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Perturbative evaluation of universal constants for a quartic map

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Abstract. We discuss a perturbative scheme for the determination of the bifurcation rate δ for a specific map, by extending Virendra Singh's method of evaluating the scaling factor α. The method is applied to a quartic map and the values obtained, $\alpha = 1{\cdot}690781026$ and $\delta = 7{\cdot}23682924$ are in good agreement with the numerically computed values reported in the literature. The perturbative approach is found to be more efficient than other existing methods.

Keywords. Dynamical systems; chaos; Feigenbaum scenario; perturbative approach.

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1. Introduction

Many nonlinear dynamical systems have been found to exhibit a transition from regular to chaotic behaviour when some control parameter is slowly varied. For special values of the parameter, the nature of the asymptotic motion may change into a complicated sequence of bifurcations. One of the most common routes to chaos is a period-doubling bifurcation cascade, known in the literature as the Feigenbaum scenario. This scenario is studied using discrete maps (Collet and Eckmann 1980; Feigenbaum 1980, 1983) which describe the Poincaré surfaces of a section of nonlinear differential flows. Even in one dimension, such maps exhibit a richness of behaviour like limit cycles, strange attractors, hysteresis, ergodic regions etc.

One-dimensional logistic maps imitate strongly dissipative systems as well as most experimental situations (Lauterborn and Cramer 1981; Eckmann 1981; Gibbs *et al* 1981; Linsay 1981; Turner *et al* 1981). Typical among these are maps of the form,

$$x_{t+1} = 1 - a|x_t|^z, \tag{1}$$

which represent a simple deterministic system exhibiting chaotic behaviour. The bifurcations of such maps follow a universal pattern and two constants δ and α are usually associated with them. The bifurcation rate δ is defined as

$$\delta = \lim_{n\to\infty} \frac{a_{n+1} - a_n}{a_{n+2} - a_{n+1}}, \tag{2}$$

where a_n is the value of the parameter a at which the nth bifurcation occurs. The scaling factor α is defined by

$$\alpha = \lim_{n\to\infty} \frac{\Delta x(2^n)}{\Delta x(2^{n+1})}, \tag{3}$$

where $\Delta x(2^n)$ is half the separation between the two adjacent fixed points in the 2^n cycle (Feigenbaum 1980). For a quadratic map, these constants were first evaluated by Feigenbaum (1978) as $\delta = 4{\cdot}66920\ldots$ and $\alpha = 2{\cdot}5029\ldots$. It is now well established that δ and α depend on the order of the local maximum, that is on the value of z (Mendes 1981; Hu and Mao 1982; Hu and Satija 1983; Hauser *et al* 1985). Hence the determination of these constants for higher order maps is a problem of current interest.

Feigenbaum (1979) and Cvitanovic and Myrheim (1983) have shown that there exists a limiting universal function $g(x)$ that satisfies the equations

$$g(g(x/\alpha)) + \frac{1}{\alpha} g(x) = 0; \quad g(0) = 1, \tag{4}$$

and

$$-\alpha[g'(g(x/\alpha))h(x/\alpha) + h(g(x/\alpha))] = \delta h(x). \tag{5}$$

These functional renormalization group equations are most suited for an analytic determination of δ and α.

There have been many analytic and numerical attempts to evaluate these constants. Among the analytic approaches we mention the renormalization schemes (Derrida *et al* 1979; Helleman 1980; Hauser *et al* 1985; Hu and Mao 1985). Recently Virendra Singh (1985) has developed a perturbative expansion scheme for evaluating α in which he used (4) directly and applied it to a quadratic map. The agreement with the numerical value (Feigenbaum 1978) is excellent. In this paper we extend his method to enable an analytic determination of δ also and specifically evaluate the δ and α values for a quartic map. From the results obtained, we come to the conclusion that the perturbative expansion method yields comparatively more accurate results with less computational effort.

This paper is organised as follows. A resumé of the perturbative approach to the calculation of the constant α is presented in §2. In §3, we discuss an extension of this method to the determination of δ for maps (1) with arbitrary z. Details of a computation of δ and α for a quartic map are given in §4 and our comments on the power of the perturbative method are offered in §5.

2. Resumé of perturbative expansion method

In this section, we summarize the essentials of the perturbative scheme for the evaluation of α (Virendra Singh 1985).

Equation (4) can be rewritten as

$$g(\alpha x) + \alpha g(g(x)) = 0 \tag{6}$$

with the normalization $g(0) = 1$.

The function $g(x)$ is expanded as a power series in $|x|^z$ as

$$g(x) = 1 + p(x); \tag{7}$$

where
$$p(x) = \sum_{n=1}^{\infty} p_n |x|^{nz}. \tag{8}$$

Let

$$p(\alpha x) = \sum_{n=1}^{\infty} C_n [p(x)]^n. \tag{9}$$

Inserting (7) and (9) into (6) we get

$$1 + \sum_{n=1}^{\infty} C_n [p(x)]^n + \alpha g(1 + p(x)) = 0. \tag{10}$$

Equating coefficients of various powers of $p(x)$ to zero, we get

$$1+\alpha g(1)=0, \tag{11}$$

and
$$n!\,C_n+\alpha f^{(n)}(1)=0, \quad (n=1,2,3\ldots), \tag{12}$$

where
$$f^{(l)}(1)=\sum_{r=1}^{\infty}\frac{(rz)!}{l!(rz-l)!}p_r; \quad (l=1,2,3\ldots), \tag{13}$$

and

$$f(1)=1+\sum_{r=1}^{\infty}p_r. \tag{14}$$

Expanding both sides of (10) in powers of $|x|^z$ and equating the coefficients

$$p_n|\alpha|^{nz}=\sum_{l,m_1,m_2\cdots m_l\geqslant 1}C_l p_{m_1}p_{m_2}\cdots p_{ml}\,\delta_{m_1+m_2+\cdots+m_l,n}. \tag{15}$$

Further, the coefficients S_n are defined as

$$p_n\alpha^n=S_n|\alpha|^z \qquad (n=1,2,3\cdots), \tag{16}$$

where z is an even integer. We arrive at the following set of equations, combining (11), (13), (14), (15) and (16).

$$\frac{1}{\alpha}+1+|\alpha|^z\sum_{r=1}^{\infty}\frac{S_r}{\alpha^r}=0, \tag{17}$$

$$\frac{1}{z}+\sum_{r\geqslant 1}^{\infty}\frac{rS_r}{\alpha^{r-1}}=0 \qquad (\text{for } n=1), \tag{18}$$

and
$$S_n\left[1-\frac{1}{|\alpha|^{z(n-1)}}\right]+\sum_{l\geqslant 2}^{n}\sum_{r\geqslant 1}^{\infty}\binom{rz}{l}\frac{S_r}{\alpha^{r-1}}$$
$$\times\sum_{m_1\geqslant 1,m_2\geqslant 1\cdots m_l\geqslant 1}\frac{S_{m_1}S_{m_2}S_{m_3}\cdots S_{m_l}\delta_{m_1+m_2+\cdots m_l,n}}{|\alpha|^{z(n-l)}}=0. \tag{19}$$

(for $n=2,3,4\ldots$)

For solving the above system of coupled nonlinear equations, the coefficients S_n are expanded in inverse powers of α;

$$S_n(\alpha)=\sum_{m=0}^{\infty}\frac{S_{nm}}{\alpha^m}. \tag{20}$$

Substituting this in (18) and (19) and equating coefficients of powers of $1/\alpha$ to zero, one can solve successively for the expansion coefficients S_{nm}. Then (17) gives α and (7) can be used to compute the function $g(x)$.

3. Evaluation of δ

The perturbation method summarized in the preceding section can be directly extended to a determination of the constant δ. This consists in expanding the function $h(x)$ also as a series in $|x|^z$. Accordingly we write $h(x)$ in the form,

$$h(x) = h_0\left[1 + \sum_{n=1}^{\infty} h_n|x|^{nz}\right], \tag{21}$$

where h_0 is a normalization factor. From (7)

$$g'(x) = \sum_{n=1}^{\infty} nz\, p_n |x|^{nz-1}, \tag{22}$$

where z is an even integer.

$$g'(g(x))h(x) = \sum_{n=1}^{\infty} nz\, p_n|g(x)|^{nz-1} h_0\left[1 + \sum_{m=1}^{\infty} h_m|x|^{mz}\right]. \tag{23}$$

Similarly,

$$h(g(x)) = h_0\left[1 + \sum_{m=1}^{\infty} h_m|g(x)|^{mz}\right]. \tag{24}$$

Substituting (23) and (24) in (5)

$$-\alpha\left[h_0\left\{1 + \sum_{m=1}^{\infty} h_m|g(x)|^{mz}\right\} + \sum_{n=1}^{\infty} nz\, p_n|g(x)|^{nz-1} h_0\left\{1 + \sum_{m=1}^{\infty} h_m|x|^{mz}\right\}\right]$$

$$= \delta h_0\left[1 + \sum_{m=1}^{\infty} h_m|\alpha|^{mz}|x|^{mz}\right]. \tag{25}$$

Equating the coefficients of $|x|^{nz}$ on both sides of (25) we get a set of equations. Introducing the coefficients S_n as defined in (16) these equations are

$$-\alpha\left[1 + \sum_{r\geq 1}^{\infty} h_r + |\alpha|^z \sum_{r\geq 1}^{\infty} \frac{rzS_r}{\alpha^r}\right] = \delta, \tag{26}$$

$$-\alpha\left[\sum_{r\geq 1}^{\infty}\binom{rz}{1} h_r \frac{S_1}{\alpha} + \sum_{r\geq 1}^{\infty} 2\binom{rz}{2}\frac{|\alpha|^{z-1}S_r S_1}{\alpha^r}\right.$$

$$\left. + h_1 \sum_{r\geq 1}^{\infty}\binom{rz}{1}\frac{S_r}{\alpha^r}\right] = \delta h_1, \qquad (\text{for } n = 1) \tag{27}$$

and

$$-\alpha\left[\sum_{l\geq 1}^{n}\sum_{r\geq 1}^{\infty}\left\{\binom{rz}{l}\frac{h_r}{\alpha^n} + (l+1)\binom{rz}{l+1}\frac{S_r}{\alpha^{n-z+r}}\right\}\right.$$

$$\times \sum_{m_1\geq 1,\ m_2\geq 1,\cdots m_l\geq 1} \frac{S_{m_1}S_{m_2}\cdots S_{m_l}}{\alpha^{z(n-l)}}\delta_{m_1+m_2+\cdots m_l, n}$$

$$+ \sum_{n'\geq 1}^{n-1}\sum_{l\geq 1}^{n-n'}\sum_{r\geq 1}^{\infty}\left\{h_{n'}(l+1)\binom{rz}{l+1}\frac{S_r}{\alpha^{n-n'-z+r}}\right\}$$

$$\times \sum_{m_1\geq 1, m_2\geq 1\cdots m_l\geq 1} \frac{S_{m_1}S_{m_2}\cdots S_{m_l}}{\alpha^{z(n-l)}}\delta_{m_1+m_2+\cdots m_l, n'}$$

$$\left. + h_n \sum_{r\geq 1}^{\infty}\frac{rzS_r}{\alpha^{r+(n-1)z}}\right] = \delta h_n. \tag{28}$$

(for $n = 2, 3, 4\cdots$).

These equations can be solved for δ and the coefficients h_n. Since δ multiplies h_n, it is convenient to write the above system of equations in matrix form,

$$[A][h] = \delta[h], \tag{29}$$

where $[h]$ is a column matrix defined by

$$[h] = \begin{pmatrix} 1 \\ h_1 \\ h_2 \\ \vdots \end{pmatrix}.$$

The elements of $[A]$ are obtained from (26) to (28). Then δ is the largest real eigenvalue of $[A]$, which is incidentally the only eigenvalue greater than one in the case we have studied. The other eigenvalues are apparently irrelevant in the present context (Chang and McCown 1985; Hu 1982).

In actual practice, one cannot include all the infinite number of coefficients in S_r and h_r. So the series has to be truncated. We use the following algorithm for this. The series in S_r is truncated by retaining all terms containing a definite power of $1/\alpha$. This procedure should be consistent with truncation employed in evaluating α. Then the maximum number of coefficients h_r that can be retained after the truncation in S_r, are included in the calculation. Such a systematic procedure is found to give better values for α and δ in successive approximations, as pointed out at the end of the paper.

4. Computation of α and δ for a quartic map

We use the expansion (20) in (18) and (19) and equate the coefficients of equal powers of $1/\alpha$, to arrive at the following sets of equations:

$$S_{10} = -1/z;$$

$$S_{11} + 2S_{20} = 0;$$

$$S_{12} + 2S_{21} + 3S_{30} = 0;$$

$$S_{13} + 2S_{22} + 3S_{31} + 4S_{40} = 0;$$

$$S_{14} + 2S_{23} + 3S_{32} + 4S_{41} + 5S_{50} = 0;$$

$$S_{15} + 2S_{24} + 3S_{33} + 4S_{42} + 5S_{51} + 6S_{60} = 0;$$

$$S_{16} + 2S_{25} + 3S_{34} + 4S_{43} + 5S_{52} + 6S_{61} + 7S_{70} = 0 \ldots$$

(for $n = 1$)

$$S_{20} + \frac{z(z-1)}{2!} S_{10}^3 = 0;$$

$$S_{21} + \frac{z(z-1)}{2!} 3S_{10}^2 S_{11} + \frac{2z(2z-1)}{2!} S_{20} S_{10}^2 = 0;$$

$$S_{22} + \frac{z(z-1)}{2!} (3S_{10}^2 S_{12} + 3S_{10} S_{11}^2) + \frac{2z(2z-1)}{2!} \times (S_{10}^2 S_{21} + 2S_{20} S_{11} S_{10}) + \frac{3z(3z-1)}{2!} S_{30} S_{10}^2 = 0;$$

$$S_{23}+\frac{z(z-1)}{2!}(3S_{10}^2S_{13}+6S_{10}S_{11}S_{12}+S_{11}^3)+\frac{2z(2z-1)}{2!}$$
$$\times(2S_{20}S_{10}S_{12}+S_{20}S_{11}^2+2S_{21}S_{10}S_{11}+S_{22}S_{10}^2)$$
$$+\frac{3z(3z-1)}{2!}(S_{10}^2S_{31}+2S_{30}S_{10}S_{11})+\frac{4z(4z-1)}{2!}S_{40}S_{10}^2=0;$$

$$S_{24}-S_{20}+\frac{z(z-1)}{2!}(3S_{10}^2S_{14}+6S_{10}S_{11}S_{13}+3S_{10}S_{12}^2+3S_{11}^2S_{12})$$
$$+\frac{2z(2z-1)}{2!}(2S_{20}S_{10}S_{13}+2S_{20}S_{11}S_{12}+2S_{21}S_{10}S_{12}+S_{21}S_{11}^2$$
$$+2S_{22}S_{10}S_{11}+S_{23}S_{10}^2)+\frac{3z(3z-1)}{2!}(2S_{31}S_{10}S_{11}+S_{30}S_{11}^2$$
$$+S_{32}S_{10}^2+2S_{30}S_{10}S_{12})+\frac{4z(4z-1)}{2!}(2S_{40}S_{10}S_{11}+S_{41}S_{10}^2)$$
$$+\frac{5z(5z-1)}{2!}S_{50}S_{10}^2=0;$$

$$S_{25}-S_{21}+\frac{z(z-1)}{2!}(3S_{10}^2S_{15}+6S_{10}S_{11}S_{14}+3S_{11}^2S_{13}+6S_{10}S_{12}S_{13}$$
$$+3S_{11}S_{12}^2)+\frac{2z(2z-1)}{2!}(2S_{20}S_{10}S_{14}+2S_{20}S_{11}S_{13}+S_{20}S_{12}^2$$
$$+2S_{21}S_{10}S_{13}+2S_{21}S_{11}S_{12}+2S_{22}S_{10}S_{12}+S_{22}S_{11}^2+2S_{23}S_{10}S_{11}$$
$$+S_{24}S_{10}^2)+\frac{3z(3z-1)}{2!}(2S_{30}S_{10}S_{13}+2S_{30}S_{11}S_{12}+2S_{31}S_{10}S_{12}$$
$$+S_{31}S_{11}^2+2S_{32}S_{10}S_{11}+S_{33}S_{10}^2)+\frac{4z(4z-1)}{2!}(2S_{40}S_{10}S_{12}$$
$$+S_{40}S_{11}^2+2S_{41}S_{10}S_{11}+S_{42}S_{10}^2)+\frac{5z(5z-1)}{2!}(2S_{50}S_{10}S_{11}$$
$$+S_{51}S_{10}^2)+\frac{6z(6z-1)}{2!}S_{60}S_{10}^2=0\ldots$$

(for $n=2$)

$$S_{30}+\frac{z(z-1)(z-2)}{3!}S_{10}^4=0;$$

$$S_{31}+\frac{z(z-1)(z-2)}{3!}4S_{10}^3S_{11}+\frac{2z(2z-1)(2z-2)}{3!}S_{20}S_{10}^3=0;$$

$$S_{32}+\frac{z(z-1)(z-2)}{3!}(4S_{10}^3S_{12}+6S_{10}^2S_{11}^2)+\frac{2z(2z-1)(2z-2)}{3!}$$
$$\times(3S_{20}S_{10}^2S_{11}+S_{21}S_{10}^3)+\frac{3z(3z-1)(3z-2)}{3!}S_{30}S_{10}^3=0;$$

$$S_{33}+\frac{z(z-1)(z-2)}{3!}(4S_{10}^3S_{13}+12S_{10}^2S_{11}S_{12}+4S_{10}S_{11}^3)$$
$$+\frac{2z(2z-1)(2z-2)}{3!}(3S_{20}S_{10}^2S_{12}+3S_{20}S_{10}S_{11}^2+3S_{21}S_{10}^2S_{11}$$
$$+S_{22}S_{10}^3)+\frac{3z(3z-1)(3z-2)}{3!}(S_{31}S_{10}^3+3S_{30}S_{10}^2S_{11})$$
$$+\frac{4z(4z-1)(4z-2)}{3!}S_{40}S_{10}^3=0;$$

$$S_{34}+\frac{z(z-1)(z-2)}{3!}(4S_{10}^3S_{14}+12S_{10}^2S_{11}S_{13}+12S_{10}S_{11}^2S_{12}$$
$$+6S_{10}^2S_{12}^2+S_{11}^4)+\frac{2z(2z-1)(2z-2)}{3!}(3S_{20}S_{10}^2S_{13}$$
$$+6S_{20}S_{10}S_{11}S_{12}+S_{20}S_{11}^3+3S_{21}S_{10}^2S_{12}+3S_{21}S_{10}S_{11}^2$$
$$+3S_{22}S_{10}^2S_{11}+S_{23}S_{10}^3)+\frac{3z(3z-1)(3z-2)}{3!}(3S_{30}S_{10}^2S_{12}$$
$$+3S_{30}S_{10}S_{11}^2+3S_{31}S_{10}^2S_{11}+S_{30}S_{10}^3)+\frac{4z(4z-1)(4z-2)}{3!}$$
$$\times(3S_{40}S_{10}^2S_{11}+S_{41}S_{10}^3)+\frac{5z(5z-1)(5z-2)}{3!}S_{50}S_{10}^3$$
$$+\frac{z(z-1)}{2!}2S_{20}S_{10}^2=0\ldots$$

(for $n=3$)

$$S_{40}+\frac{z(z-1)(z-2)(z-3)}{4!}S_{10}^5=0;$$

$$S_{41}+\frac{z(z-1)(z-2)(z-3)}{4!}5S_{10}^4S_{11}+\frac{2z(2z-1)(2z-2)(2z-3)}{4!}S_{20}S_{10}^4=0;$$

$$S_{42}+\frac{z(z-1)(z-2)(z-3)}{4!}(5S_{10}^4S_{12}+10S_{10}^3S_{11}^2)$$
$$+\frac{2z(2z-1)(2z-2)(2z-3)}{4!}(4S_{20}S_{10}^3S_{11}+S_{21}S_{10}^4)$$
$$+\frac{3z(3z-1)(3z-2)(3z-3)}{4!}S_{30}S_{10}^4=0;$$

$$S_{43}+\frac{z(z-1)(z-2)(z-3)}{4!}(5S_{10}^4S_{13}+20S_{10}^3S_{11}S_{12}+10S_{10}^2S_{11}^3)$$
$$+\frac{2z(2z-1)(2z-2)(2z-3)}{4!}(4S_{20}S_{10}^3S_{12}+6S_{20}S_{10}^2S_{11}^2+4S_{21}S_{10}^3S_{11}$$

$$+S_{22}S_{10}^4)+\frac{3z(3z-1)(3z-2)(3z-3)}{4!}(4S_{30}S_{10}^3S_{11}+S_{31}S_{10}^4)$$

$$+\frac{4z(4z-1)(4z-2)(4z-3)}{4!}S_{40}S_{10}^4=0\ldots$$

(for $n=4$)

$$S_{50}=0;$$

$$S_{51}+\frac{2z(2z-1)(2z-2)(2z-3)(2z-4)}{5!}S_{20}S_{10}^5=0;$$

$$S_{52}+\frac{2z(2z-1)(2z-2)(2z-3)(2z-4)}{5!}(5S_{10}^4S_{11}S_{20}+S_{21}S_{10}^5)$$

$$+\frac{3z(3z-1)(3z-2)(3z-3)(3z-4)}{5!}S_{30}S_{10}^5=0\ldots$$

(for $n=5$)

$$S_{60}=0;$$

$$S_{61}+\frac{2z(2z-1)(2z-2)(2z-3)(2z-4)(2z-5)}{6!}S_{20}S_{10}^6=0\ldots$$

(for $n=6$)

and

$$S_{70}=0\ldots \qquad \text{(for } n=7) \qquad (31)$$

For $z=4$, these equations are consistently solved to yield the following coefficients,

$$S_{10}=-1/4;\ S_{11}=-3/4^2;\ S_{12}=-3/4^3;\ S_{13}=-1/4^4;\ S_{14}=0$$
$$S_{15}=-3/4^2;\ S_{16}=489/4^7\ldots$$
$$S_{20}=6/4^3;\ S_{21}=3/4^3;\ S_{22}=-54/4^5;\ S_{23}=-30/4^4;$$
$$S_{24}=-1494/4^7;\ S_{25}=-12030/4^8\ldots$$
$$S_{30}=-1/4^3;\ S_{31}=9/4^4;\ S_{32}=110/4^5;\ S_{33}=190/4^5;$$
$$S_{34}=3938/4^7\ldots$$
$$S_{40}=1/4^5;\ S_{41}=-90/4^6;\ S_{42}=-870/4^7;\ S_{43}=-1755/4^7\ldots$$
$$S_{50}=0;\ S_{51}=21/4^6;\ S_{52}=159/4^7\ldots$$
$$S_{60}=0;\ S_{61}=-42/4^8\ldots$$
$$S_{70}=0. \qquad (32)$$

Equation (17) can be written in the form

$$\frac{1}{\alpha}+1+\alpha^3\sum_{r=0}^{\infty}\left[\frac{S_{1r}}{\alpha^r}+\frac{S_{2r}}{\alpha^{r+1}}+\frac{S_{3r}}{\alpha^{r+2}}+\frac{S_{4r}}{\alpha^{r+3}}+\frac{S_{5r}}{\alpha^{r+4}}+\ldots\right]=0. \qquad (33)$$

Substituting for the coefficients in (33) and retaining terms upto $1/\alpha^2$, we get

$$\alpha^5+0{\cdot}375\,\alpha^4+0{\cdot}0625\,\alpha^3-3{\cdot}91796875\,\alpha^2-3{\cdot}873046875\,\alpha$$
$$+0{\cdot}564453125=0. \qquad (34)$$

This equation is solved by the Newton-Raphson method and the solution works out to be $\alpha = 1{\cdot}690781026$. We observe that this value is in very good agreement with the numerical value, 1·69 (Mendes 1981).

Using the coefficients obtained in (32), the function $g(x)$ is written as,

$$g(x) = 1 - 1{\cdot}893139806\,x^4 + 0{\cdot}193320775\,x^8 + 0{\cdot}137176782\,x^{12} - 0{\cdot}030593823\,x^{16} + 0{\cdot}001793430346\,x^{20}\ldots \tag{35}$$

To evaluate δ, we use the coefficients S_{nm} in (26), (27) and (28) and retain terms up to $1/\alpha^2$. Thus we arrive at a set of equations, to be solved for δ. We give below the relevant equations for general z.

$$-\alpha(h_1+h_2+h_3+\cdots)-\alpha-z\alpha^z\left[S_{10}+\frac{S_{11}}{\alpha}+\frac{S_{12}}{\alpha^2}+\frac{S_{13}}{\alpha^3}+\frac{S_{14}}{\alpha^4}+\frac{S_{15}}{\alpha^5}+\frac{S_{16}}{\alpha^6}\right]-2z\alpha^{z-1}\left[S_{20}+\frac{S_{21}}{\alpha}+\frac{S_{22}}{\alpha^2}+\frac{S_{23}}{\alpha^3}+\frac{S_{24}}{\alpha^4}+\frac{S_{25}}{\alpha^5}\right]$$
$$-3z\alpha^{z-2}\left[S_{30}+\frac{S_{31}}{\alpha}+\frac{S_{32}}{\alpha^2}+\frac{S_{33}}{\alpha^3}+\frac{S_{34}}{\alpha^4}\right]-4z\alpha^{z-3}\times\left[S_{40}+\frac{S_{41}}{\alpha}+\frac{S_{42}}{\alpha^2}+\frac{S_{43}}{\alpha^3}\right]-5z\alpha^{z-4}\left[S_{50}+\frac{S_{51}}{\alpha}+\frac{S_{52}}{\alpha^2}\right]$$
$$-6z\alpha^{z-5}\left[S_{60}+\frac{S_{61}}{\alpha}\right]-7z\alpha^{z-6}S_{70}=\delta, \tag{36}$$

$$-[zh_1+2zh_2+3zh_3+\cdots]\left[S_{10}+\frac{S_{11}}{\alpha}+\frac{S_{12}}{\alpha^2}\right]-z(z-1)\alpha^{z-1}$$
$$\times\left[S_{10}^2+\frac{1}{\alpha}2S_{10}S_{11}+\frac{1}{\alpha^2}(2S_{10}S_{12}+S_{11}^2)+\frac{1}{\alpha^3}(2S_{10}S_{13}+2S_{11}S_{12})\right.$$
$$+\frac{1}{\alpha^4}(2S_{10}S_{14}+2S_{11}S_{13}+S_{12}^2)+\frac{1}{\alpha^5}(2S_{10}S_{15}+2S_{11}S_{14}$$
$$\left.+2S_{12}S_{13})\right]-2z(2z-1)\alpha^{z-2}\left[S_{20}S_{10}+\frac{1}{\alpha}(S_{20}S_{11}+S_{10}S_{21})\right.$$
$$+\frac{1}{\alpha^2}(S_{12}S_{20}+S_{21}S_{11}+S_{22}S_{10})+\frac{1}{\alpha^3}(S_{20}S_{13}+S_{21}S_{12}$$
$$+S_{22}S_{11}+S_{23}S_{10})+\frac{1}{\alpha^4}(S_{20}S_{14}+S_{21}S_{13}+S_{22}S_{12}+S_{23}S_{11}$$
$$\left.+S_{24}S_{10})\right]-3z(3z-1)\alpha^{z-3}\left[S_{30}S_{10}+\frac{1}{\alpha}(S_{30}S_{11}+S_{31}S_{10})\right.$$
$$+\frac{1}{\alpha^2}(S_{30}S_{12}+S_{31}S_{11}+S_{32}S_{10})+\frac{1}{\alpha^3}(S_{30}S_{13}+S_{31}S_{12}+S_{32}S_{11}$$
$$\left.+S_{33}S_{10})\right]-4z(4z-1)\alpha^{z-4}\left[S_{40}S_{10}+\frac{1}{\alpha}(S_{40}S_{11}+S_{41}S_{10})\right.$$

$$+\frac{1}{\alpha^2}(S_{40}S_{12}+S_{41}S_{11}+S_{42}S_{10})\Big]-5z(5z-1)\alpha^{z-5}\Big[S_{50}S_{10}$$

$$+\frac{1}{\alpha}(S_{50}S_{11}+S_{51}S_{10})\Big]-6z(6z-1)\alpha^{z-6}S_{60}S_{10}$$

$$-h_1\left[z\left(S_{10}+\frac{S_{11}}{\alpha}+\frac{S_{12}}{\alpha^2}\right)+2z\left(\frac{S_{20}}{\alpha}+\frac{S_{21}}{\alpha^2}\right)+3z\frac{S_{30}}{\alpha^2}\right]=\delta h_1,$$

$$\text{(for } n=1)\ldots \qquad (37)$$

$$-\frac{z(z-1)}{\alpha^2}S_{20}S_{10}-\left[\frac{z(z-1)}{2!}h_1+\frac{2z(2z-1)}{2!}h_2+\frac{3z(3z-1)}{2!}h_3+\ldots\right]$$

$$\times\left[\frac{S_{10}^2}{\alpha}+\frac{2S_{10}S_{11}}{\alpha^2}\right]-\frac{z(z-1)(z-2)}{2!}\alpha^{z-2}\Big[S_{10}^3+\frac{3S_{10}^2S_{11}}{\alpha}$$

$$+\frac{1}{\alpha^2}(3S_{10}^2S_{12}+3S_{10}S_{11}^2)+\frac{1}{\alpha^3}(3S_{10}^2S_{13}+6S_{10}S_{11}S_{12}+S_{11}^3)$$

$$+\frac{1}{\alpha^4}(3S_{10}^2S_{14}+6S_{10}S_{11}S_{13}+3S_{11}^2S_{12}+3S_{10}S_{12}^2)\Big]$$

$$-\frac{2z(2z-1)(2z-2)}{2!}\alpha^{z-3}\Big[S_{20}S_{10}^2+\frac{1}{\alpha}(2S_{20}S_{10}S_{11}+S_{21}S_{10}^2)$$

$$+\frac{1}{\alpha^2}(2S_{20}S_{10}S_{12}+S_{20}S_{11}^2+2S_{21}S_{10}S_{11}$$

$$+S_{22}S_{10}^2)+\frac{1}{\alpha^3}(2S_{20}S_{10}S_{13}$$

$$+2S_{20}S_{11}S_{12}+2S_{21}S_{10}S_{12}+S_{21}S_{11}^2+2S_{22}S_{10}S_{11}+S_{23}S_{10}^2)\Big]$$

$$-\frac{3z(3z-1)(3z-2)}{2!}\alpha^{z-4}\Big[S_{30}S_{10}^2+\frac{1}{\alpha}(2S_{30}S_{10}S_{11}+S_{31}S_{10}^2)$$

$$+\frac{1}{\alpha^2}(2S_{30}S_{10}S_{12}+S_{30}S_{11}^2+2S_{31}S_{10}S_{11}+S_{32}S_{10}^2)\Big]$$

$$-\frac{4z(4z-1)(4z-2)}{2!}\alpha^{z-5}\left[S_{40}S_{10}^2+\frac{1}{\alpha}(2S_{40}S_{10}S_{11}+S_{41}S_{10}^2)\right]$$

$$-\frac{5z(5z-1)(5z-2)}{2!}\alpha^{z-6}\cdot S_{50}S_{10}^2-\Big[\frac{z(z-1)}{\alpha}\left(S_{10}^2+\frac{2S_{10}S_{11}}{\alpha}\right)$$

$$+\frac{2z(2z-1)}{\alpha^2}S_{20}S_{10}\Big]h_1=\delta h_2$$

$$\text{(for } n=2) \qquad (38)$$

$$-\Big[\frac{z(z-1)(z-2)}{3!}h_1+\frac{2z(2z-1)(2z-2)}{3!}h_2$$

$$+\frac{3z(3z-1)(3z-2)}{3!}h_3+\cdots\Bigg]\frac{S_{10}^3}{\alpha^2}$$

$$-\frac{z(z-1)(z-2)(z-3)}{3!}\alpha^{z-3}\Bigg[S_{10}^4+\frac{4S_{10}^3S_{11}}{\alpha}+\frac{1}{\alpha^2}(4S_{10}^3S_{12}+6S_{10}^2S_{11}^2)$$

$$+\frac{1}{\alpha^3}(4S_{10}^3S_{13}+12S_{10}^2S_{11}S_{12}+4S_{10}S_{11}^3)\Bigg]-\frac{2z(2z-1)(2z-2)(2z-3)}{3!}$$

$$\times\alpha^{z-4}\Bigg[S_{20}S_{10}^3+\frac{1}{\alpha}(3S_{20}S_{10}^2S_{11}+S_{21}S_{10}^3)+\frac{1}{\alpha^2}(3S_{20}S_{10}^2S_{12}$$

$$+3S_{20}S_{10}S_{11}^2+3S_{21}S_{10}^2S_{11}+S_{22}S_{10}^3)\Bigg]-\frac{3z(3z-1)(3z-2)(3z-3)}{3!}$$

$$\times\alpha^{z-5}\Bigg[S_{30}S_{10}^3+\frac{1}{\alpha}(3S_{30}S_{10}^2S_{11}+S_{31}S_{10}^3)\Bigg]$$

$$-\frac{4z(4z-1)(4z-2)(4z-3)}{3!}S_{40}S_{10}^3\alpha^{z-6}-\frac{z(z-1)(z-2)}{2!}\frac{S_{10}^3}{\alpha^{z-2}}h_1$$

$$=\delta h_3$$

$$(\text{for } n=3). \tag{39}$$

Putting $z = 4$ in the above four equations we get,

$$\begin{bmatrix} 6{\cdot}48161599 & -1{\cdot}690781026 & -1{\cdot}690781026 & -1{\cdot}690781026 \\ -0{\cdot}67462368 & 2{\cdot}509170305 & 3{\cdot}018340608 & 4{\cdot}527510912 \\ -2{\cdot}93711342 & -0{\cdot}796549616 & -1{\cdot}953261165 & -4{\cdot}604115602 \\ 1{\cdot}13037874 & 0{\cdot}0874511 & 0{\cdot}30607885 & 1{\cdot}202452628 \end{bmatrix}\begin{bmatrix}1\\h_1\\h_2\\h_3\end{bmatrix}=\delta\begin{bmatrix}1\\h_1\\h_2\\h_3\end{bmatrix} \tag{40}$$

We find the coefficient matrix on the right hand side of the above equation has only one eigenvalue greater than 1. This is the acceptable value of δ, which works out to be 7·23682924. It is interesting to note that this value is in good agreement with the numerically computed value, 7·284 (Mendes 1981).

5. Discussion

We have extended Virendra Singh's perturbative method to calculate the constant δ for general nonlinear maps. The constant α and δ for a quartic map have been determined. We have also used this approach to determine the δ value for a quadratic map and found that it is equal to 4·6602965.

It has come to our notice that in the calculation of δ using the series (20), the choice of a cut-off is crucial. The contribution of the different terms does not decrease uniformly with the order of the term as it should in the case of a convergent perturbation series. On the contrary the first few terms show a decrease followed by an increase at the next stage, a behaviour characteristic of asymptotic series. Hence the selection of the truncation point may be dictated by the requirement that the best value of δ relative to the numerically available value is obtained. There is apparently no self-consistent criterion for choosing the cut-off stage. With the increase in the order in the asymptotic series (20), the value may in all probability oscillate about the actual value.

Even with the imposition of a definite cut-off in terms of a power in $1/\alpha$ in the series (20), there is considerable latitude in building the column matrix $[h]$. Thus different approximations result by taking different components of $[h]$ successively from the top (see equation (30)). In four different runs, we have taken $[h]$ as (1), $(1, h_1)$, $(1, h_1, h_2)$ and $(1, h_1, h_2, h_3)$ where the brackets represent columns. The corresponding δ values obtained are; 6·48161599, 6·757592744, 7·48537771 and 7·23682924, the last value being the closest to the numerically computed one, namely 7·284 (Mendes 1981). It would be interesting to compare this behaviour to that found in a renormalization group calculation (Hauser *et al* 1984) wherein the values 9·31426, 8·08956 and 6·99948 are obtained corresponding to 1-2 cycles, 1-4 cycles and 2-4 cycles renormalizations. We find that the perturbative method herein employed is far less cumbersome than the renormalization group approach and yields the best value in a sufficiently small number of steps.

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