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One-parameter Lie groups admitted by time-dependent Schrödinger equation: Atoms, molecules and nucleons in harmonic oscillator field

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Abstract. Lie's method of differential equation is used to obtain the one-parameter Lie groups admitted by the time-dependent Schrödinger equations for atoms, molecules and nucleons in harmonic oscillator field. This group for atoms and molecules is isomorphic to 10-parameter inhomogeneous orthogonal group in 4 dimensions, irrespective of the numbers of nuclei and electrons. For Z protons and N neutrons in a harmonic oscillator field, both isotropic and anisotropic, the r-parameter Lie groups are semidirect products of an invariant subgroup and a factor group. In the case of isotropic oscillator field r is $\frac{1}{2}[3Z(3Z-1)+3N(3N-1)+2]$, while for the anisotropic oscillator field r is $\frac{1}{2}[3Z(Z+1)+3N(N+1)+2]$.

Keywords. One-parameter Lie groups; atoms; molecules; harmonic oscillator field.

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1. Introduction

The symmetry group of an equation of motion of a physical problem classifies the energy eigenfunctions of the system. Knowledge of the largest symmetry group of the problem gives a clear insight of the degeneracies of the system (McIntosh 1971). In the case of atom Fock (1935) showed that the homogeneous orthogonal group $O(4)$ in 4 dimensions is the symmetry group of one-electron hydrogen atom, and Bargmann (1936) showed that this larger group comes into play because, besides the angular momentum operator, the Runge-Lenz vector is required for complete description of the orbit. Many authors have investigated the role of $O(4)$ in many-electron atom (Alper 1969; Biedenharn 1961; Butler and Wybourne 1970; Wulfman 1971). In the case of molecules, Shibuyu and Wulfman (1965) investigated $O(5)$, the orthogonal group in 5 dimensions, as the dynamical non-invariance group for one-electron in the field of any number of fixed nuclei. In nuclear structure calculations (Preston and Bhaduri 1975) one pictures the nucleons moving in an average harmonic oscillator field. For spherical nuclei the field is isotropic, while for distorted nuclei this is an anisotropic field (Nilsson 1955). In realistic calculations the characteristic frequencies for neutrons and protons are taken to be different. Symmetry groups for nuclear structure have been extensively studied (Baker 1956; Kramer and Moshinsky 1968; Kretschmar 1960) and different groups have been considered as possible candidates. These three systems are at the base of atomic, molecular and nuclear problems.

Lie developed a method (Eisenhart 1961) by which one can generate the solutions of a complete system of differential equations from one solution. This was based on the concept of one-parameter Lie groups admitted by a complete system of differential equations. If ψ is a solution of the system of differential equations, then all functions $X_\alpha\psi$ for a set of linear operators X_α $(\alpha = 1, \ldots, r)$ are also solutions when X_α's satisfy a particular condition. The one-parameter Lie groups generated by the X_α's are said to be admitted by the given system of differential equations. Moreover, if the set of equations admits X_α and X_β then it also admits the commutator $[X_\alpha, X_\beta]$. A key theorem states that for a system consisting of a single equation, either it admits a r-parameter Lie group or the solutions of the equation are obtained by direct processes. It is thus clear that the one-parameter Lie groups admitted by the set of differential equations are all subgroups of the dynamical group.

In §2 we have expressed Lie's method for a second order differential equation, and in particular that of the time-dependent Schrödinger type. In §§3 and 4 we have used this method to obtain the one-parameter Lie groups admitted by the time-dependent Schrödinger equations for atoms, molecules and nucleons in harmonic oscillator field. For atoms and molecules this group is the 10-parameter inhomogeneous orthogonal group $IO(4)$ in 4 dimensions, irrespective of the number of nuclei and electrons. In the case of Z protons and N neutrons in an isotropic harmonic oscillator field with different characteristic frequencies for protons and neutrons. This is a $\frac{1}{2}[3Z(3Z-1) + 3N(3N-1)+2]$-parameter Lie group, which is a semi-direct product of an invariant subgroup and a proper subgroup. In the anisotropic case this is a Lie group with $\frac{1}{2}[3Z(Z+1)+3N(N+1)+2]$ parameters, and is again a semi-direct product.

2. One-parameter Lie groups

In this section we describe Lie's method for obtaining the one-parameter groups admitted by a complete system of linear differential equations

$$\Delta^\alpha \equiv A^\alpha\psi = 0, \quad \alpha = 1, \ldots, p, \tag{1}$$

where ψ is a s-component dependent variable, in n independent variables q^i, $i = 1, \ldots, n$, and A^α's are $s \times s$ matrices. If ψ is a solution of (1), then X^ψ, where the $s \times s$ linear operators X is given in terms of the $s \times s$ velocity vectors $\xi^m(q)$'s by

$$X = \sum_m \xi^m(q) X^m, \quad \text{with} \quad X^m = -i\partial/\partial q^m, \tag{2}$$

is also a solution of the system of equation (1) if and only if

$$[X, A^\alpha] = \sum_\beta \Lambda^\alpha_\beta(q) A^\beta, \quad \text{for all } \alpha = 1, \ldots, p, \tag{3}$$

where Λ^α_β's are $s \times s$ matrices. The necessary part of the condition exists if there are no other linear operators satisfying (1). This is ensured by the completeness of the system of equations. The one-parameter Lie group generated by X is said to be admitted by the system of equations Δ^α.

For a second-order matrix system

$$A^\alpha \equiv a^\alpha_0(q) + \sum_i a^\alpha_i(q)\, \partial/\partial q^i + \sum_{ij} a^\alpha_{ij}(q)\, \partial^2/\partial q^i\, \partial q^j,$$

$$a^\alpha_{ij} = a^\alpha_{ji}, \tag{4}$$

the necessary and sufficient conditions become (Rudra 1984)

$$\sum_m \xi^m \, \partial a_0^\alpha/\partial q^m = \sum_\beta \Lambda^\alpha_\beta a_0^\beta,$$

$$[\xi^i, a_0^\alpha] + \sum_m \xi^m \partial a_i^\alpha/\partial q^m - \sum_m a_m^\alpha \, \partial \xi^i/\partial q^m - \sum_{km} a_{km}^\alpha \, \partial^2 \xi^i/\partial q^k \partial q^m = \sum_\beta \Lambda^\alpha_\beta \, a_i^\beta,$$

$$\tfrac{1}{2}[\xi^j, a_i^\alpha] + \tfrac{1}{2}[\xi^i, a_j^\alpha] + \sum_m \xi^m \, \partial a_{ij}^\alpha/\partial q^m$$

$$- \sum_m (a_{im}^\alpha \, \partial \xi^j/\partial q^m + a_{mj}^\alpha \, \partial \xi^i/\partial q^m) = \sum_\beta \Lambda^\alpha_\beta \, a_{ij}^\beta,$$

$$[\xi^k, a_{ij}^\alpha] + [\xi^i, a_{jk}^\alpha] + [\xi^j, a_{ki}^\alpha] = 0. \tag{5}$$

Solutions of these sets of partial differential equations for the velocity vectors ξ's give the velocity vectors and hence the generators for the one-parameter Lie groups admitted by the system Δ^α.

For a one-component Schrödinger equation

$$[i\partial/\partial t + \sum_{m\alpha} a_m (\partial/\partial q^{m\alpha})^2 - v]\psi = 0, \tag{6}$$

for a N-particle system with masses $M_m, m = 1, \ldots, N$ and components $\alpha = 1, 2, 3$ in a potential $V = v\hbar$, the non-vanishing parameters of (4) are $a_0 = -v$, $a_t = i$, $a_{m\alpha, n\beta} = a_m \delta_{m\alpha, n\beta}$. Equation (5) now reduces to

$$\xi^t \partial v/\partial t + \sum_{m\alpha} \xi^{m\alpha} \, \partial v/\partial q^{m\alpha} = \Lambda v, \quad a_m \partial \xi^t/\partial q^{m\alpha} = 0,$$

$$i\partial \xi^{m\alpha}/\partial t + \sum_{n\beta} a_n \partial^2 \xi^{m\alpha}/(\partial q^{n\beta})^2 = 0,$$

$$i\partial \xi^t/\partial t + \sum_{n\beta} a_n \partial^2 \xi^t/(\partial q^{n\beta})^2 + i\Lambda = 0,$$

$$a_m \partial \xi^{n\beta}/\partial q^{m\alpha} + a_n \partial \xi^{m\alpha}/\partial q^{n\beta}$$

$$+ \delta_{m\alpha, n\beta}\left(\Lambda a_m - \xi^t \partial a_m/\partial t + \sum_{s\sigma} \xi^{s\sigma} \partial a_m/\partial q^{s\sigma}\right) = 0. \tag{7}$$

In the Cartesian components a_m's are all constants. It then immediately follows that

$$\xi^t \equiv \xi^t(t), \quad \Lambda \equiv \Lambda(t), \quad 2\partial \xi^{m\alpha}/\partial q^{m\alpha} = \partial \xi^t/\partial t = -\Lambda.$$

It also follows that $\partial^2 \xi^{m\alpha}/\partial q^{n\beta} \partial q^{k\gamma} = 0$ for all sets of $m\alpha$, $n\beta$, $k\gamma$. Thus, $\partial \xi^{m\alpha}/\partial t = 0$, i.e. $\xi^{m\alpha} \equiv \xi^{m\alpha}(q)$. Thus we get

$$\xi^{m\alpha} = b_0^{m\alpha} + \sum_\beta b_{m\beta}^{m\alpha} \, q^{m\beta} + a_m \sum_{n \neq m} \sum_\beta b_{n\beta}^{m\alpha} \, q^{n\beta},$$

and $\qquad \partial\xi^t/\partial t = -\Lambda = 2\partial\xi^{m\alpha}/\partial q^{m\alpha} = b^{m\alpha}_{m\alpha}, \qquad$ independent of α

and hence $\xi^t = b_0 + 2bt$. From $b^{m\alpha}_{n\beta} + b^{n\beta}_{m\alpha} = 0$, we finally obtain

$$\xi^{m\alpha} = b_0^{m\alpha} + bq^{m\alpha} + \sum_{\beta\gamma} e_{\alpha\beta\gamma} b_1^{m\gamma} q^{m\beta} + a_m \sum_{n<m}\sum_{\beta} b^{m\alpha}_{n\beta} q^{n\beta} - a_m \sum_{n>m}\sum_{\beta} b^{n\beta}_{m\alpha} q^{n\beta},$$

$$\xi^t = b_0 + 2bt, \quad \Lambda = -2b,$$

$$\xi^t \partial v/\partial t + \sum_{m\alpha} \xi^{m\alpha} \partial v/\partial q^{m\alpha} + 2bv = 0. \tag{8}$$

In the next two sections we apply these equations to obtain the one-parameter Lie groups admitted by the time-dependent Schrödinger equations for atoms, molecules and nucleons in harmonic oscillator field.

3. Atoms and molecules

For atoms and molecules we consider the k_nth nucleus of type $n (k_n = 1, \ldots, N_n$, $n = 1, \ldots, T)$ at $\mathbf{r}_{nk_n}$ and the k_eth electron of mass m at $\mathbf{r}_{k_e}$. The n-type nucleus has mass M_n and atomic number Z_n. The parameters of (8) are

$$a_n = \hbar/2M_n, \qquad a_e = \hbar/2m,$$

$$v = -\sum_{nk_n k_e} Z_n e^2/\hbar|\mathbf{r}_{nk_n} - \mathbf{r}_{k_e}| + \sum_{(mk_m)\neq(nk_n)} Z_m Z_n e^2/2\hbar|\mathbf{r}_{mk_m} - \mathbf{r}_{nk_n}|$$
$$+ \sum_{k_e \neq k'_e} e^2/2\hbar|\mathbf{r}_{k_e} - \mathbf{r}_{k'_e}|. \tag{9}$$

Equation (8) now becomes

$$\xi^{mk_m\alpha} = b_0^{mk_m\alpha} + br_{mk_m\alpha} + \sum_{\beta\gamma} e_{\alpha\beta\gamma} b_1^{mk_m\gamma} r_{mk_m\beta} + a_m \sum_{n<m}\sum_{k_n}\sum_{\beta} b^{mk_m\alpha}_{nk_n\beta} r_{nk_n\beta}$$
$$+ a_m \sum_{k_n<k_m}\sum_{\beta} b^{mk_m\alpha}_{mk_n\beta} r_{mk_n\beta} - a_m \sum_{k_n>k_m}\sum_{\beta} b^{mk_n\beta}_{mk_m\alpha} r_{mk_n\beta}$$
$$- a_m \sum_{n>m}\sum_{k_n}\sum_{\beta} b^{nk_n\beta}_{mk_m\alpha} r_{nk_n\beta} - a_m \sum_{k_e}\sum_{\beta} b^{k_e\beta}_{mk_m\alpha} r_{k_e\beta},$$

$$\xi^{k_e\alpha} = b_0^{k_e\alpha} + br_{k_e\alpha} + \sum_{\beta\gamma} e_{\alpha\beta\gamma} b_1^{k_e\gamma} r_{k_e\beta} + a_e \sum_{nk_n}\sum_{\beta} b^{k_e\alpha}_{nk_n\beta} r_{nk_n\beta}$$
$$+ a_e \sum_{k'_e<k_e}\sum_{\beta} b^{k_e\alpha}_{k'_e\beta} r_{k'_e\beta} - a_e \sum_{k'_e>k_e}\sum_{\beta} b^{k'_e\beta}_{k_e\alpha} r_{k'_e\beta},$$

$$\xi^t = b_0 + 2bt,$$

$$2bv + \sum_{mk_m\alpha} \xi^{mk_m\alpha} \partial v/\partial r_{mk_m\alpha} + \sum_{k_e\alpha} \xi^{k_e\alpha} \partial v/\partial r_{k_e\alpha} = 0, \tag{10}$$

and the admissible generator is

$$X = \xi^t \partial/\partial t + \sum_{mk_m\alpha} \xi^{mk_m\alpha}\, \partial/\partial r_{mk_m\alpha} + \sum_{k_e\alpha} \xi^{k_e\alpha}\, \partial/\partial r_{k_e\alpha}.$$

Inserting (9) in the last of (10), we get

$$\begin{aligned}
&b_0^{mk_m\alpha} = b_0^{nk_n\alpha} \quad b_0^{k_e\alpha} = b_0^{k'_e\alpha}, \quad b_0^{mk_m\alpha} = b_0^{k_e\alpha},\\
&a_m b^{k_e\beta}_{mk_m\alpha} = a_e b^{k_e\beta}_{k'_e\alpha} (k'_e < k_e),\, b^{k'_e\beta}_{mk_m\alpha} = b^{k_e\beta}_{mk_m\alpha},\, b^{k_e\beta}_{mk_m\alpha} + b^{k_e\alpha}_{mk_m\beta} = 0,\\
&a_m b^{mk_m\alpha}_{mk_n\beta} = a_e b^{k_e\alpha}_{mk_n\beta}\ (k_n < k_m),\, a_m b^{mk_m\alpha}_{nk_n\beta} = a_e b^{k_e\alpha}_{nk_n\beta}\ (n < m),\\
&\sum_\gamma e_{\alpha\beta\gamma}(b_1^{k_e\gamma} - b_1^{k'_e\gamma}) = 0 = \sum_\gamma e_{\alpha\beta\gamma}(b_1^{mk_m\gamma} - b_1^{nk_n\gamma}),\\
&\sum_\gamma e_{\alpha\beta\gamma}(b_1^{mk_m\gamma} - b_1^{k_e\gamma}) + (a_m - a_e) b^{k_e\alpha}_{mk_m\beta} = 0.
\end{aligned} \tag{11}$$

The general solution is thus

$$\begin{aligned}
&b = 0, \quad b_0^{mk_m\alpha} = b_0^{k_e\alpha} = b_0^{0\alpha},\\
&a_n b^{k_e\alpha}_{nk_n\beta} = a_e b^{k_e\alpha}_{k'_e\beta} = \sum_\gamma e_{\alpha\beta\gamma} b_0^\gamma\ (k'_e < k_e),\\
&a_m b^{mk_m\alpha}_{mk_n\beta} = a_e b^{k_e\alpha}_{mk_n\beta} = (a_e/a_m) \sum_\gamma e_{\alpha\beta\gamma}\, b_0^\gamma \quad (k_n < k_m),\\
&a_m b^{mk_m\alpha}_{nk_n\beta} = a_e b^{k_e\alpha}_{nk_n\beta} = (a_e/a_n) \sum_\gamma e_{\alpha\beta\gamma}\, b_0^\gamma \quad (n < m),\\
&b_1^{k_e\alpha} = b_1^\alpha, \quad b_1^{mk_m\alpha} = b_1^\alpha + (a_e/a_m - 1)\, b_0^\alpha.
\end{aligned} \tag{11}$$

The generators of the admissible one-parameter Lie groups are then

$$\begin{aligned}
&X^{0\alpha} = -i\left(\sum_{k_e} \partial/\partial r_{k_e\alpha} + \sum_{mk_m} \partial/\partial r_{mk_m\alpha}\right), \quad X^t = -i\partial/\partial t,\\
&X_1^\alpha = -i \sum_{\beta\gamma} e_{\alpha\beta\gamma}\left(\sum_{k_e} r_{k_e\beta}\partial/\partial r_{k_e\gamma} + \sum_{mk_m} r_{mk_m\beta}\partial/\partial r_{mk_m\gamma}\right),\\
&X_0^\alpha = (1/M_0) \sum_{\beta\gamma} e_{\alpha\beta\gamma}\left(m \sum_{k_e} r_{k_e\beta} + \sum_{mk_m} M_m r_{mk_m\beta}\right) X^{0\gamma},
\end{aligned}$$

where $$M_0 = mN_e + \sum_n M_n N_n, \tag{12}$$

with non-vanishing commutators

$$\begin{aligned}
&[X^{0\alpha}, X_1^\beta] = i \sum_\gamma e_{\alpha\beta\gamma}\, X^{0\gamma}, [X^{0\alpha}, X_0^\beta] = i \sum_\gamma e_{\alpha\beta\gamma}\, X^{0\gamma},\\
&[X_1^\alpha, X_1^\beta] = i \sum_\gamma e_{\alpha\beta\gamma}\, X_1^\gamma, [X_1^\alpha, X_0^\beta] = i \sum_\gamma e_{\alpha\beta\gamma}\, X_0^\gamma, [X_0^\alpha, X_0^\beta] = i \sum_\gamma e_{\alpha\beta\gamma}\, X_0^\gamma.
\end{aligned} \tag{13}$$

$X^{0\alpha}$ denotes total space translation, X_1^α denotes total rotation of the system, while X^t denotes time translation. To see clearly the presence of the inhomogeneous orthogonal group *IO* (4) in 4 dimensions we take the linear combinations

$$L^\alpha = X_1^\alpha, \quad A^\alpha = -X_1^\alpha + 2X_0^\alpha,$$

and get the non-vanishing commutation relations

$$[X^{0\alpha}, L^\beta] = i\sum_\gamma e_{\alpha\beta\gamma} X^{0\gamma}, \quad [X^{0\alpha}, A^\beta] = i\sum_\gamma e_{\alpha\beta\gamma} X^{0\gamma},$$

$$[L^\alpha, L^\beta] = i\sum_\gamma e_{\alpha\beta\gamma} L^\gamma, \quad [L^\alpha, A^\beta] = i\sum_\gamma e_{\alpha\beta\gamma} A^\gamma, \quad [A^\alpha, A^\beta] = i\sum_\gamma e_{\alpha\beta\gamma} L^\gamma. \tag{14}$$

These ten admissible one-parameter Lie groups form a 10-parameter Lie group G having an invariant subgroup $N = \{X^{0\alpha}, X^t\}$ and a subgroup $H = \{L^\alpha, A^\beta\}$, such that $G/N \approx H$. Thus G is a semi-direct product $G = N \circledS H$. It is to be noted that both atoms and molecules have the same symmetry group irrespective of the number of nuclei and electrons.

4. Nucleons in harmonic oscillator field

We first consider N particles of masses $M_m, m = 1, \ldots, N$ in the harmonic oscillator field of characteristic frequencies $\omega_{m\alpha}$ corresponding to the mth particle in the αth Cartesian component. Thus $a_m = \hbar/2M_m, v = \sum_{m\alpha} \omega_{m\alpha}^2 r_{m\alpha}^2/4a_m$. The last of (8) gives us

$$\sum_{m\alpha} [b_0^{m\alpha}\, \omega_{m\alpha}^2\, r_{m\alpha}/2a_m + b\omega_{m\alpha}^2\, r_{m\alpha}^2/a_m$$

$$+ \sum_{n<m} \sum_\beta b_{n\beta}^{m\alpha} (\omega_{m\alpha}^2 - \omega_{n\beta}^2)\, r_{m\alpha}\, r_{n\beta}/2] = 0. \tag{15}$$

Equating the coefficients of different monomials to zero, we get

$$b = b_0^{m\alpha} = 0, \quad b_{n\beta}^{m\alpha} = \begin{cases} 0, \text{ if } \omega_{m\alpha}^2 - \omega_{n\beta}^2 \neq 0, \\ \text{arbitrary if } \omega_{m\alpha}^2 - \omega_{n\beta}^2 = 0. \end{cases} \quad (n < m) \tag{16}$$

For Z protons and N neutrons denoted by $P_k (k = 1, \ldots, Z)$ and $N_m (m = 1, \ldots, N)$ in an isotropic harmonic oscillator field with characteristic frequencies ω_p and ω_n for protons and neutrons respectively, we get the generators for admissible one-parameter groups

$$X^t = -i\partial/\partial t, \quad X_1^{P_k\alpha} = -i\sum_{\beta\gamma} e_{\alpha\beta\gamma}\, r_{P_k\beta}\, \partial/\partial r_{P_k\gamma},$$

$$X_{P_l\beta}^{P_k\alpha} = -i\,(r_{P_l\beta}\, \partial/\partial r_{P_k\alpha} - r_{P_k\alpha}\, \partial/\partial r_{P_l\beta}), \quad k > l, k, l = 1, \ldots, Z,$$

$$X_1^{N_n\alpha} = -i\sum_{\beta\gamma} e_{\alpha\beta\gamma}\, r_{N_n\beta}\, \partial/\partial r_{N_n\gamma},$$

$$X_{N_n\beta}^{N_m\alpha} = -i(r_{N_n\beta}\, \partial/\partial r_{N_m\alpha} - r_{N_m\alpha}\, \partial/\partial r_{N_n\beta}), \quad m > n, m, n = 1, \ldots, N, \tag{17}$$

with non-vanishing commutators

$$
\begin{aligned}
&[X_1^{P_k\alpha}, X_1^{P_m\beta}] = i\delta_{km} \sum_\gamma e_{\alpha\beta\gamma} X_1^{P_k\gamma},\\
&[X_1^{P_k\alpha}, X_{P_n\beta}^{P_m\mu}] = i\delta_{km} \sum_\gamma e_{\alpha_\mu\gamma}\ X_{P_n\beta}^{P_m\gamma} + i\delta_{kn} \sum_\gamma e_{\alpha\beta\gamma} X_{P_n\gamma}^{P_m\mu}, \quad m > n,\\
&[X_{P_l\beta}^{P_k\alpha}, X_{P_n\gamma}^{P_m\mu}] = i\delta_{lm}\, \delta_{\beta\mu}\, X_{P_n\gamma}^{P_k\alpha} - i\delta_{kn}\, \delta_{\alpha\gamma}\, X_{P_l\beta}^{P_m\mu} + i\delta_{km}\, \delta_{\alpha\mu} (X_{P_l\beta}^{P_n\gamma} - X_{P_n\gamma}^{P_l\beta})\\
&\qquad\qquad + i\delta_{ln}\delta_{\beta\mu} (X_{P_n\alpha}^{P_m\mu} - X_{P_m\mu}^{P_n\alpha}), \quad k > l, \quad m > n,\\
&k, l, m, n = 1, \ldots, Z,
\end{aligned} \tag{18}
$$

and similar relations between neutron operators among themselves. If in the last two terms on the right side of the last of (18) any of the upper indices is less than the lower index, then that operator is absent. These generators form a $[3Z(3Z-1)+3N(3N-1)+2]/2$-parameter Lie group G having an invariant subgroup

$$N = \{X^t, X_{P_l\beta}^{P_k\alpha}, X_{N_n}^{N_m\mu}\}, \quad k > l, \quad m > n, \text{ and a subgroup}$$

$$H = \{X_1^{P_k\alpha}, X_1^{N_m\mu}\}, \quad k, l = 1, \ldots, Z, \quad m, n = 1, \ldots, N, \text{ such that}$$

$G/N \approx H$. Thus G is a semi-direct product $G = N \circledS H$.

Physically X^t denotes time-translation; $X_1^{P_k\alpha}$ and $X_1^{N_m\alpha}$ denote rotation about α-axis of only the coordinates of the P_kth proton and the N_mth neutron respectively; $X_{P_l\beta}^{P_k\alpha}$ and $X_{N_n\beta}^{N_m\alpha}$ denote rotation only in the $(r_{P_k\alpha}, r_{P_l\beta})$-plane and $(r_{N_m\alpha}, r_{N_n\beta})$-plane respectively.

In the case of anisotropic oscillator with characteristic frequencies

$$\omega_{p\alpha}, \text{ and } \omega_{n\alpha} \text{ only } X_{P_l\alpha}^{P_k\alpha} \text{ and } X_{N_n\alpha}^{N_m\alpha}, \quad k > l, \quad m > n,$$

$k, l = 1, \ldots, Z, m, n = 1, \ldots, N$ exist; the generators with $\alpha \neq \beta$ are absent. In the commutation relations this has to be taken into account. The resulting group G is a $[3Z(Z+1)+3N(N+1)+2]/2$-parameter Lie group with the corresponding semi-direct product structure.

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