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The three faces of Maxwell's equations

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Abstract. In dealing with electromagnetic phenomena and in particular the phenomena of optics, despite the recognition of the quanta of light people tend to talk of the amplitudes and field strengths, as if the electromagnetic field were a classical field. For example we measure the wavelength of light by studying interference fringes. In this paper we study the interrelationship of three ways of looking at the problem: in terms of classical wave fields, wave function of the photon; and the quantized wave field. The comparison and contrasts of these three modes of description are carefully analyzed in this paper. The ways in which these different modes complement our intuition and insight are also discussed.

Keywords. Maxwell's equations; photon wavefunction; coherent states; quantized Maxwell field.

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1. Introduction

The concept of the field as an autonomous dynamical entity independent of ponderable matter, first intuitively realized by Michael Faraday, found precise expression at the hands of James Clerk Maxwell. With the advent of the equations of classical electromagnetism, it was possible to endow the field with the dynamical attributes of energy and momentum; and electromagnetic radiation could be seen as the direct expression of its independent degrees of freedom.

Maxwell's electromagnetic theory has served as the window through which both special relativity and quantum theory were first seen. These were later realized to be general features of physical phenomena. While the special theory of relativity springs from a physical understanding of the invariance group of Maxwell's equations, and its clash with Galilean-Newtonian relativity, quantum theory arose from a study of the properties of black body radiation which indicated a failure of classical statistical mechanics when combined with Maxwell's theory. Thus the former has its origins in a property of the Maxwell equations, and the latter in a breakdown of its classical interpretation.

The existence of the quantum of action was thus first discerned by Planck via the statistical properties of radiation in 1900, and it was soon put to use by Einstein in 1905 to arrive at the concept of quanta of radiation. Its relevance for matter came later, through Einstein's inadequate theory of specific heats in 1907 followed by Bohr's

The authors felicitate Prof. D S Kothari on his eightieth birthday and dedicate this paper to him on this occasion.

theory of the atom in 1913 and Debye's theory of specific heats in 1916. This order was however reversed in the development of a mathematically cogent quantum mechanics, it being first achieved for matter by Heisenberg and Schrödinger in 1925 and 1926 respectively, and only later for radiation by Dirac in 1927. There are good physical reasons that account for this situation. For many purposes it is adequate to treat the mechanics of matter in a non-relativistic manner, when in fact the number of particles can be regarded as a conserved quantity. It then happens that in spite of the special puzzling features of quantum mechanics, it is possible to visualize to some degree what happens to matter in terms of ordinary space and time, by using the Schrödinger wave function. Thus localization in space remains meaningful for quantized non-relativistic matter. Electromagnetic radiation is however intrinsically relativistic on account of the masslessness of the photon; furthermore the existence of polarization makes localization in space an ill-defined concept for photons. Connected with the masslessness of the photon we have gauge invariance of the electromagnetic field. It is the need to contend with both gauge invariance and relativity in the case of radiation that makes its quantum theory, involving infinitely many degrees of freedom, more subtle than in the case of non-relativistic matter.

A related comment on the physical side is perhaps relevant. Einstein's conception of 1905 took an exceptionally long time to be generally accepted, notable among its early opponents being Planck and Bohr. It was only sometime after the discovery of the Compton effect in 1923 that the reality of radiation quanta won widespread acceptance. The original name of needle quanta (or rather its German equivalent) was replaced by the term photon by G N Lewis as late as 1926.

Maxwell's equations display considerable beauty of form and structure when written in the relativistic notation natural to them. Their gauge invariance properties have inspired generalization to other symmetries of great importance. When following through the programme of quantization and moving over to the photon picture, there is a tendency to lose sight of the starting point of the whole development, namely the classical Maxwell equations and the space of their solutions. Common experience suggests that there are several aspects here that are not as well and widely understood as one often assumes to be the case. Our aim in this essay is to clarify some of these points and to stress the continuing relevance of the classical Maxwell equations and their solutions in understanding the quantum mechanics of single photons as well as of the entire field.

2. Maxwell's equations and the classical manifold $\mathcal{M}$

Classical electrodynamics deal with real electric and magnetic fields $\mathbf{E}(x)$, $\mathbf{B}(x)$ obeying the vacuum field equations

$$\partial_0 \mathbf{E} = \nabla \times \mathbf{B}, \quad \partial_0 \mathbf{B} = -\nabla \times \mathbf{E},$$
$$\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0. \tag{1}$$

(We deal only with the source-free Maxwell equations. Our metric is $g_{00} = -1$, with $x^0 = ct$ and $\partial_\mu = \partial/\partial x^\mu$. Factors of c and $\hbar$ are explicitly retained wherever they belong. Much of our study might be termed 'kinematics', but it is wise to remember the old adage 'The dynamics of today is the kinematics of tomorrow'). For such fields a conserved real energy-momentum four-vector P^μ (all components having the dimensions of energy) is defined by

$$P^0 = \frac{1}{8\pi}\int d^3\mathbf{x}(\mathbf{E}(x)^2 + \mathbf{B}(x)^2),$$
$$\mathbf{P} = \frac{1}{4\pi}\int d^3\mathbf{x}\,\mathbf{E}(x)\times\mathbf{B}(x), \tag{2}$$

the spatial integrations being carried out at any fixed time. (For simplicity we omit reference to the angular momentum). On account of the wave equation, both **E** and **B** separate in a Lorentz-invariant way into positive and negative frequency parts. Developments in statistical optics (Born and Wolf 1980) have shown the usefulness of dealing directly with the former, which are called analytic signals. We shall follow this practice and hereafter use **E** and **B** to denote complex positive frequency analytic signal electric and magnetic fields obeying (1). The real fields are then the combinations $\mathbf{E}+\mathbf{E}^*$, $\mathbf{B}+\mathbf{B}^*$. Taking advantage of the gauge freedom in the introduction of potentials, **E** and **B** can be represented in terms of a complex vector potential $\mathbf{V}(x)$ chosen in the radiation gauge, and which is likewise an analytic signal:

$$\mathbf{E}(x) = \partial^0\mathbf{V}(x),\quad \mathbf{B}(x) = \nabla\times\mathbf{V}(x),$$
$$\nabla\cdot\mathbf{V}(x) = 0,\quad \Box^2\mathbf{V}(x) = 0. \tag{3}$$

(The usual symbol **A** is reserved for the hermitian vector potential operator obtained after quantization). In terms of **V**, the expression (2) for the energy-momentum takes the compact form

$$P^\mu(\mathbf{V}) = \frac{1}{2\pi}\int d^3\mathbf{x}\,\partial^0\mathbf{V}(x)^*\cdot\partial^\mu\mathbf{V}(x). \tag{4}$$

We wish to define in a precise way what we would like to call the manifold of solutions of (1). For this purpose and also to identify the elements of the canonical formalism it is useful to express the general analytic signal $\mathbf{V}(x)$ as a Fourier integral:

$$\mathbf{V}(x) = \frac{1}{2\pi}\int d^3k\left(\frac{c}{k^0}\right)^{1/2}\mathbf{v}(\mathbf{k})\exp(ik\cdot x),$$
$$\mathbf{k}\cdot\mathbf{v}(\mathbf{k}) = 0,\quad k^0 = |\mathbf{k}| = \omega/c > 0. \tag{5}$$

If for each **k** a pair of mutually orthogonal unit polarization vectors $\mathbf{e}_\alpha(k)$, $\alpha = 1, 2$, are chosen to satisfy

$$\mathbf{k}\cdot\mathbf{e}_\alpha(\mathbf{k}) = 0,\quad \mathbf{e}_\alpha(\mathbf{k})^*\cdot\mathbf{e}_\beta(\mathbf{k}) = \delta_{\alpha\beta},\quad \alpha,\beta = 1, 2, \tag{6}$$

the amplitude $\mathbf{v}(\mathbf{k})$ can be expanded as

$$\mathbf{v}(\mathbf{k}) = \sum_{\alpha=1}^{2} v_\alpha(\mathbf{k})\,\mathbf{e}_\alpha(\mathbf{k}). \tag{7}$$

The four-vector $P^\mu(\mathbf{V})$ then becomes

$$P^\mu(\mathbf{V}) = c\int \mathrm{d}^3k\, k^\mu\,|\mathbf{v}(\mathbf{k})|^2$$
$$= c\int \mathrm{d}^3k\, k^\mu \sum_\alpha |v_\alpha(\mathbf{k})|^2. \tag{8}$$

The structure of this expression motivates the definition of a norm $\|\mathbf{V}\|$, (at this point an object without an immediate classical interpretation), to be associated with each $\mathbf{V}(x)$ as follows:

$$\|\mathbf{V}\|^2 = \frac{i}{2\pi c}\int \mathrm{d}^3\mathbf{x}\,\partial^0\mathbf{V}(x)^*\cdot\mathbf{V}(x) = \int \mathrm{d}^3k\,|\mathbf{v}(\mathbf{k})|^2. \tag{9}$$

This is evidently positive definite; moreover it is easily checked that the physical dimension of $\|\mathbf{V}\|^2$ is that of action. It is amusing to express it in terms of the 'observable' electric and magnetic fields:

$$\|\mathbf{V}\|^2 = \frac{i}{8\pi^2 c}\int \mathrm{d}^3\mathbf{x}\int \mathrm{d}^3\mathbf{x}'\,\frac{\mathbf{E}(\mathbf{x},t)^*\cdot\nabla'\times\mathbf{B}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}. \tag{10}$$

It is worth emphasizing that this (non-local) expression is both Lorentz invariant and conserved, apart from being positive definite. Though it is a purely classical expression with the dimensions of action, it is local in time and must at least formally be distinguished from the other classically important action, namely, the time integral of a Lagrangian.

We now define the classical manifold $\mathscr{M}$ as the set of all solutions $\mathbf{V}(x)$ of (3) with finite norm:

$$\mathscr{M} = \{\text{analytic signals } \mathbf{V}(x)\,|\,\nabla\cdot\mathbf{V} = \Box^2\mathbf{V} = 0,\ \|\mathbf{V}\| < \infty\}. \tag{11}$$

Thus even classically $\mathscr{M}$ has the structure of a Hilbert space (though of course we recognize that $\|\mathbf{V}\|^2$ is an action and is not dimensionless). It is not hard to see that finiteness of $\|\mathbf{V}\|$ does not guarantee finiteness of $P^0(\mathbf{V})$, because of the possibility of an ultraviolet divergence in the latter. It can be shown that there are field configurations outside $\mathscr{M}$ that admit Poincaré transformations. We shall devote another paper for a discussion of these questions. (Conversely, finiteness of $P^0(\mathbf{V})$ cannot guarantee finiteness of $\|\mathbf{V}\|$, due to a possible infrared divergence in $\|\mathbf{V}\|$.) The usefulness of the norm $\|\mathbf{V}\|$ and the classical space $\mathscr{M}$ of solutions to Maxwell's equations will be clear when we discuss the quantum mechanics of single photons and of the quantized field. Having chosen the finiteness of $\|\mathbf{V}\|$ to define $\mathscr{M}$, we can say in the language of quantum mechanics that $P^0(\mathbf{V})$ is an unbounded functional on $\mathscr{M}$.

In order to reproduce Maxwell's equations within the canonical formalism, $P^0(\mathbf{V})$ must be used as the Hamiltonian and the Fourier amplitudes $v_\alpha(\mathbf{k})$ must be subject to the fundamental Poisson Bracket (PB) relations

$$\{v_\alpha(\mathbf{k}), v_\beta(\mathbf{k}')^*\} = -i\delta_{\alpha\beta}\,\delta^{(3)}(\mathbf{k}-\mathbf{k}'),$$
$$\{v_\alpha(\mathbf{k}), v_\beta(\mathbf{k}')\} = \{v_\alpha(\mathbf{k})^*, v_\beta(\mathbf{k}')^*\} = 0. \tag{12}$$

In coordinate space the analytic signal $\mathbf{V}(x)$ obeys non-local PB relations

$$\{V_j(x), V_l(x')^*\} = \frac{-ic}{4\pi^2}\left(\delta_{jl} - \frac{\partial_j\partial_l}{\nabla^2}\right)\int\frac{d^3k}{k^0}\exp[ik\cdot(x-x')],$$
$$\{V_j(x), V_l(x')\} = \{V_j(x)^*, V_l(x')^*\} = 0, \tag{13}$$

which of course lead to local equal time PB relations between $\mathbf{E}+\mathbf{E}^*$ and $\mathbf{B}+\mathbf{B}^*$.

The formal significance of $\|\mathbf{V}\|^2$ within classical theory can be grasped by referring to the action-angle formalism of canonical dynamics. As the expression for the field Hamiltonian $P^0(\mathbf{V})$ shows,

$$P^0(\mathbf{V}) = \int d^3k \sum_\alpha \omega\, v_\alpha(\mathbf{k})^*\, v_\alpha(\mathbf{k}), \tag{14}$$

the Maxwell field is a collection of independent harmonic oscillators. For a single oscillator, whose Hamiltonian in terms of a canonical pair of variables q, p is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\,\omega_0^2 q^2, \tag{15}$$

the canonical transformation to action-angle variables J, Θ is performed by the equations

$$q = (J/m\pi\omega_0)^{1/2}\sin 2\pi\Theta,$$
$$p = (m\omega_0 J/\pi)^{1/2}\cos 2\pi\Theta. \tag{16}$$

If complex dynamical variables v, v^* are defined by

$$v = (m\omega_0 q + ip)/(2m\omega_0)^{1/2}, \tag{17}$$

we have the relations

$$H = \omega_0 v^* v,$$
$$J = 2\pi H/\omega_0 = 2\pi v^* v,$$
$$\{v, v^*\} = -i. \tag{18}$$

Comparison of (9) and (14) with these expressions makes it clear that (save for a factor of 2π) the norm $\|\mathbf{V}\|^2$ is just the sum of all the action variables J corresponding to all the oscillators that go to make up the classical Maxwell field.

3. Quantized Maxwell field

Quantization promotes classical dynamical variables to linear operators on a Hilbert space and replaces the fundamental classical PB's by $(i\hbar)^{-1}$ times the commutators among the corresponding operators. The operator that $\mathbf{V}(x)$ is thus promoted to will be written $\mathbf{A}^{(+)}(x)$: it is the positive frequency (annihilation) part of the hermitian vector potential $\mathbf{A}(x)$. To avoid the explicit appearance of $\hbar$ in the basic commutation relations, the classical Fourier amplitudes $v_\alpha(\mathbf{k})$ will be replaced by $(\hbar)^{1/2}$ times annihilation operators $a_\alpha(\mathbf{k})$. Thus the expression for the vector potential operator and the basic non-vanishing commutation relations are:

$$\mathbf{A}(x) = \mathbf{A}^{(+)}(x) + \mathbf{A}^{(-)}(x),$$

$$\mathbf{A}^{(-)}(x) = \mathbf{A}^{(+)}(x)^\dagger,$$

$$\mathbf{A}^{(+)}(x) = \frac{(\hbar c)^{1/2}}{2\pi} \int \frac{\mathrm{d}^3 k}{\sqrt{k^0}} \sum_\alpha a_\alpha(\mathbf{k}) \mathbf{e}_\alpha(\mathbf{k}) \exp(ik \cdot x),$$

$$[a_\alpha(\mathbf{k}), a_\beta(\mathbf{k}')^\dagger] = \delta_{\alpha\beta}\, \delta^{(3)}(\mathbf{k} - \mathbf{k}'). \tag{19}$$

The (positive frequen annihilation parts of the) electric and magnetic field operators are of course given by

$$\mathbf{E}(x) = \partial^0 \mathbf{A}^{(+)}(x), \quad \mathbf{B}(x) = \nabla \times \mathbf{A}^{(+)}(x). \tag{20}$$

Thus $\mathbf{A}^{(+)}$, $\mathbf{E}$ and $\mathbf{B}$ are the quantum analogues of the corresponding classical analytic signals, and Maxwell's equations (1) continue to hold in operator form. After normal ordering, the hermitian energy momentum operator is

$$P^\mu = \hbar c \int \mathrm{d}^3 k\, k^\mu \sum_\alpha a_\alpha(\mathbf{k})^\dagger a_\alpha(\mathbf{k}). \tag{21}$$

From the point of view of the photon picture, the most natural description of the Hilbert space $\mathscr{H}$ on which these operators act and realize the above commutation relations is along the following lines. First one has a normalized vacuum or no-photon state $|0\rangle$ annihilated by all $a_\alpha(\mathbf{k})$:

$$a_\alpha(\mathbf{k})|0\rangle = 0, \quad \langle 0|0\rangle = 1. \tag{22}$$

This vector spans a one-dimensional subspace $\mathscr{H}_0$ in $\mathscr{H}$. Then one has subspaces $\mathscr{H}_1$, $\mathscr{H}_2, \ldots, \mathscr{H}_n, \ldots$ in $\mathscr{H}$ corresponding to states with one, two, $\ldots, n, \ldots$ photons obtained by applying one, two, $\ldots, n, \ldots$ creation operators $a^\dagger$ to $|0\rangle$. The inner products among such vectors are determined by the commutation relations among the a's and $a^\dagger$'s and the properties (22) of the vacuum state, and $\mathscr{H}$ appears as the direct sum

$$\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \ldots \oplus \mathscr{H}_n \oplus \ldots. \tag{23}$$

General states of the quantized field have components belonging to each of the subspaces $\mathscr{H}_n$. This photon number description of $\mathscr{H}$ is complementary to one in which the quantum fields are given more prominence. The photon description given here is sometimes called the 'myriotic' representation or the Fock representation (Friedrichs 1952). They are the only ones which possess a vacuum state $|0\rangle$. Only such states can arise as long as $\|\mathbf{V}\| < \infty$.

We now proceed to explore the ways in which the space $\mathscr{M}$ of solutions to the classical Maxwell equations retains its relevance and usefulness after quantization and the introduction of the photon language have been carried out.

4. The relation between $\mathscr{M}$ and $\mathscr{H}_1$

Let us first examine the relationship between $\mathscr{M}$ and $\mathscr{H}_1$, the subspace of one photon states in $\mathscr{H}$. A general vector in $\mathscr{H}_1$ is obtained by applying a 'linear combination' of the various creation operators $a_\alpha(\mathbf{k})^\dagger$ to $|0\rangle$. If some $\mathbf{V}(x) \in \mathscr{M}$ is given, we can ask if in a natural way a corresponding vector of $\mathscr{H}_1$ is determined. This is indeed so, and writing $|\mathbf{V}\rangle$ for it, we can define it by the property

$$\langle 0|\, \mathbf{A}^{(+)}(x)\, |\mathbf{V}\rangle = \mathbf{V}(x). \tag{24}$$

This is a matrix element, and not an expectation or eigenvalue condition. The vector $|\mathbf{V}\rangle$ is easily constructed:

$$\begin{aligned} |\mathbf{V}\rangle &= a(\mathbf{V})^\dagger\, |0\rangle, \\ a(\mathbf{V})^\dagger &= \frac{i}{\hbar c}\int \mathrm{d}^3x\, \mathbf{V}(x)\cdot \partial^0 \mathbf{A}^{(-)}(x) \\ &= (\hbar)^{-1/2}\int \mathrm{d}^3k \sum_\alpha v_\alpha(\mathbf{k})\, a_\alpha(\mathbf{k})^\dagger. \end{aligned} \tag{25}$$

According to this definition, $a(\mathbf{V})^\dagger$ is linear, and $a(\mathbf{V})$ antilinear, in $\mathbf{V}$. The norm of this vector, as well as the commutation relation obeyed by $a(\mathbf{V})^\dagger$ and its adjoint, involve the classical norm $\|\mathbf{V}\|$ used in defining $\mathscr{M}$:

$$\langle \mathbf{V}|\mathbf{V}\rangle = [a(\mathbf{V}), a(\mathbf{V})^\dagger] = \|\mathbf{V}\|^2/\hbar. \tag{26}$$

More generally, we have the commutation relation

$$[a(\mathbf{V}), a(\mathbf{V}')^\dagger] = (\mathbf{V}, \mathbf{V}')/\hbar$$

where the inner product $(\mathbf{V}, \mathbf{V}')$ among elements of $\mathscr{M}$ is obtained from $\|\mathbf{V}\|^2$ by polarization. (This result is of course dimensionless). The matrix element of the operator P^μ of (21) between $\langle \mathbf{V}|$ and $|\mathbf{V}\rangle$ coincides with the actual value of the energy-momentum of the classical field in the state $\mathbf{V}$, given in (4) and (8):

$$\langle \mathbf{V}|\, P^\mu\, |\mathbf{V}\rangle = P^\mu(\mathbf{V}). \tag{27}$$

The expectation value of energy and momentum for a photon in the state $|\mathbf{V}\rangle$ is thus

$$\frac{\langle \mathbf{V}|P^\mu|\mathbf{V}\rangle}{\langle \mathbf{V}|\mathbf{V}\rangle} = \frac{\hbar}{\|\mathbf{V}\|^2}\, P^\mu(\mathbf{V}). \tag{28}$$

Since both $P^\mu(\mathbf{V})$ and $\|\mathbf{V}\|^2$ are classically defined quantities, one sees in this expression an explicit and simple signature of quantum theory in the presence of the single factor $\hbar$ on the right hand side. Naturally multiplication of $\mathbf{V}$ by a numerical factor causes no change in this expectation value.

On account of the direct proportionality of the norms in $\mathscr{M}$ and in $\mathscr{H}_1$, exhibited by (26), we see a natural one-to-one correspondence between $\mathscr{M}$ and $\mathscr{H}_1$, i.e. between classical solutions of the Maxwell equations and general *unnormalized* single photon states. This however in no way means that the one-photon state $|\mathbf{V}\rangle$ is the quantum analogue of the state V of the classical field. Among other things, as we have just seen, if λ is any complex number, we have

$$|\lambda \mathbf{V}\rangle = \lambda|\mathbf{V}\rangle \tag{29}$$

and this causes no change in the physical one-photon state.

The association $\mathbf{V} \in \mathscr{M} \leftrightarrow |\mathbf{V}\rangle \in \mathscr{H}_1$ naturally prompts the question whether $\mathbf{V}(x)$ can be regarded as the space-time wave function of the photon, in the sense of quantum mechanics, when it is in the state $|\mathbf{V}\rangle$. The answer is that this is not so, partly for reasons of relativistic kinematics and partly because, strictly speaking, there is no such thing as a spatial wave function for a photon (Newton and Wigner 1949; Acharya and Sudarshan 1960; Mandel 1963).

In wave-vector or momentum space we have a quantum mechanical probability interpretation: from (25) we conclude that in the state $|\mathbf{V}\rangle$ there is a probability amplitude $v_\alpha(\mathbf{k}) / \|\mathbf{V}\|$ and a probability density

$$|v_\alpha(\mathbf{k})|^2/\|\mathbf{V}\|^2 \tag{30}$$

for finding the photon with momentum $\hbar\mathbf{k}$ and polarization α. By 'plain' Fourier transformation we may construct a wave-function in coordinate space as follows:

$$\psi^{(\mathbf{V})}(\mathbf{x}, t) = (1/(2\pi)^{3/2}\sqrt{\hbar}) \int \mathrm{d}^3k\ \mathbf{v}(\mathbf{k}) \exp(ik\cdot x). \tag{31}$$

Like $\mathbf{V}(x)$, $\psi^{(\mathbf{V})}(x)$ too is a transverse analytic signal solution of the wave equation; while they do determine each other uniquely, they are nonlocally related in space:

$$\psi^{(\mathbf{V})}(x) = (1/\sqrt{\hbar c})(-\nabla^2)^{1/4}\mathbf{V}(x). \tag{32}$$

Thus elements of $\mathscr{H}_1$, and of $\mathscr{M}$, can as well be specified by $\psi^{(\mathbf{V})}$ as by $\mathbf{V}$. That $\psi^{(\mathbf{V})}$ is very much like a single-particle wave function in nonrelativistic quantum mechanics is shown by the forms taken by the norm and the expectation value of P^μ:

$$\langle \mathbf{V}|\mathbf{V}\rangle = \int \mathrm{d}^3\mathbf{x}\ |\psi^{(\mathbf{V})}(x)|^2,$$

$$\langle \mathbf{V}|P^\mu|\mathbf{V}\rangle = \int \mathrm{d}^3\mathbf{x}\ \psi^{(\mathbf{V})}(x)^* \cdot (-i\hbar c\,\partial^\mu)\psi^{(\mathbf{V})}(x). \tag{33}$$

Nevertheless we cannot interpret $\psi^{(\mathbf{V})}(x)$ as a quantum mechanical probability amplitude for the photon in configuration space. The group theoretical reason for this was discovered by Newton and Wigner in a well-known analysis. They showed that in any finite (non-zero) mass and finite spin unitary representation of the Poincaré group it is possible to set up physically reasonable position operators with suitable properties. These representations survive as ray representations in the Galilean limit and lead to a well-defined concept of localization in non-relativistic quantum mechanics. However in the zero mass and non-zero ($\geqslant 1$) helicity representations of the Poincaré group no reasonable position operators can be defined and of course these representations have Galilean limits which cannot be identified with Galilean particle wave functions. We can see this quite explicitly in the present case as follows. While it is true that the wave functions $\psi^{(\mathbf{V})}(x)$ give an acceptable representation of the possible states of a single photon, the correspondence between elements of $\mathscr{H}_1$ and the ψ's being one-to-one, multiplication of a ψ by (any component of) $\mathbf{x}$ is not a bonafide operator on $\mathscr{H}_1$. This is because the transversality property of $\psi^{(\mathbf{V})}(x)$ is not shared by $\mathbf{x}_j\psi^{(\mathbf{V})}(x)$, where $j = 1$, 2 or 3. Thus the $\mathbf{x}_j$ are not observables for a photon.

Thus there is a limitation in principle to the concept of localization for a photon. In spite of this, the amplitude $\psi^{(\mathbf{V})}(x)$ is the closest one can get to a configuration space wave function, and as long as we do not attempt to localize the photon to an accuracy better than the (mean) wavelength, we can say in an approximate way that in the state $|\mathbf{V}\rangle$,

$$|\psi^{(\mathbf{V})}(x)|^2 / \langle \mathbf{V}|\mathbf{V}\rangle = \frac{\hbar}{\|\mathbf{V}\|^2}|\psi^{(\mathbf{V})}(x)|^2 \tag{34}$$

is the probability density for the photon to be found at position x at time t. In particular, because of the non-local relationships between $\psi^{(\mathbf{V})}$ and $\mathbf{V}$, the vector potential $\mathbf{V}(x)$ cannot in general be given even this approximate probabilistic interpretation. There are however states strictly localized in any compact measurable region however small!

These non-localities can be neglected, and the expressions simplify if we have the quasi-monochromatic case. This occurs when the amplitudes $v_\alpha(\mathbf{k})$ are non-zero or sizable only for a narrow band of values of $|\mathbf{k}|$. If the corresponding mean frequency is ω_0, we have the approximate equalities

$$\begin{aligned}
\psi^{(\mathbf{V})} &\approx \frac{-i}{(\hbar\omega_0)^{1/2}}\,\mathbf{E} \approx \frac{1}{c}\,(\omega_0/\hbar)^{1/2}\,\mathbf{V},\\
P^0(\mathbf{V}) &\approx \omega_0\,\|\mathbf{V}\|^2,\\
\langle \mathbf{V}|\mathbf{V}\rangle &\approx \frac{\omega_0}{\hbar c^2}\int \mathrm{d}^3\mathbf{x}\;\mathbf{V}(x)^*\cdot\mathbf{V}(x)\\
&\approx \frac{1}{\hbar\omega_0}\int \mathrm{d}^3\mathbf{x}\;\mathbf{E}(x)^*\cdot\mathbf{E}(x).
\end{aligned} \tag{35}$$

For such a single-photon state $|\mathbf{V}\rangle$, the (approximate) position probability density for the photon becomes proportional to $|\mathbf{E}(x)|^2$ or $|\mathbf{V}(x)|^2$. If we have a true wave packet state, so that $v_\alpha(\mathbf{k})$ is non-zero or appreciable only for a small range of values of the wave vector $\mathbf{k}$ in the vicinity of some vector $\mathbf{k}^{(0)}$, all four components of $P^\mu(\mathbf{V})$ simplify to

$$P^\mu(\mathbf{V}) \approx \|\mathbf{V}\|^2 k^{(0)\mu} = \langle \mathbf{V}|\mathbf{V}\rangle \hbar k^{(0)\mu}. \tag{36}$$

5. The relation between $\mathscr{M}$ and $\mathscr{H}$-coherent states

Now we go on to consider a special family of states of the quantized fields, which are in close physical correspondence with elements of the classical state space $\mathscr{M}$. As the commutation relations (19) show, the quantized field is subject to infinitely many independent uncertainty relations, which become more transparent if the a's and $a^\dagger$'s of each mode are separated into fictitious 'positions' and 'momenta'. One can ask: is there a state of the complete quantized field which in its physical properties is as close as is permitted by these uncertainty relations, to a given classical state corresponding to an analytic signal $\mathbf{V}(x) \in \mathscr{M}$? The idea is to look for a state in which all the independent uncertainty products are simultaneously brought down to their minimum values, and stay that way for all time. The answer is that such a state does exist, for each $\mathbf{V}(x) \in \mathscr{M}$, and is a generalization to the field of the coherent states familiar in the study of the oscillator. We now develop these states and briefly describe their significant properties.

For a given $\mathbf{V}(x) \in \mathscr{M}$, the corresponding coherent state of the field will be denoted by $|\{\mathbf{V}\}\rangle$. It is to be distinguished from the single photon state $|\mathbf{V}\rangle$, though the same creation operator $a(\mathbf{V})^\dagger$ is involved in its construction. The state $|\{\mathbf{V}\}\rangle$ is defined by (Klauder and Sudarshan 1968)

$$|\{\mathbf{V}\}\rangle = \exp\left[-\frac{\|\mathbf{V}\|^2}{2\hbar} + a(\mathbf{V})^\dagger\right]|0\rangle$$
$$= \exp\left[a(\mathbf{V})^\dagger - a(\mathbf{V})\right]|0\rangle \in \mathscr{H}. \tag{37}$$

Note that, in contrast to $|\mathbf{V}\rangle \in \mathscr{H}_1$, this state is explicitly normalized to unity. The most familiar mathematical properties of these states are:

$$\langle\{\mathbf{V}'\}|\{\mathbf{V}\}\rangle = \exp[\{(\mathbf{V}', \mathbf{V}) - \tfrac{1}{2}\|\mathbf{V}'\|^2 - \tfrac{1}{2}\|\mathbf{V}\|^2\}/\hbar], \tag{38a}$$

$$\mathbf{A}^{(+)}(x)|\{\mathbf{V}\}\rangle = \mathbf{V}(x)|\{\mathbf{V}\}\rangle, \tag{38b}$$

$$\langle\{\mathbf{V}\}|P^\mu|\{\mathbf{V}\}\rangle = P^\mu(\mathbf{V}). \tag{38c}$$

There is another convenient way to express this property (38b):

$$a(\mathbf{V}')|\{\mathbf{V}\}\rangle = \frac{1}{\hbar}(\mathbf{V}', \mathbf{V})|\{\mathbf{V}\}\rangle.$$

Thus these states are eigenstates of the annihilation part of the operator vector potential (and not of the hermitian vector potential). No two of them are mutually orthogonal; and the expectation value of energy-momentum for the quantum field in state $|\{\mathbf{V}\}\rangle$ equals its numerical value in the classical state $\mathbf{V}$. Each $|\{\mathbf{V}\}\rangle$ is a superposition of states with definite photon numbers. Such states are therefore complementary to the states in individual subspaces $\mathscr{H}_n$ of $\mathscr{H}$, in that the fields rather than photon numbers have a direct meaning in them.

The n-photon part of $|\{\mathbf{V}\}\rangle$ is

$$\exp(-\|\mathbf{V}\|^2/2\hbar)\cdot\frac{(a(\mathbf{V})^\dagger)^n}{n!}|0\rangle. \tag{39}$$

All the n photons here are in the same single photon state $|\mathbf{V}\rangle$ of $\mathscr{H}_1$, but their total wave function is (Bose)-symmetrized. In fact $|\{\mathbf{V}\}\rangle$ is a superposition of states with various numbers of photons all of the same "type", since the same photon mode keeps appearing again and again. From (39) we see that the following exact statements can be made: the probability of finding n photons when the field is in the state $|\{\mathbf{V}\}\rangle$ is

$$P(n) = \exp(-\|\mathbf{V}\|^2/\hbar)\cdot\frac{1}{n!}\cdot\left(\frac{\|\mathbf{V}\|^2}{\hbar}\right)^n. \tag{40}$$

This is a Poisson distribution, so the expected number of photons in this state of the field is

$$\sum_{n=0}^{\infty} nP(n) = \|\mathbf{V}\|^2/\hbar. \tag{41}$$

The physical importance of the classical norm $\|\mathbf{V}\|$ is clearly seen: it tells us how many photons there are on the average if we try to interpret the classical state **V** in terms of photons.

We already saw that subject to certain limitations and apart from a normalising factor, $\psi^{(\mathbf{V})}(x)$ can be used as a configuration space probability amplitude for a photon in the state $|\mathbf{V}\rangle \in \mathscr{H}_1$. Since all the photons in the field state $|\{\mathbf{V}\}\rangle$ are in the same single photon state, it follows that with the help of $\psi^{(\mathbf{V})}(x)$ and subject to the same limitations, a more detailed physical description of $|\{\mathbf{V}\}\rangle$ can be given. In fact the following approximate statements can be made: the probability of finding n photons in a spatial volume $\mathscr{V}$ at time t (and any number outside $\mathscr{V}$) follows a Poisson distribution:

$$P(n, \mathscr{V}, t) \approx \exp\left(- \int_{\mathscr{V}} \mathrm{d}^3\mathbf{x} |\psi^{(\mathbf{V})}(x)|^2 \right) \cdot \frac{1}{n!} \left(\int_{\mathscr{V}} \mathrm{d}^3\mathbf{x}\, |\psi^{(\mathbf{V})}(x)|^2 \right)^n. \tag{42}$$

Therefore the expected number of photons in $\mathscr{V}$ at time t is

$$\sum_{n=0}^{\infty} n\, P(n, \mathscr{V}, t) \approx \int_{\mathscr{V}} \mathrm{d}^3\mathbf{x} |\psi^{(\mathbf{V})}(x)|^2, \tag{43}$$

leading to an expected density of photons given by (approximately)

$$|\psi^{(\mathbf{V})}(x)|^2 \tag{44}$$

at time t.

All these configuration space characterizations of $|\{\mathbf{V}\}\rangle$ involve $\psi^{(\mathbf{V})}$ and not **V** directly. Therefore these local properties of such a state are not immediately proportional to corresponding local properties of $\mathbf{V}(x)$ at all. However, in the quasimonochromatic limit there is some simplification and a satisfying agreement with simple-minded expectations. Thus for a quasimonochromatic $\mathbf{V}(x)$ with mean frequency ω_0, the spatial density of photons becomes time independent and is given by

$$|\psi^{(\mathbf{V})}(x)|^2 \approx \frac{\mathbf{E}(x)^* \cdot \mathbf{E}(x)}{h\omega_0} = \frac{1}{2\pi} \cdot \mathbf{E}(x)^* \cdot \mathbf{E}(x) \cdot \frac{1}{\hbar\omega_0}. \tag{45}$$

Remembering that we are representing classical fields as analytic signals, we make contact here in an approximate way and in the quasimonochromatic case with the energy density for radiation in the classical Maxwell theory, and the energy associated with a single photon of frequency ω_0. At least in this limiting case we can see in a clear and physical way why the quantized field state $|\{\mathbf{V}\}\rangle$ is the closest we can get to the classical state **V**.

In tracing the correspondence between $\mathscr{M}$ and $\mathscr{H}_1$, we noted that multiplication of $\mathbf{V}(x)$ by a constant factor causes no physical change in the associated one photon state; this is shown by (29). More generally, linear superposition of classsical solutions **V** of the Maxwell equations goes over into the quantum mechanical superposition of one photon states. The situation at the level of coherent states is, however, quite different because $|\{\mathbf{V}\}\rangle$ has a decidedly nonlinear dependence on $\mathbf{V}$. In particular the state $|\{\lambda_1 \mathbf{V}_1 + \lambda_2 \mathbf{V}_2\}\rangle$ is far from a linear superposition of the states $|\{\mathbf{V}_1\}\rangle$ and $|\{\mathbf{V}_2\}\rangle$. One sees a direct illustration of all this by examining how the coherent state $|\{\mathbf{V}\}\rangle$ changes if one multiplies $\mathbf{V}$ by a steadily diminishing numerical factor λ. As we see from (40), in the state $|\{\lambda \mathbf{V}\}\rangle$ it is the probabilities $P(n)$, $n \geqslant 1$, of finding various numbers of photons that steadily decrease with λ, while the photons themselves when found have unchanging properties independent of λ. In a sense, this is an example of the remark of Dirac on superposition in quantum mechanics in general to the effect that "the intermediate character of the state formed by superposition (thus) expresses itself through the probability of a particular result for an observation being intermediate between the corresponding probabilities for the original states, not through the result itself being intermediate between the corresponding results for the original states".

Each pure state $\mathbf{V}$ of the classical field has been seen to closely correspond to the state $|\{\mathbf{V}\}\rangle$ of the quantized field. While the classical pure states are all counted as $\mathbf{V}$ runs over all of $\mathscr{M}$, the possible quantum pure states are by no means exhausted by the set $|\{\mathbf{V}\}\rangle$. This is a result of the fact that the state space of a quantum mechanical system is unimaginably richer than the state space of the corresponding classical system. At the level of pure states this is again caused by the possibility of quantum mechanical superposition, which has no classical analogue. The "number" of dynamical variables for a quantum system, on the other hand is about the same as for the corresponding classical system. In a way we can regard this richness of states as a compensation for the non-commutativity of dynamical variables, as a result of which all of them cannot simultaneously possess precise numerical values. At the level of general (statistical as well as pure) quantum states the diagonal coherent state representation (Sudarshan 1962; Mukunda and Sudarshan 1978) makes this richness quite explicit, since 'classical' non-negative weight functions are an 'infinitely small' subset of all possible quantum weight functions. For the case at hand, $\mathscr{M}$ as a set coincides with the set of all pure states of the classical Maxwell field, but for the quantized field the photon picture has also to be accommodated, leading to many more pure states than the set $|\{\mathbf{V}\}\rangle$. Nevertheless every pure state in $\mathscr{H}$, and more generally even impure ones, can be expressed in terms of the coherent states, a point best discussed after explaining the close connection between $\mathscr{H}_1$ and $\mathscr{H}$.

6. The relation between $\mathscr{H}_1$ and $\mathscr{H}$

It is well known that $\mathscr{H}$ is the Fock space built over $\mathscr{H}_1$. Mathematically what is involved is the process of exponentiation of one Hilbert space to get another:

$$\mathscr{H} = \mathrm{Exp}\,(\mathscr{H}_1). \tag{46}$$

This process can be described briefly as follows. Let V denote any complex separable Hilbert space of finite or infinite dimension, with elements ψ, φ, . . . and inner product $(\ ,\ )$. The n-fold tensor product of V with itself will be denoted

$$\mathsf{V}^n = \mathsf{V} \otimes \mathsf{V} \otimes \ldots \otimes \mathsf{V} \quad (n \text{ factors}). \tag{47}$$

Elements of V^n are linear combinations of special elements of the form

$$\psi_1 \otimes \psi_2 \otimes \ldots \otimes \psi_n, \quad \psi_j \in \mathsf{V}. \tag{48}$$

Inner products in V^n are determined by inner products among such special elements:

$$(\varphi_1 \otimes \varphi_2 \otimes \ldots \otimes \varphi_n, \psi_1 \otimes \psi_2 \otimes \ldots \otimes \psi_n) = \prod_{j=1}^{n} (\varphi_j, \psi_j). \tag{49}$$

(Restriction to linear combinations with finite norm and the process of completion in the norm are both understood). By restricting oneself to those linear combinations of the products (48) that are symmetric under permutations of the factors one obtains a subspace V_n of V^n called the symmetric n-fold tensor product of V with itself:

$$\mathsf{V}_n = \{\mathsf{V} \otimes \mathsf{V} \otimes \ldots \otimes \mathsf{V}\}_{\text{sym}} \subset \mathsf{V}^n. \tag{50}$$

The inner product in V_n is therefore determined by that in V^n. A special class of elements of V_n arises from elements of V in this way:

$$\psi \in \mathsf{V} \Rightarrow \psi \otimes \psi \otimes \ldots \psi \ (n \text{ times}) \in \mathsf{V}_n. \tag{51}$$

We shall later use the fact that for finite dimensional V, as ψ runs over V these special elements of V_n certainly span V_n; formally this result extends also for the infinite dimensional case. For $n = 1$ we have the coincidences

$$\mathsf{V}^1 = \mathsf{V}_1 = \mathsf{V}. \tag{52}$$

The exponential of V, Exp (V), is now defined to be the Hilbert space obtained by taking the direct sum of all the symmetrized tensor products of V with itself various numbers of times:

$$\begin{aligned}\operatorname{Exp}(\mathsf{V}) &= \mathbf{C} \oplus \mathsf{V}_1 \oplus \mathsf{V}_2 \oplus \ldots \oplus \mathsf{V}_n \oplus \ldots \\ &= \sum_{n=0}^{\infty} \oplus \mathsf{V}_n,\end{aligned} \tag{53}$$

if it is understood that $\mathsf{V}_0 = \mathbf{C}$.

As an example, consider the one-dimensional case $\mathsf{V} = \mathbf{C}$, with the norm of $z \in \mathbf{C}$ being $(z^* z)^{1/2}$. All the spaces V_n ($= \mathsf{V}^n$ in this case) are one-dimensional and reduce to $\mathbf{C}$; and in the sense of (51), $z \in \mathsf{V} = \mathbf{C}$ leads to the special element $z^n \in \mathsf{V}_n = \mathbf{C}$. The norm in V_n is the same as in V, and one sees that Exp (V) is the infinite dimensional space l_2 of square-summable sequences of complex numbers.

As another example we can choose V to be the infinite dimensional space $\mathscr{H}_1$ of one-photon states. Then V_n coincides with the n-photon subspace $\mathscr{H}_n$, and $\mathscr{H}$ is the exponential of $\mathscr{H}_1$ as claimed in (46).

Now we turn to the idea of coherent states. In the case of a general V, for each $\psi \in \mathsf{V}$ we define a special vector $\Psi(\psi) \in \operatorname{Exp}(\mathsf{V})$ with components in the subspaces V_n as follows:

$$\Psi(\psi) = \exp[-\tfrac{1}{2}(\psi, \psi)]\left(1, \frac{\psi}{\sqrt{1!}}, \frac{\psi \otimes \psi}{\sqrt{2!}}, \frac{\psi \otimes \psi \otimes \psi}{\sqrt{3!}}, \ldots\right). \tag{54}$$

In the direct sum notation this may alternatively be written as

$$\Psi(\psi) = \exp[-\tfrac{1}{2}(\psi, \psi)]\left(1 + \frac{\psi}{\sqrt{1!}} + \frac{\psi \otimes \psi}{\sqrt{2!}} + \ldots\right). \tag{55}$$

There are 'as many' coherent states in Exp(V) as there are vectors in V. Inner products among them are easily computed:

$$(\Psi(\varphi), \Psi(\psi)) = \exp[(\varphi, \psi) - \tfrac{1}{2}(\varphi, \varphi) - \tfrac{1}{2}(\psi, \psi)]. \tag{56}$$

Thus no two of them are mutually orthogonal. These states have remarkable properties. To express them, let us first go back to the simplest case $V = \mathbf{C}$. Then we have one coherent state for each $z \in \mathbf{C}$:

$$\Psi(z) = \exp(-\tfrac{1}{2}|z|^2)\left(1, \frac{z}{\sqrt{1!}}, \frac{z^2}{\sqrt{2!}}, \dots\right). \tag{57}$$

In the conventional notation where $|n\rangle$ stands for a unit vector in V_n (= $\mathbf{C}$ here), $\Psi(z)$ takes the familiar form

$$\begin{aligned}\Psi(z) \to |z\rangle &= \exp(-\tfrac{1}{2}|z|^2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle \\ &= \exp[-\tfrac{1}{2}|z|^2 + za^\dagger]\,|0\rangle \\ &= \exp(za^\dagger - z^* a)\,|0\rangle,\end{aligned} \tag{58}$$

where $a^\dagger$ is the creation operator taking $|n\rangle \in V_n$ to $(n+1)^{1/2}\,|n+1\rangle \in V_{n+1}$. As is well known in this case, $|z\rangle$ is an eigenvector of the annihilation operator a, the hermitian conjugate of $a^\dagger$, with eigenvalue z. Moreover the family of vectors $|z\rangle$ for all $z \in \mathbf{C}$ is an overcomplete system in l_2. This means that any vector in l_2 can be expanded in terms of coherent states in infinitely many ways, though there is a preferred expansion obtained by using the resolution of the identity

$$\int \frac{d^2 z}{\pi} |z\rangle\langle z| = \mathbf{1} \text{ on } l_2. \tag{59}$$

As a further result of this overcompleteness, one has the diagonal coherent state representation as a result of which any density operator ρ on l_2 can be expanded in terms of projections on to coherent states with a weight function which may be a distribution:

$$\rho = \int d^2 z\, f(z)\,|z\rangle\langle z|. \tag{60}$$

These properties of coherent states generalize easily to a V of finite dimension, and then formally to V of infinite dimension. Let us indicate briefly some of the key expressions for a V of finite dimension. (The Hilbert space Exp (V) then corresponds to the space of states of a quantum-mechanical system with as many degrees of freedom as the dimension of V). Taking advantage of the fact that the special elements in V_n of the form appearing in (51) do span V_n, the annihilation and creation operators are fully defined by their actions on such vectors in all V_n. These operators may be written $a(\varphi)$ and $a(\varphi)^\dagger$ with φ varying over V; they are respectively antilinear and linear in φ. The operator $a(\varphi)$ is defined by

$$a(\varphi)\Psi(\psi) = (\varphi, \psi)\Psi(\psi) \text{ for all } \psi \in V. \tag{61}$$

(One must in principle check that this is a consistent definition since the vectors $\Psi(\psi)$ are linearly dependent, but this causes no difficulties. The same comment applies to (62) and (63) as well). By using (54) and (55) here and equating terms of the same 'power' of ψ on both sides, we find:

$$a(\varphi)\Psi(0) = 0;$$

$$a(\varphi)\psi\otimes\psi\otimes\ldots\otimes\psi \quad (n \text{ factors})$$

$$= \sqrt{n}(\varphi, \psi)\psi\otimes\psi\otimes\ldots\otimes\psi \quad (n-1 \text{ factors}),\ n = 1, 2, \ldots. \tag{62}$$

Thus for each $\varphi \in \mathrm{V}$, $a(\varphi)$ maps V_n into V_{n-1}. $\Psi(0)$ is the vacuum or ground state, analogous to $|0\rangle$ appearing in (58). The adjoint of (62) leads to the action of $a(\varphi)^\dagger$:

$$a(\varphi)^\dagger\Psi(0) = (0, \varphi, 0, \ldots, 0, \ldots);$$

$$a(\varphi)^\dagger\psi\otimes\psi\otimes\ldots\otimes\psi\ (n \text{ factors}) = \frac{1}{(n+1)^{1/2}}(\varphi\otimes\psi\otimes\ldots\otimes\psi$$

$$+\psi\otimes\varphi\otimes\psi\otimes\ldots\otimes\psi + \ldots$$

$$+\psi\otimes\psi\otimes\ldots\otimes\psi\otimes\varphi),$$

$$(n+1) \text{ terms with } (n+1) \text{ factors each},\ n = 1, 2, \ldots. \tag{63}$$

In particular we can set $\varphi = \psi$ here to get

$$a(\psi)^\dagger\psi\otimes\psi\otimes\ldots\otimes\psi\ (n \text{ factors}) = (n+1)^{1/2}\,\psi\otimes\psi\otimes\ldots\otimes\psi \quad (n+1 \text{ factors}),$$

so that

$$\psi\otimes\psi\otimes\ldots\otimes\psi\ (n \text{ factors}) = \frac{1}{\sqrt{n!}}(a(\psi)^\dagger)^n\Psi(0). \tag{64}$$

Using this in the definition (55), we can write the coherent state as

$$\Psi(\psi) = \exp[-\tfrac{1}{2}(\psi, \psi) + a(\psi)^\dagger]\Psi(0). \tag{65}$$

[With the help of these formulae one easily finds the relation

$$[a(\varphi), a(\psi)^\dagger] = [(\varphi, \psi)].$$

This generalizes the familiar (58) to the case of any V of finite dimension; if formally one proceeds also to a V of infinite dimension, such as $\mathscr{H}_1$, one arrives at the expression (37) for the coherent states of the quantized Maxwell field.

The resolution of the identity given by (59) in the case V = C extends, if V is of finite dimension, to

$$\int_{\mathrm{V}} d\psi\, d\bar{\psi}\ \Psi(\psi)\Psi(\psi)^\dagger = 1 \text{ on Exp (V)}, \tag{66}$$

where the integration measure is a unitary invariant volume element on V. Likewise for the diagonal coherent state expansion of density operators on Exp (V). Proceeding to the case $V = \mathscr{H}_1$ we generalize (59) and (66) to a formal resolution of the identity in the form

$$\int_{\mathscr{M}} \mathscr{D}[\mathbf{V}]\, \mathscr{D}[\mathbf{V}^*] |\{\mathbf{V}\}\rangle \langle\{\mathbf{V}\}| = \mathbf{1} \text{ on } \mathscr{H}, \tag{67}$$

where the left hand side must be understood as a functional integral over $\mathscr{M}$.

These developments show that the relationship between $\mathscr{H}_1$ and $\mathscr{H}$ is a particular instance of a quite general mathematical connection between a Hilbert space and its exponential. The structure of coherent states is also a very general one. What is perhaps surprising is that the coherent states $|\{\mathbf{V}\}\rangle$ of the Maxwell field, each of which is made up of photons all of the same type, and of which there are 'only as many' as single photon states, nevertheless form an *overcomplete* set of states for the *entire* quantized field!

7. The action of relativity transformations

The Maxwell field strengths **E** and **B** are the time-space and space-space components of the second rank antisymmetric field tensor $F_{\mu\nu}$ which transforms in a local covariant manner under inhomogeneous Lorentz, i.e. Poincaré, transformations. The field equations (1) are manifestly Poincaré-covariant since they take the neat form

$$\partial^\mu F_{\mu\nu}(x) = 0,$$
$$\partial_\lambda F_{\mu\nu}(x) + \partial_\mu F_{\nu\lambda}(x) + \partial_\nu F_{\lambda\mu}(x) = 0. \tag{68}$$

Turning to the vector potential, its Poincaré transformation law depends on the gauge in which it is chosen. In the radiation gauge used in this paper, the Euclidean group $E(3)$ and time translations act in a simple way on $\mathbf{V}(x)$, while pure Lorentz transformations have a non-local and more complicated effect. But what is important to realize, in the spirit of this paper, is that the action of the Poincaré group $\mathscr{P}$ on the classical manifold $\mathscr{M}$ determines completely its action on the states of the quantized field.

Consider at first an infinitesimal element of $\mathscr{P}$, corresponding to the coordinate changes

$$x^\mu \to x'^\mu \simeq x^\mu + \omega^{\mu\nu} x_\nu + a^\mu,$$
$$\omega^{\mu\nu} = -\omega^{\nu\mu}. \tag{69}$$

Here $\omega^{\mu\nu}$ and a^μ are infinitesimal parameters. The change in functional form of $\mathbf{V}(x)$ induced by this element of $\mathscr{P}$ can be expressed via a generator linear in ω and a:

$$\mathbf{V} \to \mathbf{V}' \simeq \mathbf{V} + \delta\mathbf{V},$$
$$\delta\mathbf{V} = -iG(\omega, a)\mathbf{V},$$
$$G(\omega, a) = a^\mu P_\mu + \omega_{0j} K_j - \tfrac{1}{2}\omega_{jk}\varepsilon_{jkl} J_l. \tag{70}$$

For the moment, the ten independent generators P_μ, K_j, J_j may be thought of as purely geometrical objects. On a general $\mathbf{V}(x) \in \mathscr{M}$ they act as follows:

$$P_\mu \mathbf{V}(x) = -i\partial_\mu \mathbf{V}(x),$$

$$(J_j \mathbf{V})_k(x) = -i[(\mathbf{x}\times\nabla)_j \mathbf{V}_k(x) + \varepsilon_{jkl}\mathbf{V}_l(x)],$$
$$(K_j \mathbf{V})_k(x) = -i\left[(x_0\partial_j - x_j\partial_0)V_k(x) - \frac{\partial_k}{4\pi}\int d^3x' \frac{1}{|\mathbf{x}-\mathbf{x}'|}\partial_0 V_j(\mathbf{x}', x^0)\right]. \tag{71}$$

In principle these infinitesimal transformations can be exponentiated to get the effect of a general finite element of $\mathscr{P}$, whose space-time action is given by

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \tag{72}$$

on any analytic signal $\mathbf{V}\in\mathscr{M}$. Let us denote the image of $\mathbf{V}$ under the action of $(\Lambda, a)\in\mathscr{P}$ by $\mathbf{V}'$:

$$(\Lambda, a)\in\mathscr{P}: \mathbf{V}(x)\in\mathscr{M} \to \mathbf{V}'(x)\in\mathscr{M}. \tag{73}$$

The significant classical statements are these: (a) the passage $\mathbf{V}\to\mathbf{V}'$ is a canonical transformation, in which context P_μ changes its character from being merely the geometric object appearing in (71) to a dynamical quantity [(2) and (4)] generating the transformation through the PB (and analogously for J_j and K_j); (b) the classical norm is preserved,

$$\|\mathbf{V}'\| = \|\mathbf{V}\|. \tag{74}$$

Even in cases where the infinitesimal transformations are unbounded and take states in $\mathscr{M}$ to those outside, the finite transformations (Λ, a) are well defined on $\mathscr{M}$.

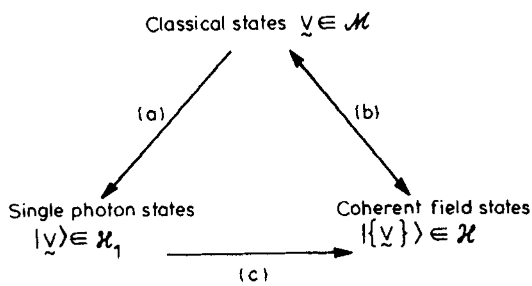
Upon quantization, we obtain a unitary action of $\mathscr{P}$ on the Hilbert space $\mathscr{H}$ carrying the states of the photon field. The unitary operators $\bar{U}(\Lambda, a)$ representing elements of $\mathscr{P}$ realise the composition law of the group. But the action of $\bar{U}(\Lambda, a)$ on the states $|\mathbf{V}\rangle$ and $|\{\mathbf{V}\}\rangle$ we have introduced is already determined by the classical action of $\mathscr{P}$ on $\mathscr{M}$: in a sense this action is 'lifted' from classical to quantum theory and then reinterpreted. The classical (canonical) change translates into the unitary changes

$$\bar{U}(\Lambda, a)|\mathbf{V}\rangle = |\mathbf{V}'\rangle,$$
$$\bar{U}(\Lambda, a)|\{\mathbf{V}\}\rangle = |\{\mathbf{V}'\}\rangle, \tag{75}$$

on one photon states in $\mathscr{H}_1$ and on coherent states in $\mathscr{H}$ respectively.

8. Conclusion

The various relationships we have traced can be conveyed by a three-cornered diagram:



Link (a) is a linear unitary mapping connecting the classical norm$\|\cdot\|$ in $\mathcal{M}$ with the quantum mechanical inner product $\langle\cdot|\cdot\rangle$ among single-photon states; link (b) takes each classical state $\mathbf{V}$ to its closest physical analogue in quantum theory, $|\{\mathbf{V}\}\rangle$ in $\mathcal{H}$. Link (c) is the exponential mapping producing $\mathcal{H}$ out of $\mathcal{H}_1$ and associating one coherent state $|\{\mathbf{V}\}\rangle$ with each one-photon state $|\mathbf{V}\rangle$. These interconnections 'commute with' the Poincaré group actions; the classical action given on $\mathcal{M}$ is carried faithfully by the Links (a) and (b) to the other corners of the diagram, and these results are respected by (c).

All this should bring home quite clearly the relevance and indispensability of the classical equations and their solutions in understanding the physical properties of photons and the quantized field. One is reminded in this context of the words of Heinrich Hertz:

'One cannot escape the feeling that these mathematical formulae have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than was originally put into them'.

Among typically quantum concepts and phenomena, the photon has remained a most elusive entity with a truly fairly-like intangibility. It can justifiably be said that the superposition principle is a gift offered by quantum mechanics to compensate for the loss of classical visualizability caused by the non-commutativity of dynamical variables. In a similar spirit one might say that the mathematical elegance and cogency of the formalism of the quantum mechanics of the Maxwell field, facets of which we have tried to expose, are gifts to compensate for our inability to really grasp the 'ultimate nature' of light. One can do no better at this point than recall Einstein's words:

'All the fifty years of conscious brooding have brought me no closer to the answer to the question. 'What are light quanta?' Of course today every rascal thinks he knows the answer, but he is deluding himself':

Dedication

Dr D S Kothari is the grand sire of Indian physics today. He has been a teacher and guide to several generations of physicists in the country, all of whom remember him with gratitude and affection. His pedagogical skills and commitment to excellence have been unmatched in our times; and his philosophical concerns with relativity and quantum theory, against the background of a thorough knowledge of our own traditions, have been deep and profound. On the occasion of his eightieth birthday, it is with respect and admiration that we offer this pedagogical essay to him, with the hope that at the very least it might amuse him.

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