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Physical approach to cosmological homogeneity II

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Abstract. The question of cosmological homogeneity was earlier studied from the physical point of view, the consideration being limited to the case where the velocity vector is orthogonal to the pressure constant hypersurfaces. The present paper extends the investigation where this condition may not hold true. However the proof of homogeneity requires the introduction of some additional assumptions.

Keywords. Cosmological homogeneity; general relativity.

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1. Introduction

In cosmological investigation spatial homogeneity is a very commonly used idea. One may conceive of this either in terms of uniformity of physical variables over a suitably defined three space or as a geometric equivalence of different points in a three space as obtains with the existence of a suitable group of isometrics. The field equations of general relativity link up these two ideas and in the past the usual practice had been to start with the existence of a transitive group of motions G_3 as this allows an elegant mathematical framework. Recently this problem was inverted (Raychaudhuri and Modak 1985) and starting with the assumptions of a functional relation between the pressure p and density ρ of the fluid, it was shown that if the velocity vector of the fluid be orthogonal to the p-constant hypersurfaces (which are because of functional relation also ρ-constant) and the eigenvectors of the shear tensor be parallely transported along the velocity vector, they could deduce the existence of a three-parameter group of motions admitted by the p-constant hypersurfaces. However the condition of orthogonality of the velocity vector to the p-constant hypersurfaces left out the cases where vorticity is present or the velocity vector although hypersurface orthogonal, is not orthogonal to the orbits of the group (the so-called tilted universe case). In the present paper we take up these cases and show that the existence of a group of motions with three linearly independent Killing vectors can be demonstrated if the following conditions hold good: (a) The matter is a perfect fluid with an equation of state of the form $p = p(\rho)$. (b) The scalar product of the velocity vector v^μ and the unit normal K^μ to the ρ-constant hypersurfaces is constant over these hypersurfaces i.e.

$$v_\mu K^\mu = f(\rho). \tag{1}$$

(c) The eigenvectors of the shear tensor $S_{\mu\nu}$ corresponding to the vector K_μ are parallely propagated along K^μ. The shear tensor

$$S_{\mu\nu} = \tfrac{1}{2}(K_{\mu;\nu} + K_{\nu;\mu}) - \frac{\bar{\theta}}{3}(g_{\mu\nu} - K_\mu K_\nu) - \tfrac{1}{2}[(K_{\mu;\alpha}K^\alpha)K_\nu + K_\mu(K_{\nu;\alpha}K^\alpha)] \quad (2)$$

has K_μ as an eigenvector. We are assuming this property of parallel transport about the other three eigenvectors i.e.

$$\xi^\mu_{a;\alpha}K^\alpha = 0 \quad (3)$$

with a suitable normalization of ξ^μ_a. Here **a** numbers the eigenvectors of $S_{\mu\nu}$, thus **a** runs over 1, 2, 3.

(d) The velocity vector v^μ lies in the two space spanned by the vector K^μ and one of the eigenvector ξ^μ_a, say ξ^μ_1 i.e.

$$v^\mu = \alpha_0 K^\mu + \alpha_1 \xi^\mu_1 \quad (4)$$

(e) The divergence of the vector $K^\mu_{;\mu}(\bar{\theta})$ is constant over the hypersurfaces i.e.

$$\bar{\theta}_{,\mu}(\delta^\mu_\alpha - K^\mu K_\alpha) = 0 \quad (5)$$

we are taking the signature $(+ - - -)$.

2. Some consequences of our assumptions

In the case of non-degenerate eigenvalues of $S_{\mu\nu}$ the eigenvectors will be mutually orthogonal; otherwise we can choose them to be mutually orthogonal so that in particular

$$K_\mu \xi^\mu_a = 0. \quad (6a)$$

Equation (6a) along with the assumptions (c) leads to

$$(K_{\mu;\alpha}K^\alpha)\xi^\mu_a = 0. \quad (6b)$$

Again $$(K_{\mu;\alpha}K^\alpha)K^\mu = 0. \quad (6c)$$

Thus from (6a, b, c) one has

$$K_{\mu;\alpha}K^\alpha = 0. \quad (7a)$$

Since K_μ is hypersurface orthogonal it must be rotation-free i.e.

$$K_{[\mu}K_{\nu;\alpha]} = 0. \quad (7b)$$

Equation (7a) shows that the hypersurfaces are geodesically parallel, then one can write

$$dS^2 = dx^{0^2} + g_{ik}\,dx^i\,dx^k, \quad (8)$$

where the vector K^μ is tangential to the x^0 lines $K^\mu = \delta^\mu_0$. The x^0 constant three spaces being the ρ-constant hypersurfaces, ρ and p are functions of x^0 alone.

From the vanishing of the divergence of the energy-momentum tensor one has

$$\frac{p_{,\mu}}{\rho+p} = \frac{(p_{,\alpha}v^{\alpha})}{\rho+p} v_{\mu} + \dot{v}_{\mu}, \tag{9a}$$

and
$$(\rho+p)\theta = -\rho_{,\mu}v^{\mu}, \tag{9b}$$

where $\theta = v^{\mu}_{;\mu}$. Obviously $\rho_{,\mu}$ is in the same direction as K_{μ}, hence we may write $\rho_{,\mu}/(\rho+p) = \lambda K_{\mu}$, where λ is a function of x^0 alone. We then have from (9a, b)

$$K_{\mu} = -\frac{\theta}{\lambda} v_{\mu} + \frac{1}{\lambda}(\mathrm{d}p/\mathrm{d}\rho)^{-1}\dot{v}_{\mu} \tag{9c}$$

and
$$K_{\mu}v^{\mu} = -\frac{\theta}{\lambda}. \tag{9d}$$

Hence from (1) θ is function of x^0 alone. Again from (5) $\bar{\theta}$ is function of x^0 alone. The Raychaudhuri equation for the normal congruences K^{μ} reads

$$\bar{\theta}_{,\mu}K^{\mu} + \bar{\theta}^2/3 + 2S^2 + 8\pi(\rho+p)(v_{\alpha}K^{\alpha})^2 - 4\pi(\rho-p) = 0 \tag{10}$$

shows that S^2 is function of x^0 alone as $\bar{\theta}$, ρ, p, $(v_{\mu}K^{\mu})$ are all functions of x^0 alone, which means

$$S_1^2 + S_2^2 + S_3^2 = f(x^0), \tag{10a}$$

where S_1, S_2, S_3 are the eigenvalues of $S_{\mu\nu}$ corresponding to ξ_1^{μ}, ξ_2^{μ}, ξ_3^{μ}; the other eigenvalue being zero. Since the trace of $S_{\mu\nu} = 0$, then

$$S_1 + S_2 + S_3 = 0. \tag{10b}$$

Then from (9d) and (4)

$$\alpha_0 = -\frac{\theta}{\lambda},\ \alpha_1^2 = \alpha_0^2 - 1 = \frac{\theta^2-\lambda^2}{\lambda^2}. \tag{11}$$

Thus α_0 and α_1 are functions of x^0 alone. From (4) we get

$$\dot{v}_{\mu} = v_{\mu;\alpha}v^{\alpha} = (\alpha_{0,\alpha}v^{\alpha})K_{\mu} + (\alpha_{1,\alpha}v^{\alpha})\xi_{1\mu} + \alpha_0\alpha_1 K_{\mu;\alpha}\xi_1^{\alpha} + \alpha_1^2\xi_{1\mu;\alpha}\xi_1^{\alpha}. \tag{12}$$

Now introducing the Ricci rotation coefficients with the tetrad (K^{μ}, ξ_a^{μ})

$$\gamma_{ABC} = \xi_{A_{\mu;\alpha}}\xi_B^{\mu}\xi_C^{\alpha},$$

where A, B, C runs from 0 to 3 and a, b, c runs from 1 to 3; ξ_0^{μ} is just K^{μ}. From (9c) and (4) $\dot{v}_{\mu}$ is orthogonal to ξ_2^{μ} and ξ_3^{μ}, hence from (12)

$$\gamma_{211} = 0 = \gamma_{311}. \tag{12a}$$

Also, contracting (12) with K^{μ}

$$\gamma_{011} = \frac{\alpha_0\dot{\alpha}_0}{\alpha_1^2} - \frac{(\dot{v}_{\mu}K^{\mu})}{\alpha_1^2} \tag{12b}$$

From (9c) $\dot{v}_\mu K^\mu = \lambda\left(1-\frac{\theta^2}{\lambda^2}\right)\cdot\frac{\mathrm{d}p}{\mathrm{d}\rho}.$ (12c)

From (2) and (7a, b)

$$\gamma_{0ab} = -\left(S_a+\frac{\bar{\theta}}{3}\right)\delta_{ab}. \tag{13}$$

Now using (12c) and (13) in (12b) we get

$$S_1+\frac{\bar{\theta}}{3} = \frac{\alpha_0\dot{\alpha}_0}{1-\alpha_0^2}+\frac{\theta}{\alpha_0}\cdot\frac{\mathrm{d}p}{\mathrm{d}\rho}. \tag{14}$$

In view of (5) S_1 is function of x^0 alone, then from (10a, b) one can show S_2, S_3 are also functions of x^0 alone.

3. Proof of the theorem

We shall now proceed to show that the following results hold:

$$-X_bX_a+X_aX_b = C^e_{ab}X_e, \tag{15a}$$

$$C^e_{ab}C^d_{ef}+C^e_{bf}C^d_{ea}+C^e_{fa}C^d_{eb} = 0, \tag{15b}$$

where

$$X_a = \xi^i_a\frac{\partial}{\partial x^i},$$

and C^a_{bc}'s are functions of x^0 alone. The C^a_{bc}'s are related to the Ricci rotation coefficient γ_{abc}. In orthonormal tetrad

$$C^a_{bc} = \gamma_{acb}-\gamma_{abc}. \tag{16}$$

We now consider field equations. The G_{0i}'s equations are

$$S^\alpha_{\mu;\alpha}-\tfrac{2}{3}\bar{\theta}_{,\alpha}(\delta^\alpha_\mu-K^\alpha K_\mu)+2S^2K_\mu = 8\pi(\rho+p)(v_\alpha K^\alpha)[v_\mu-(v_\alpha K^\alpha)K_\mu]. \tag{17}$$

Contracting (17) with ξ^μ_a and using (5) and constancy of S_a's we get

$$\sum_b(S_a-S_b)\gamma_{abb} = -8\pi(\rho+p)\alpha_0\alpha_a, \tag{18}$$

where a, b runs from 1 to 3 and $\alpha_1 \neq 0$, $\alpha_2 = \alpha_3 = 0$.
Then we have

$$(S_1-S_2)\gamma_{122}+(S_1-S_3)\gamma_{133} = -8\pi(\rho+p)\alpha_0\alpha_1, \tag{19a}$$

$$\gamma_{233} = \gamma_{322} = 0. \tag{19b}$$

Again taking divergence of (4) we have

$$\xi^\mu_{1;\mu} = \gamma_{122}+\gamma_{133} = \text{function of } x^0 \text{ alone}, \tag{19c}$$

as $\theta, \bar{\theta}, \alpha_0, \alpha_1$ are functions of x^0 alone. From (19a,c) γ_{122} and γ_{133} are functions of x^0 alone

i.e. $$\gamma_{122,K} = \gamma_{133,K} = 0. \tag{20}$$

The other six field equations are

$$R^*_{iK} = -\bar{\theta}S_{iK} - S_{iK;\alpha}K^\alpha - 8\pi(\rho+p)v_i v_K + \frac{1}{3}g_{iK}\left[8\pi(\rho-p) + 8\pi(\rho+p)(v_\alpha K^\alpha)^2 + 2S^2 - \frac{2\bar{\theta}^2}{3}\right] \tag{21}$$

where R^*_{iK} are the three space Ricci tensor. Introducing γ_{abcd} defined by

$$\gamma_{abcd} = R^*_{ijkl}\xi^i_a \xi^j_b \xi^K_c \xi^l_d.$$

We have from (21)

$$\sum_a \gamma_{abca} = \Theta_b \delta_{bc} - 8\pi(\rho+p)\alpha_b\alpha_c, \tag{22}$$

where Θ_b's are functions of x^0 alone and $\alpha_1 \neq 0$, $\alpha_2 = \alpha_3 = 0$. Using symmetry property of γ_{abcd} we have

$$\gamma_{abba} = \text{function of } x^0 \text{ alone}, \tag{22a}$$

$$\gamma_{abca} = \gamma_{acba} \quad (b \neq c), \tag{22b}$$

where $a = 1, 2, 3$
$b = 1, 2, 3$ and $a \neq b$.

Using (12a), (19b) and (20) in (22a) we can show that $\gamma_{123}, \gamma_{231}$ and γ_{312} are functions of x^0 alone. Thus all non-vanishing γ_{abc}'s are functions of x^0 alone. The relation (22b) can be simplified for $a = 1$ to

$$\sum_m (C^1_{2m}C^m_{31} + C^1_{1m}C^m_{23} + C^1_{3m}C^m_{12}) = 0,$$

which is one of the Jacobi identities (15b). The other Jacobi identities are obtained by taking $a = 2$ and 3 in (22b).

We have defined γ_{abc} hence C^a_{bc} (from (16)) with respect to ξ^μ_a and shown that they are constant and satisfy the Jacobi identity. Hence ξ^i_a's are the generators of a three-parameter group.

Now using (21) one can show that ξ^i_a's are the eigenvectors of R^*_{iK} with constant eigenvalues. Then one can show that ξ^i_a's are the reciprocal group vectors of a transitive group as in the paper (Raychaudhuri and Modak 1985) (see also Eisenhert 1945). It is now easy to show the existence of the three linearly independent spacelike Killing vectors admitted by the ρ-constant hypersurfaces.

4. Concluding remarks

It is pertinent to ask how far the assumptions that we have introduced are necessary. It is easy to see that all conditions other than (d) are necessary for the homogeneity of the ρ-constant hypersurfaces. Regarding (d) we may note that as far as the present author is aware, this condition is invariably satisfied in all the homogeneous rotating (or tilted) solutions presented in the literature. Apparently the reason for this as well as our introduction of this assumption is that otherwise the situation becomes too complicated. However we have not been able to show that this is a necessary condition.

In general, dropping the assumption (d), if one replaces the assumption (b) by (b*)

$$v_\mu \xi^\mu_a = f_a(\rho),$$

where a runs from 1 to 3, i.e. the scalar product of the velocity vector with all the three eigenvectors of the $S_{\mu\nu}$ are functions of x^0 alone, the eigenvectors being normalized as in our previous consideration, then we can construct three linearly independent spacelike eigenvectors of R^*_{iK} which are linear combination of ξ^i_a's with constant coefficient.

Once this is proved, we can show the existence of spatial homogeneity under the assumptions (a) (b*) and (c) in the case of tilted universe (i.e. the vorticity $\omega = 0$). However the result cannot apparently be generalized to the case where $\omega \neq 0$.

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