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# Spectral asymmetry of Dirac Hamiltonian in a five-dimensional Kaluza-Klein theory

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**Abstract.** It is shown that the fermion number in a five-dimensional Kaluza-Klein theory ($M^4 \times S^1$) in which the fermion is interacting with a monopole field, is quantized in units of $(\phi R)^2$ where the scalar $\phi$ is asymptotically constant and $R$ is the radius of $S^1$.

**Keywords.** Kaluza-Klein theory; spectral symmetry monopole field; fermion number.

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In a system where fermions interact with classical background fields, the induced fermion number $N$ is the spectral asymmetry of the pertinent Dirac Hamiltonian. There exists in literature a number of ways of computing $N$ (Goldstone and Wilczek 1981; Bardeen *et al* 1983). A new quantum field theoretical method to evaluate $N$ has been proposed by Niemi and Semenoff (1983). This method uses relevant trace identities. The final expression for $N$ here involves only the topological properties of the background field. Paranjape and Semenoff (1983) used this approach to compute the fermion number of a magnetic monopole-fermion system. It is to be noted that the advantage of this method is the computation of spectral asymmetry for Dirac Hamiltonians admitting continuum eigenvalues.

It is the purpose of this paper to evaluate the spectral asymmetry of the Dirac Hamiltonian in a five-dimensional Kaluza-Klein theory. The motivation is to examine whether the infinite tower of fermion massive modes arising in the process of compactification of the fifth dimension do really contribute to the spectral asymmetry.

The ground state of 5-dimensional Kaluza-Klein theory is taken to be $M^4 \times S^1$. The four-dimensional space $M^4$ is taken to be flat. The Dirac particle is assumed to move in a background field $A_\mu(x)$ which could be external or arising from the theory itself as the five-dimensional theory unifies gravitation and electromagnetism. In the latter case, the effect of the background field after compactification can be thought of as a perturbation to the ground state configuration, whose stability has been studied by Appelquist and Chodos (1983), Tsokos (1983) and Rubin and Roth (1983).

The harmonic expansion for the fermion field, namely

$$\psi(x, x^5) = \sum_{n=-\infty}^{\infty} \psi_n(x) \exp(-inx^5/R), \tag{1}$$

yields the following Dirac equation

$$(i\gamma^\mu\partial_\mu - \phi_n(x) - i\gamma^5 n/R)\psi_n(x) = 0, \tag{2}$$

where $R$ is the radius of $S^1$ and $\phi_n(x)$ is the scalar field existing in the ground state of the Kaluza-Klein theory which could be taken to be a constant. In (2), $\mu = 1, 2, 3, 4$. Let the background field be $A_\mu(x)$. Then (2) becomes

$$(i\gamma^\mu\partial_\mu - \gamma^\mu A_\mu - \phi_n(x) - i\gamma^5 n/R)\psi_n(x) = 0. \tag{3}$$

The well-known difficulty with the dimensional reduction is the occurrence of mass terms $n/R$. For $n \neq 0$, one encounters an infinite tower of massive modes. Such modes are usually neglected. However, they are retained here.

Consider the background field $A_\mu(x)$. This is taken to be static ($A_0(x) = 0$). $A_i(x)$ is assumed $\leqslant 1/r$ as $r \to \infty$ i.e. monopole configurations. We rewrite $n/R$ as $m$. Equation (3) can be rewritten as

$$\begin{bmatrix} -m & D \\ D^\dagger & m \end{bmatrix}\begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} = E\begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} \tag{4}$$

by using the following representation of $\gamma$-matrices

$$\gamma_0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \gamma_k = \begin{pmatrix} -i\sigma_k & 0 \\ 0 & i\sigma_k \end{pmatrix}; \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and

$$D = i\sigma^k\partial_k + \sigma^k A_k + i\phi.$$

Equation (4) is of the form

$$H_m\psi_n = \lambda_n\psi_m, \tag{5}$$

where

$$H_m = H + m\Gamma^c; \quad \Gamma^c = i\gamma_0\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$H = \begin{bmatrix} 0 & D \\ D^\dagger & 0 \end{bmatrix},$$

$$\{\Gamma^c, H\} = 0. \tag{6}$$

Further $H_m^2 = H^2 + m^2 \geqslant m^2$. Hence all the eigenvalues in (5) are non-zero. Since $H_m$ is a Dirac Hamiltonian it has both positive and negative eigenvalues. These are asymmetric in the present case around $\lambda = 0$. This spectral asymmetry is given by

$$\eta_m(s) = \sum_n \operatorname{sign}\lambda_n |\lambda_n|^{-s} \tag{7}$$

in the limit $s \to 0$.

Introduce the Mellin transform to write

$$\eta_m(s) = \frac{2}{\pi}\cos(s\pi/2)\sum_n \int_0^\infty d\omega\, \omega^{-s}\frac{\lambda_n}{(\lambda_n^2 + \omega^2)}. \tag{8}$$

On account of (5), this can be written as

$$= \frac{2}{\pi}\cos(s\pi/2)\int_0^\infty \mathrm{d}\omega\,\omega^{-s}\int \mathrm{d}^3x \text{ trace } \left\langle x \left| \frac{H_m}{(H_m^2+\omega^2)} \right| x \right\rangle. \tag{9}$$

We make use of the trace identity of Niemi and Semenoff (1984) to write the above equation as

$$\frac{1}{\pi}\cos(s\pi/2)\int_0^\infty \mathrm{d}\omega\,\omega^{-s}\left(\frac{m}{\omega^2+m^2}\right)\oint \mathrm{d}s^i \,\mathrm{Tr}\,\left\langle x \left| \gamma^i\gamma^5 \frac{1}{H+i\sigma} \right| x \right\rangle \tag{10}$$

where $\sigma = (\omega^2+m^2)^{1/2}$.

The Green function for $(H+i\sigma)^{-1}$ satisfies

$$(-\mathbf{D}^2+\phi^2+\omega^2+m^2)S(x,y) = (-i\gamma\cdot\mathbf{D}-\phi-i\gamma^0\sigma)\,\delta(\mathbf{x}-\mathbf{y}) - (\tfrac{1}{2}\mathbf{\Sigma}\cdot\mathbf{F}+i\gamma\cdot[\mathbf{D},\phi])\,S(x,y), \tag{11}$$

where $\mathbf{D} = \nabla + i\mathbf{A}$; $\Sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]$ and $[D_\mu, D_\nu] = iF_{\mu\nu}$. Evaluating $S(x,y)$ iteratively and taking the trace over $\gamma$-matrices, we arrive at

$$\eta_m(s) = \frac{4}{\pi}\cos(s\pi/2)\int_0^\infty \mathrm{d}\omega\,\omega^{-s}\left(\frac{m}{\omega^2+m^2}\right)\int\frac{\mathrm{d}^3k}{(2\pi)^3} \oint \mathrm{d}s^i\, B_i(\mathbf{x})\,(k^2+\omega^2+m^2+\phi^2)^{-2}\phi, \tag{12}$$

where

$$B_i = \tfrac{1}{2}\varepsilon_{ijk}F^{jk}.$$

Taking the limit $s\to 0$, and carrying out the integration

$$\eta_m = \lim_{s\to 0}\eta_m(s) = \frac{1}{2\pi^2}\{\tan^{-1}(\phi/m)\}\oint \mathrm{d}s^i B_i. \tag{13}$$

Thus the spectral asymmetry gets a transcendental form. Now the external field $A_i$ is chosen for the sake of illustration to be a magnetic monopole field. Then $\oint \mathrm{d}s^i B_i = 4\pi N$ (gauge-invariant). Recalling $m = n/R$, the spectral asymmetry for the Kaluza-Klein mode $\psi_n(x)$ is

$$\eta_n = \frac{2N}{\pi}\tan^{-1}[\phi/(n/R)]. \tag{14}$$

Now the following points are in order. First of all one notices that the spectral asymmetry is given by the asymptotic behaviour of the external fields. Therefore at these regions $\phi$ can be assumed to become constant. The $n$th spinor has a charge proportional to $n$ or alternately the monopole it interacts with has magnetic charge $ng$ where $g$ is the fundamental Dirac value. Therefore the spectral asymmetry is given by

$$\eta_n = n\frac{2N}{\pi}\tan^{-1}(\phi R/n).$$

Summing over all Kaluza-Klein modes ($n$) we have

$$\eta = \sum_{n=-\infty}^{\infty} \eta_n = \frac{2N}{\pi} \sum_{-\infty}^{\infty} n \tan^{-1}(\phi R/n).$$

This sum is approximated to the integral

$$\int_{-\infty}^{\infty} n \tan^{-1}(\phi R/n)\,\mathrm{d}n.$$

The finite part of this integral is found to be $(\pi/2)(\phi R)^2$. This can be seen by rewriting the integral as $\int_0^\infty \tan^{-1}(\phi R/n)\,\mathrm{d}(n^2)$ and doing the integration by parts. The divergent part is $[n^2 \tan^{-1}(\phi R/n)]_0^\infty$ and the finite part is $\phi R \int_0^\infty \frac{n^2\,\mathrm{d}n}{(\phi R)^2 + n^2}$ which gives $\frac{\pi}{2}(\phi R)^2$. Therefore the full spectral asymmetry summing over all the Kaluza-Klein modes in a 5-dimensional theory is given by

$$\eta = N(\phi R)^2.$$

Since $\eta$ gives the fermion number, in a five-dimensional theory with fermion interacting with an external monopole field, it is quantized in units of $(\phi R)^2$.

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