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Realization of a unique time evolution unitary operator in Klein Gordon theory

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Abstract. The scattering theory for the Klein Gordon equation, with time-dependent potential and in a non-static space-time, is considered. Using the Klein Gordon equation formulated in the Hilbert space $L^2(\mathbf{R}^3)$ and the Einstein's relativistic equation in the space $L^2(\mathbf{R}^3, \mathrm{dx})$ and establishing the equivalence of the vacuum states of their linearized forms in the Hilbert space $L^2(\mathbf{R}^3)$ with the help of unique symmetric symplectic operator, the time evolution unitary operator $U(t)$ has been fixed for the Klein Gordon equation, incorporating either the positive or negative frequencies, in the infinite dimensional Hilbert space $L^2(\mathbf{R}^3)$.

Keywords. Physical vacuum states; linearized Klein Gordon equation; linearized Einstein's relativistic equation; symplectic operator.

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1. Introduction

In this paper our aim is to find a unique time evolution unitary operator for the Klein Gordon equation in the infinite dimensional Hilbert space $L^2(\mathbf{R}^3)$. It is known that the Klein Gordon equation, being second order in time, does not by itself warrant the existence of such an operator, mainly due to the simultaneous presence of the positive and negative frequencies in its solutions. A way, then, has to be found to weed out one of these frequencies from the solutions, say for definiteness, the negative frequencies. Hopefully, we could do so and ultimately go to obtain the time evolution operator by considering the scattering theory for the Klein Gordon equation with the time-dependent potential and in a nonstatic space-time. The case, in favour, is that both the Klein Gordon equation formulated in the Hilbert space $L^2(\mathbf{R}^3)$ and the Einstein's relativistic equation of flat space formulated in the Hilbert space $L^2(\mathbf{R}^3, \mathrm{dx})$ are second order differential equations. The first problem is to find out some equivalent solution of the two equations with a worthwhile operator. The equivalent solution works out to be the physical vacuum states of the linearized Klein Gordon equation (LKGE) and the linearized form of the Einstein's relativistic equation (LERE) respectively. The worthwhile operator is the symplectic operator.

In § 2, the exact vacuum solutions of LERE and LKGE and their symplectic operators Ω_{LERE} and Ω_{LKGE} are determined. Note that the symplectic operator exists, if $A(1)$: it be symmetric, corresponds to singular perturbations and $A(2)$: the order of approximation considered is $O(\varepsilon^2)$, where ε is a small positive expansion parameter of the background metric g_{ab} and the Killing vector k_a

$$g_{ab} = g_{ab} + \varepsilon h_{1ab} + \varepsilon^2 h_{2ab},$$
$$k_a = k_a + \varepsilon k_{1a} + \varepsilon^2 k_{2a}, \tag{1}$$

where $\underset{1}{h}_{ab}$, $\underset{2}{h}_{ab}$, are the first and second order changes of the metric along the family $g_{ab}(\lambda)$ evaluated at $|\lambda| = 0$ and $\underset{1}{k}_a$, $\underset{2}{k}_a$ are the corresponding first and second order changes of the Killing vector k_a.

In §3, we show that the symplectic operators Ω_{LERE} and Ω_{LKGE} are equivalent through the equality of their energy momentum tensors T_{ab} and the equivalence of the respective spaces of formulation of LERE and LKGE viz $L^2(\mathbf{R}^3, \text{dx})$ and $L^2(\mathbf{R}^3)$.

As the energy momentum represents the same scalar field, be it either formulated in the Hilbert space $L^2(\mathbf{R}^3, \text{dx})$ with respect to LERE or in the Hilbert space $L^2(\mathbf{R}^3)$ with respect to LKGE, their equality, under the mathematical assumptions A(3), A(4), leads to part of the requirements (equation (19)) for the equivalence of the symplectic operators Ω. The other part of the equivalence of the two spaces is established by expanding the space $L^2(\mathbf{R}^3)$ to $L^2(\mathbf{R}^3, \text{dx})$.

In §4, we have formulated the unitary time evolution operator for KG in terms of the physical vacuum solution in the Hilbert space $L^2(\mathbf{R}^3)$. For this, we have extended our Hilbert space $L^2(\mathbf{R}^3)$, which is in fact the real Hilbert space $\mathscr{H}_c(\mathscr{H}_c = \mathscr{H}_R \oplus i\mathscr{H}_R)$, for decomposing the positive and negative frequencies in the complex Hilbert space $\mathscr{H}_c$. First we have decomposed the frequencies of KG and shown that they are unique. For the existence of the symplectic operator Ω, in the extended Hilbert space $L^2(\mathbf{R}^3, \text{dx})$ of $L^2(\mathbf{R}^3)$, we have considered the inequality (25), which stipulates the existence of the scattering operators. Subsequently, we have, in the interaction picture in the extended Hilbert space $L^2(\mathbf{R}^3, \text{dx})$, constructed the time evolution unitary operator $U(t)$, expression (35) that contains only one type of frequencies (the positive ones). This $U(t)$ is unique because the S operator is identified as the symplectic operator Ω in the interaction picture in (31). This symplectic operator Ω is orthogonal to any arbitrary vector in $\mathscr{H}_c$.

2. Physical vacuum states of LERE and LKGE and their symplectic operators

2.1 *Physical vacuum states*

To obtain the physical vacuum states we have to linearize Einstein's relativistic and Klein Gordon equations. Einstein's relativistic equation, formulated in the Hilbert space $L^2(\mathbf{R}^3, \text{dx})$, is of the form

$$R_{ab}[g_{ab}(\lambda)] = 0, \tag{2}$$

where λ is the parameter. To linearize it, differentiate (2) with respect to λ and put $|\lambda| = 0$, i.e.

$$(\text{d}/\text{d}\lambda)\,(R_{ab}(g_{ab}(\lambda))_{|\lambda|=0} = 0. \tag{3}$$

We, then, get the following LERE

$$\nabla^m \nabla_m h_{ab} - 2\nabla^m \nabla(ahb)m + \nabla_a \nabla_b (g^{mn} h_{mn}) = 0, \tag{4}$$

where, $g_{ab} = g_{ab}(0)$ is the background metric:

$$\begin{aligned} g_{ab} &= \left[1 + \frac{\text{d}}{\text{d}\lambda}(g_{ab}(\lambda))_{|\lambda|=0}(\text{t}, \mathbf{r})\,\text{d}s^2\right] \\ &= [1 + h_{ab}(\text{t}, \mathbf{r})\,\text{d}s^2] \end{aligned} \tag{5}$$

on using the expansion (1) and $ds^2 = dt^2 - dx^2$. Consider now the Klein Gordon equation in the Hilbert space $L^2(\mathbf{R}^3)$

$$(\Box + m^2)\phi = 0. \tag{6}$$

Compare (6) with the linear equation of the form

$$D^{AB}\phi^B = 0, \tag{7}$$

where, D^{AB} is a second order linear differential operator, viz

$$D^{AB} = X^{AB\,ab}\nabla_a\nabla_b + Y^{AB\,a}\nabla_a + Z^{AB}. \tag{8}$$

Comparison of expressions of (8) with corresponding expressions of (6), yields the symmetric conditions (as outlined in criteria A(1) of § 1)

$$X^{AB} = G_{ab}, \quad Y^{AB} = 0, \quad Z^{AB} = m^2, \tag{9}$$

and the LKGE becomes

$$G_{ab}{}^{ab}\nabla_a\nabla_b + m^2 = 0. \tag{10}$$

Let (M', g_{ab}) be an exact vacuum solution of LERE and l_a be a null vector field, such that $h_{ab} = l_a l_b$ (h_{ab}, a linearized field in (M', g_{ab}) satisfies the LERE). Then $g'_{ab} = g_{ab} + l_a l_b$ is an exact vacuum solution of LERE (4) and its inverse is

$$g'_{ab} = g^{ab} - l_a l_b.$$

For the LKGE, (10), for (M, G_{ab}), the exact vacuum solutions, we arrive at the expression

$$\underset{\text{LKGE}}{G'_{ab}} = G_{ab} + L_a L_b.$$

To prove the uniqueness of these exact vacuum solutions, we should show that $G'_{ab} = g'_{ab} = G_{ab}$ (say), which can be achieved with the help of a unique symplectic operator Ω. For that, we have to find the symplectic operators for LKGE and LERE.

2.2 *Symplectic operator*

To construct the symplectic operator, first consider the Klein Gordon (6) and the symmetric conditions, (9). Then the inner product of ϕ_A with ϕ_B will be

$$(\phi_A, \phi_B) = \int \phi^A\phi^B \,dv \tag{11}$$

on space time (M, G_{ab}), where ϕ^A and ϕ_B are fields of the inner product space of tensor fields of valence $|A|$ with compact 4-volume v with boundary Σ_a. Using Stoke's theorem, the operator D^{AB} will be proved to be symmetrical on the domain

$$(\phi^A, D_{AB}\phi^B)_M = (D_{AB}\phi^A, \phi^B)_M, \tag{12}$$

on putting the boundary condition for the region to be compact, we find

$$\int_v (\phi^A D_A \phi_B - \phi^B D_A \phi_A)\,dv$$
$$= 0 = \int_{\Sigma_a} \{\phi^A G_{ab}{}^{ab}\nabla_a\phi^B - \phi_A G_{ab}{}^{ab}\nabla_a\phi_B\}\,d\Sigma_a, \tag{13}$$

where, (11) is the Σ_a-surface integral. Hence, the symplectic operator for LKGE reads

$$\Omega_{\text{LKGE}}(\phi, E\phi) = [\, 2 \int_{\Sigma_a} \{\nabla^a\phi\nabla_b\phi - \tfrac{1}{2}\delta_a^b(\nabla_e\phi\nabla^e\phi - m^2\phi)\}\, k^b\, d\Sigma_a], \tag{14}$$

and similarly for the LERE, it works out to be

$$\Omega_{\text{LERE}}(\psi, E\psi) = 2 \int_{\Sigma_a} -\beta[2\psi'^a\psi'_b - g_{ab}(\psi'_c\psi'^c + m^2\psi)k^b\, d\Sigma_a], \tag{15}$$

where β, m^2 are positive constants related to the mass of the field and the choice of the units respectively. Note that for arriving at (14) and (15), use is made of the criteria A(1) and A(2) of the § 1 of the existence of symplectic operator $\Omega(.,.)$. Now we have to show that the exact vacuum solutions of LKGE and LERE are equivalent and unique.

3. Equivalence of the physical vacuum states of LERE and LKGE

In this section first we have to show that the symplectic operators for LERE Ω_{LERE} and LKGE Ω_{LKGE} are equivalent though the equality of the energy momentum tensors, T_{ab}. Because the energy momentum tensors, T_{ab}, represent the same scalar fields, whether they be formulated in either the Hilbert space $L^2(\mathbf{R}^3, dx)$ with respect to the LERE or the Hilbert space $L^2(\mathbf{R}^3)$ with respect to LKGE. The question that arises here is the following: Under what conditions the tensors, T_{ab}, are identical? To answer that, consider the space time (M, g_{ab}) which admits a Killing vector k_a. Then,

$$\nabla_{[a}\nabla_{b]}k_a = \tfrac{1}{2}R^{m}_{aba}k_m$$
$$k_a = k^m_{amb}, \quad R = R^m_m. \tag{16}$$

Using (4) and (5), the energy momentum tensor satisfies, by virtue of the Killing vector, k_a, $\nabla_a T^{ab} = 0$ or equivalently the integrable equation $\nabla_a(T^a_b k^b) = 0$. Hence for the scalar field, the energy-momentum tensor for LERE becomes

$$T^{ab}_{\text{LERE}} = -\beta[-2\psi'^a\psi'^b - g_{ab}\{\psi'^c\psi'^c + m^2\psi^2\}]. \tag{17}$$

Note that the energy momentum tensorial form of the LERE becomes $T'_{ab} = 0$, if $T_{ab} = 0$, where T'_{ab} is in the space of g'_{ab}, viz $L^2(\mathbf{R}^3, dx)$. Similarly for the scalar field, the energy momentum tensor for LKGE reduces to

$$T^{ab}_{\text{LKGE}} = [\nabla^a\phi\nabla_b\phi - \tfrac{1}{2}\delta^a_b(\nabla_c\phi\nabla^c\phi - m^2\phi)]. \tag{18}$$

The expressions (17) and (18) should be identical under the mathematical assumptions: $A(3)$. For each function $f \in C_0^\infty(\mathbf{R}^3)$ taking values in C^4, associate an operator $\psi(f)$, $(\psi(f))$ corresponds to

$$\sum_\alpha \int f_\alpha(\mathbf{x})\tilde{\psi}_\alpha(\mathbf{x}, 0)\, d_3\mathbf{x},$$

and $A(4)$. The anti-commutation relation $\{\psi(f), \psi^*(g)\} = (f, g)$ holds for arbitrary f, g (the inner product is taken in $L^2(\mathbf{R}^3)\otimes C^4$). For the relativistic case considered, we have

$$\{\psi(f), \psi^*(g)\} = B(f, g)$$

and $$B(f, g) = B(f_k, g_k),$$

where k is the Lorentz transformation. Now comparing $T^{ab}_{\text{LERE}} = T^{ab}_{\text{LKGE}}$, on equating corresponding terms of the right hand sides of expressions (17) and (18) (with the help of the mathematical assumptions A(3), A(4)) would entail conditions, which, when expressed in the surface integral form, amount to the following:

$$\left.\begin{aligned} &\text{(i)} && \int_{\Sigma_a} \phi^2 \, \mathrm{d}\Sigma_a = \beta \int_{\Sigma_a} \psi^2 \, \mathrm{d}\Sigma_a \\ &\text{(ii)} && \int_{\Sigma_a} \beta(\psi'^a \psi'^b) \mathrm{d}\Sigma_a = \int_{\Sigma_a} (\nabla^a \phi \nabla_b \phi) \mathrm{d}\Sigma_a \\ & && \text{with } \delta^a_b = -g_{ab}. \end{aligned}\right\} \tag{19}$$

These are part of the requirements for the equality of the symplectic operators

$$\Omega_{\text{LERE}} \text{ and } \Omega_{\text{LKGE}},$$

$$\Omega_{\text{LERE}} = \Omega_{\text{LKGE}} = \Omega(\,.\,,.\,) \text{ (say)}. \tag{20}$$

Here, both the symplectic operators Ω_{LERE} and Ω_{LKGE} operate in different Hilbert spaces, i.e. Ω_{LERE} in $L^2(\mathbf{R}^3, \mathrm{dx})$ and Ω_{LKGE} in $L^2(\mathbf{R}^3)$ respectively. Since we need to establish the equivalence of the symplectic operators in the single Hilbert space $L^2(\mathbf{R}^3)$, either we have to expand the space of $L^2(\mathbf{R}^3)$ to $L^2(\mathbf{R}^3, \mathrm{dx})$ or we have to contract the Hilbert space $L^2(\mathbf{R}^3, \mathrm{dx})$ to $L^2(\mathbf{R}^3)$. We adopt the former method. Let us assume that M' is manifold equivalent to $L^2(\mathbf{R}^3, \mathrm{dx})$ and M is the manifold equivalent to $L^2(\mathbf{R}^3)$. We have already shown that (M', g_{ab}) is an asymptotically flat-space-time and is exact vacuum solution of Einstein's equation. We can say that the manifold

$$M' = M \cup \mathscr{I}, \tag{21}$$

with smooth metric and scalar symplectic function Ω in M, and the associated fields being in conformally coupled space-time and extension vacuum fields of g'_{ab}, g_{ab}. Then Ωl_a, and n_a are both null and orthogonal on $\mathscr{I}$. Then, for

$$\left.\begin{aligned} &g'_{ab} = (\Omega)^2 g_{ab} \text{ on } \mathscr{I}\Omega = 0 \\ &n_a = \nabla'_a \Omega. \end{aligned}\right\} \tag{22}$$

h_{ab} is said to preserve asymptotically the flatness of first order, if Ω^2 admits a smooth extension to $\mathscr{I}$, such that

$$\Omega h_{ab} n'_a n'^b |_{\mathscr{I}} = 0. \tag{23}$$

Now, in our case the $L^2(\mathbf{R}^3, \mathrm{dx})$ is considered as extension space of $L^2(\mathbf{R}^3)$. The new boundary conditions, on extending the manifold from M to M', are $g'_{ab} = \Omega^2 g_{ab}$ on $\mathscr{I}\Omega = 0$, $n_a = \nabla'_a \Omega$. They are non-zero, null and

$$\nabla_a n_a = 0. \tag{24}$$

Thus we see that the symplectic operators Ω_{LERE} and Ω_{LKGE} are equivalent and the vacuum solutions are also the same, i.e.

$$\underset{\text{LERE}}{g'_{ab}} = \underset{\text{LKGE}}{G'_{ab}} = G_{ab} \text{ (say)}$$

in Hilbert space $L^2(\mathbf{R}^3)$ under the the conditions (24) and mathematical assumptions A(3), A(4).

To show that g'_{ab} is unique in the Hilbert space $L^2(\mathbf{R}^3)$, consider l_a to be a null vector field, such that $h_{ab} = l_a l_b$ (h_{ab}, a linearized field on (M', g_{ab}) satisfies the LERE). Then $g_{ab} + l_a l_b$ is an exact vacuum solution of LERE. Set $x^a = l^m \nabla_m l^a$. Contracting $l^a l^b$, we obtain $x^a x_a = 0$ which implies $l_a x^a = 0$, using the fact that l^a is null. Thus l^a must be a geodesic. We can write (4) as

$$\nabla^m \nabla_m (l_a l_b) - 2\nabla^m \nabla_{(a}[l_{b)} l_m] = 0.$$

Then g'_{ab} is non-degenerate and an inverse of $g'_{ab} = g^{ab} - l_a l_b$, where $l^a = g^{am} l_m$ exists. Let ∇'_a denote the derivative operator compatible with g'_{ab}. Due to l^a being null, the connection tensor $C^m_{ab} = 0$ relates the two derivatives ∇_a and ∇'_a respectively. The connection tensor C^m_{ab} is zero, so the exact vacuum solutions g'_{ab} will be identical. The connection tensor C^m_{ab} is zero because of the nullness of l^a. This shows the uniqueness of the exact vacuum solutions of LERE and LKGE.

4. Formulation of the time evolution operator for the Klein-Gordon equation

In this section we formulate the unique unitary time evolution operator for KG. For this, we adopt the method jotted below: $A(5)$: Extend the real Hilbert space $\mathscr{H}_R$ to the complex space $\mathscr{H}_c$, viz $\mathscr{H}_c = \mathscr{H}_R \oplus i\mathscr{H}_R$, so that $\mathscr{H}_R \subset \mathscr{H}_c$. This facilitates splitting of the positive and negative frequencies.

$A(6)$: Consider the inequality, which is uniformly bounded:

$$\int_{-\alpha}^{\infty} \|F(t, .)\|_\infty \, \mathrm{d}t < \infty, \tag{25}$$

where, $\|\cdot\|_\infty$ denotes the L_∞ norm over space $\mathscr{H}_R$. This inequality ensures that the scattering operator S exists. Split the positive and negative frequencies of KG. For that, consider the KG equation in $L^2(\mathbf{R}^3)$,

$$(\Box + m^2)\phi = 0, \tag{26}$$

where, $\phi(\Phi, \dot{\Phi})$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\Phi \\ \dot{\Phi}\end{pmatrix} = P\begin{pmatrix}\Phi \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(\Phi)\end{pmatrix}, \tag{27}$$

where, $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\Phi \\ \dot{\Phi}\end{pmatrix} = \partial_t^2\phi + \alpha\partial_t\phi + \beta\phi = 0,$$

and $$P = \begin{pmatrix}0 & I \\ \beta & \alpha\end{pmatrix}, \text{ so that } PP^{-1} = I. \tag{28}$$

Now putting $B = (m^2 - \nabla)^{1/2}$ – acting in

$$L^2(\mathbf{R}^3, \mathrm{d}x)_-, \quad C = \mathscr{B}^{1/2} \text{ and}$$

$$Q = \begin{pmatrix} 0 & I \\ \mathscr{B} & 0 \end{pmatrix}, \text{ so that}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \Phi \\ \dot{\Phi} \end{pmatrix} = Q\begin{pmatrix} \Phi \\ \dot{\Phi} \end{pmatrix} \text{and } QQ^{-1} = I. \tag{29}$$

We have $P \equiv Q$, because $P = \beta I$ and $Q = \mathscr{B}^2 I$ are related to Ω and α is null since $\beta \neq 0$.
We can write (27) as

$$\begin{pmatrix} \dfrac{\mathrm{d}}{\mathrm{d}t} \end{pmatrix}\begin{pmatrix} \Phi \\ \dot{\Phi} \end{pmatrix} = \begin{pmatrix} 0 & I \\ P & 0 \end{pmatrix}\begin{pmatrix} \Phi \\ \dot{\Phi} \end{pmatrix} \tag{30}$$

and in the interaction picture, we can write it as

$$\frac{\mathrm{d}}{\mathrm{d}t}(W) = \exp(-tQ)\begin{pmatrix} 0 & 0 \\ -F & 0 \end{pmatrix}\exp(tQ)\,W = J(t)W, \tag{31}$$

where,

$$W(t) = \exp(-tQ)\begin{pmatrix} \Phi \\ \dot{\Phi} \end{pmatrix} \text{satisfies (26).}$$

Then $\qquad \exp(tQ)\,{}^{\pm}G'_{ab},$ $\qquad$ (32)

where, ${}^{\pm}G'_{ab}$ are solutions of (26), are asymptotic to $\begin{pmatrix} \Phi \\ \dot{\Phi} \end{pmatrix}$ in $\mathscr{H}_R$ as $t \to \infty$. Therefore, the scattering operator $(S\,{}^{-}G'_{ab}) = {}^{+}G'_{ab}$ satisfies

$${}^{\pm}G'_{ab} = \lim_{t \to \pm\infty} W(t). \tag{33}$$

This shows that the scattering operator S exists and bounded. In the interaction picture, S is the S-matrix that incorporates the positive and negative frequencies ${}^{\pm}G_{ab} \in \mathscr{H}_{sc}$ (the scattering space) through the W's, viz

$$S = W_+\, W_-^{\dagger} \text{ or } W_+^{\dagger}\, W_-. \tag{34}$$

We write, the time evolution operator $U(t)$ as

$$U(t) = \exp(-itH_0)\, W_+^{\dagger} \exp(-itH_0). \tag{35}$$

Using the mathematical assumptions A(3), A(4) of § 3 and the inequality (25), we found out that (31) can be solved in the closed interval $[-\infty, +\infty]$. This means that the Cauchy data for (31) can be specified separately at $t = +\infty$ or $t = -\infty$ respectively. The solutions of the KG lie in the real Hilbert space $\mathscr{H}_R$. The canonical symplectic structure on $\mathscr{H}_R$ is

$$\Omega(\,.\,,\,.\,) = \int_{\xi} (\phi_1 * \phi_2 - \phi_2 * \phi_1)$$

on the Cauchy surface ξ, and $*$ denotes the metric operators of $\mathscr{H}_R$ at the specified time $t = t_0$, we write

$$\Omega(\,.\,,\,.\,) = \int_{t=t_0} (\dot{\phi}_1\phi_2 - \dot{\phi}_2\phi_1)\mathrm{d}^3x,$$

where, ϕ_1, ϕ_2 satisfy (26). With A(5), there are no fixed vectors in $\mathscr{H}_R$. Hence there are no vectors in $\mathscr{H}_c$ fixed under $U(t)$. Suppose that for no vector x in $\mathscr{H}_R$, we have $\Omega(t)x = x$ for all t, then on $\mathscr{H}_R$ we have a complex Hilbertian structure such that the given real inner product becomes the real part of the complex inner product. It follows that there exists a new complex structure $\mathscr{H}_{oc}$ on $\mathscr{H}_c$ such that

$$\mathrm{Re}\,\langle.,.\rangle_{\mathscr{H}_{oc}} = \mathrm{Re}\,\langle.,.\rangle_{\mathscr{H}_c}$$

and that $U(t)$ has strictly positive energy on $\mathscr{H}_{oc}$, since $\mathscr{H}_R$ is invariant under $U(t)$. In fact, $\mathscr{H}_R$ becomes a complex Hilbert space with complex structure

$$\begin{pmatrix} 0 & \mathbf{B}' \\ \mathbf{B} & 0 \end{pmatrix}$$

and complex scalar product $\langle\,.\,,.\,\rangle$. Then $\exp(tQ)$ is a strongly continuous unitary group in $\mathscr{H}_R$. From (32) and (33), we found S is bounded and invertible. This means that S is also symplectic (Paneitz and Stephen 1982), since each $J(t)$ in (31) is in symplectic group.

Equation (35) defines only the positive frequency as having been defined on $\mathscr{H}_R$ and $\mathscr{H}_R(t) \subseteq \mathscr{H}_c$. Thus for any $t \in \mathbf{R}$, $U(t)$ is unitary transformation of $\mathscr{H}_c$ and $\mathscr{H}_R(t)$ with the property

$$s-\lim_{t\to-\infty} U(t) = I, \quad s-\lim_{t\to\infty} U(t) = S.$$

Here $U(t)$ as given by (35), is certainly unique, since $\mathscr{H}_0$ and $W_\pm$ are unique. It is formulated in terms of the vacuum states of LKGE by virtue of the fact that

$$W(t) = \exp(tQ)\begin{pmatrix} \Phi \\ \dot{\Phi} \end{pmatrix}.$$

It is worthwhile to note that for the necessary transformation to go from the scattering space $\mathscr{H}_{sc}$ to the interaction picture space (Xanthopoulas 1978), as embodied in (35), we have, for $\phi_1, \phi_2 \in \mathscr{H}_{sc}$ and $\tilde{S} = W_+^\dagger$,

$$\begin{aligned} \langle \phi_1, \tilde{S}\phi_2 \rangle &= \langle \phi_1, W_-^\dagger W_+ \phi_2 \rangle \\ &= \langle W_-^\dagger \phi_1, W_-^\dagger W_+ W_-^\dagger \phi_2 \rangle \\ &= \langle W_-^\dagger \phi_1, W_-^\dagger S\phi_2 \rangle \\ &= \langle \phi_1, S\phi_2 \rangle. \end{aligned}$$

Thus, we arrive at the unique time evolution unitary operator $U(t)$, (35), for the Klein Gordon equation.

5. Conclusions

Finding a unique exact vacuum solution for the Klein Gordon equation with the help of a unique symmetric symplectic operator, (for the singular perturbation problem), splitting the solution uniquely into its positive and negative parts and incorporating the respective frequencies in the scattering operator S, we have arrived at the unique unitary time evolution operator $U(t)$ (expression (35)) for Klein Gordon equation in terms of its vacuum solution in the infinite dimensional Hilbert space $L^2(\mathbf{R}^3)$.

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