{29}$$

The results given by (29) follow from the fact that the old Lagrangian and the transformed Lagrangian differ by the total time derivative of F_1, i.e.,

$$\frac{\mathrm{d}F_1}{\mathrm{d}t} = \left(-\mathscr{H} + \sum_{i=1}^{N} \beta_i\dot{\alpha}_i\right) - \left(-\mathscr{H}' + \sum_{i=1}^{N} \delta_i\dot{\gamma}_i\right). \tag{30}$$

On substituting the definitions (28a)–(28c) into (30) we also get

$$\partial_{\beta_i}F_2 = -\alpha_i, \qquad F_{2,\gamma_i} = -\delta_i, \tag{31a}$$

$$F_{3,\alpha_i} = \beta_i, \qquad \partial_{\delta_i}F_2 = \gamma_i, \tag{31b}$$

$$\partial_{\beta_i}F_4 = -\alpha_i, \qquad \partial_{\delta_i}F_4 = \gamma_i, \quad i = 1, 2 \ldots N. \tag{31c}$$

From (29) and (31a)–(31c) follow immediately the generalized 'Maxwell relations'

$$\beta_{i,\gamma_j} = -(-1)^{\pi(\gamma_j)\pi(\alpha_i)}\delta_{j,\alpha_i}, \tag{32a}$$

$$\alpha_{i,\gamma_j} = \partial_{\beta_i}\delta_j, \tag{32b}$$

$$\partial_{\delta_j}\beta_i = \gamma_{j,\alpha_i}, \tag{32c}$$

$$\partial_{\delta_j}\alpha_i = -(-1)^{\pi(\delta_j)\pi(\beta_i)}\partial_{\beta_i}\gamma_j, \quad i,j = 1, 2 \ldots N. \tag{32d}$$

Recall that $\pi(\alpha_i) = \pi(\beta_i)$ and also $\pi(\gamma_i) = \pi(\delta_i)$. These relations enable us to show the invariance of fundamental brackets. To see this let us evaluate $\{\gamma_i, \gamma_j\}$ in the α-β basis:

$$\{\gamma_i, \gamma_j\}_{\alpha\beta} \equiv \sum_{k=1}^{N} (\gamma_{i,\alpha_k})(\partial_{\beta_k}\gamma_j) - (-1)^{\pi(\gamma_i)\pi(\gamma_j)}(\gamma_{j,\alpha_k})(\partial_{\beta_k}\gamma_i)$$

$$= \sum_{k=1}^{N} (\partial_{\delta_i}\beta_k)(\partial_{\beta_k}\gamma_j) + (-1)^{\pi(\gamma_i)\pi(\gamma_j)+\pi(\delta_i)\pi(\beta_k)}(\gamma_{j,\alpha_k})(\partial_{\delta_i}\alpha_k).$$

In arriving at the last equality we have used (32c) and (32d). It is easy to verify that

$$(\gamma_{j,\alpha_k})(\partial_{\delta_i}\alpha_k) = (-1)^{\pi(\delta_i)\pi(\alpha_k)+\pi(\delta_i)\pi(\gamma_j)}(\partial_{\delta_i}\alpha_k)(\partial_{\alpha_k}\gamma_j).$$

So we immediately get

$$\{\gamma_i, \gamma_j\}_{\alpha\beta} = \partial_{\delta_i}\gamma_j.$$

Hence

$$\{\gamma_i, \gamma_j\}_{\alpha\beta} = 0. \tag{33a}$$

Similarly we can show that

$$\{\gamma_i, \delta_j\}_{\alpha\beta} = \partial_{\delta_i}\delta_j = \delta_{ij}, \tag{33b}$$

$$\{\delta_i, \delta_j\}_{\alpha\beta} = -(-1)^{\pi(\delta_i)\pi(\delta_j)}\partial_{\gamma_i}\delta_j = 0. \tag{33c}$$

In view of the invariance of fundamental bracket relations shown above, invariance of an arbitrary Poisson bracket $\{A, B\}$ under canonical transformations follows: It is straightforward to verify that

$$\begin{aligned}
\{A, B\}_{\alpha\beta} &\equiv \sum_{i=1}^{N} A_{,\alpha_i}\partial_{\beta_i}B - (-1)^{ab}B_{,\alpha_i}\partial_{\beta_i}A \\
&= \sum_{j=1}^{N}\sum_{k=1}^{N} A_{,\gamma_j}\{\gamma_j, \gamma_k\}_{\alpha\beta}\partial_{\gamma_k}B + A_{,\delta_j}\{\delta_j, \gamma_k\}_{\alpha\beta}\partial_{\gamma_k}B \\
&\quad + A_{,\gamma_j}\{\gamma_j, \delta_k\}_{\alpha\beta}\partial_{\delta_k}B + A_{,\delta_j}\{\delta_j, \delta_k\}_{\alpha\beta}\partial_{\delta_k}B \\
&= \sum A_{,\gamma_j}\partial_{\delta_j}B - (-1)^{ab}B_{,\gamma_j}\partial_{\delta_j}A \\
&\equiv \{A, B\}_{\gamma\delta}.
\end{aligned}$$

In this way we have demonstrated that the generalized Poisson bracket defined in §2 may be evaluated with respect to any convenient canonical set of coordinates and conjugate momenta. Thus this bracket enjoys all the desired properties.

5. Construction of the classical Lagrangian

In this section we shall construct the Lagrangian for supersymmetric classical mechanics which is the classical limit of the Lagrangian described in §1. This operator Lagrangian is

$$\hat{\mathscr{L}} = \tfrac{1}{2}\dot{\hat{q}}^2 + i(\hat{\Psi}^{+}\partial_t\hat{\Psi} - (\partial_t\hat{\Psi})^{\dagger}\hat{\Psi}) - \hat{V}_s \tag{34a}$$

$$\hat{V}_s = \tfrac{1}{2}\left(W^2(\hat{q}) + [\hat{\Psi}, \hat{\Psi}^{\dagger}]\frac{\mathrm{d}W}{\mathrm{d}\hat{q}}\right). \tag{34b}$$

The momentum conjugate to $\hat{\Psi}$ is given by

$$\hat{\pi} = i\hat{\Psi}^{\dagger}. \tag{35}$$

Substituting (35) into (34b), $\hat{V}_s$ assumes the form

$$\hat{V}_s = \tfrac{1}{2}\left(W^2 - i[\hat{\Psi}, \hat{\pi}]\frac{\mathrm{d}W}{\mathrm{d}\hat{q}}\right). \tag{36}$$

In the classical limit, $\hat{\Psi}$ and $\hat{\pi}$ anticommute and (36) reduces to

$$V_s = \tfrac{1}{2}\left(W^2 - 2i\Psi\pi \frac{\mathrm{d}W}{\mathrm{d}q}\right). \tag{37}$$

In the following we present the Poisson bracket derivation of (37).

The Hamiltonian for supersymmetric classical mechanics is given by

$$\mathscr{H} = \tfrac{1}{2}\{Q, \bar{Q}\} \tag{38a}$$

with

$$\{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0. \tag{38b}$$

Here $Q(\bar{Q})$ is the classical limit of the odd 'super-charge' operator which we now show is conserved. Utilizing the equation of motion in the Poisson bracket form:

$$\begin{aligned}\dot{Q} &= \{Q, \mathscr{H}\}\\ &= \tfrac{1}{2}\{Q, \{Q, \bar{Q}\}\},\end{aligned}$$

using (38a). By virtue of the Jacobi identity this expression reduces to

$$-\tfrac{1}{2}\{Q, \{Q, \bar{Q}\}\}$$

using (38b). Note that Q and $\bar{Q}$ are odd in nature. Thus

$$\dot{Q} = -\dot{Q}$$

implying that $\dot{Q}$ vanishes. Similarly the structure given by (38a) and (38b) ensures the conservation of $\bar{Q}$. So we have

$$\{Q, \mathscr{H}\} = \{\bar{Q}, \mathscr{H}\} = 0. \tag{39}$$

The expression for the classical potential given by (37) follows immediately from (38a) if the following identification for Q and $\bar{Q}$ in terms of coordinates q, θ and momenta p, π is made.

$$Q = A\theta, \tag{40a}$$

$$\bar{Q} = A^*\pi, \tag{40b}$$

$$A = p - iW(q). \tag{40c}$$

In the notation already explained in §2, θ and π are anticommuting coordinates and momenta. Using the algebraic properties of the generalized Poisson bracket, it is easy to verify that (40a)–(40c) are consistent with (38b) and that the explicit expression for $\mathscr{H}$ becomes

$$\begin{aligned}\mathscr{H} &= |A|^2\{\theta, \pi\} + \theta\pi\{A, A^*\}\\ &= \tfrac{1}{2}\left[p^2 + W^2 - 2i\theta\pi\frac{\mathrm{d}W}{\mathrm{d}q}\right].\end{aligned}$$

The potential that corresponds to this Hamiltonian is given precisely by (37). Thus we have obtained the supersymmetric classical Lagrangian which corresponds to the supersymmetric quantum mechanics.

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