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Analysis of a nonlinear stochastic model of cooperative behaviour

M C VALSAKUMAR
Materials Science Laboratory, Indira Gandhi Centre for Atomic Research, Kalpakkam 603 102, India

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Abstract. A stochastic model of cooperative behaviour is analyzed with regard to its critical properties. A cumulant expansion to fourth order is used to truncate the infinite set of coupled evolution equations for the moments. Linear stability analysis is performed around all the permissible steady states. The method is shown to be incapable of reproducing the critical boundary and the nature of the phase transition. A linearization, which respects the symmetry of the potential, is proposed which reproduces all the basic features associated with the model. The dynamics predicted by this approximation is shown to agree well with the Monte-Carlo simulation of the nonlinear Langevin equation.

Keywords. Nonlinear Langevin equation; phase transition; critical exponent; cumulant expansion; Gaussian decoupling.

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1. Introduction

The relaxation to equilibrium of physical systems far from equilibrium can often be modelled in terms of nonlinear Langevin equations (van Kampen 1976, 1981; Fox 1978). Closed form solutions of these equations are seldom possible and hence one resorts to approximations. Depending upon the type of nonlinearity and the initial condition, different scenarios are possible for the fluctuations. Each of these has to be analyzed using a method devised specifically for that case (see for example van Kampen 1961; Budgor 1976; Suzuki 1977, 1981; Caroli *et al* 1979; Dekker 1980; Valsakumar *et al* 1983; Valsakumar 1983, 1985).

Fluctuations in physical systems are small and that forms the basis for all the above mentioned approximation methods. However, there can be cases which do not fall within the scope of these methods. One such situation arises when the system undergoes a phase transition. The interest is in determining the critical value D_c of the strength of the fluctuations below which the system is ordered, the nature of the phase transition (continuous or discontinuous), the critical exponent associated with the order parameter, etc. Clearly, the fact that the fluctuations are small cannot be used for the determination of D_c.

Approximation methods for this situation do not seem to have been well developed. A canonical example of problems in this category is the model introduced by Kometani and Shimuzu (1975). The model consists of a collection of anharmonic oscillators interacting through a mean field potential. Our concern with this model will be in the limit of the number of oscillators going to infinity. In the Gaussian decoupling

(Kometani and Shimuzu 1975) of the higher order moments (to truncate the infinite set of coupled evolution equations for the moments) the model shows a phase transition. The nature of the phase transition (in this approximation) depends upon the strength θ of the mean field interaction. While the transition is continuous for $\theta \geqslant 2$, it is discontinuous for $\theta < 2$.

This was followed by the detailed analyses of Desai and Zwanzig (1978) and Dawson (1983). The analysis (Desai and Zwanzig 1978) of the exact asymptotic solution revealed the emergence of a non-zero order parameter in addition to the trivial branch, for some $D = D_c$, for a given θ (with the order parameter $\to 0$ as $D \to D_c$). A rigorous perturbation analysis (Dawson 1983) indicated the dynamical instability of the trivial branch at D_c. In other words, the point of bifurcation of the asymptotic solution and the point of dynamical instability of the trivial branch are identical in this model, for all θ. This implies a continuous phase transition. The exponent associated with the order parameter is seen to be 1/2.

Any good approximation scheme should reproduce these basic features of the model, in addition to obtaining the critical boundary accurately. Desai and Zwanzig (1978) have also obtained the dynamics by cumulant expansion of the characteristic function, to fourth and sixth orders. For the particular initial conditions chosen by them, the cumulant expansion method seems to work well. They have not, however, evaluated the critical boundary given by this approximation. Nor have they performed the stability analysis (Minorski 1962) to determine the order of the transition.

One of the objectives of the present work is to carry out a detailed stability analysis (around all the permissible steady states) of the model in the approximation of cumulant expansion to fourth order. Our analysis reveals many interesting results. First, we find that this approximation does not reproduce satisfactorily the value of D_c for small θ (such values of θ being more interesting). Second, like the Gaussian decoupling, this too shows a discontinuous transition, but for $\theta < 1{\cdot}295$ rather than for $\theta < 2$. Finally, for small θ, there exists a range of D for which the trivial branch and two nontrivial branches are simultaneously stable. This conclusion is also borne out by the numerical integration of the decoupled equations for the moments.

On a more constructive note, we have presented a linearization scheme which reproduces well the critical boundary, the order of the transition and the critical exponent. Comparison of the dynamics obtained using this scheme shows very good agreement with the Monte-Carlo simulation of the original Langevin equation, for small values of D.

The paper is organized as follows. The model and the Gaussian decoupling are discussed in detail in §2. Section 3 is devoted to a detailed analysis of the cumulant expansion. The new linearization is discussed in §4, and the conclusions in §5.

2. The model and the gaussian decoupling

2.1 *The model*

We consider the nonlinear stochastic differential equation

$$\dot{x}(t) = (1-\theta)x(t) - x^3(t) + \theta m_1(t) + \eta(t), \tag{1}$$

where the overdot denotes differentiation with respect to time t, $m_1(t)$ is the expectation value of $x(t)$ and $\eta(t)$ is a Gaussian white noise source with the normalization

$$\langle \eta(t) \rangle = 0 \quad \text{and} \quad \langle \eta(t)\eta(t+\tau) \rangle = 2D\delta(\tau). \tag{2}$$

Desai and Zwanzig (1978) and Dawson (1983) have shown that the above equations approximately represent the motion of a single oscillator in an infinite collection of anharmonic oscillators with a mean field interaction (introduced by Kometani and Shimizu (1975) to model muscle contraction). In (1) θ is the strength of the mean field interaction.

The stationary probability distribution associated with (1) is given by (Dawson 1983)

$$P_{st}(x) = Z^{-1} \exp\left\{-\frac{1}{D}\left[\frac{x^4}{4} + (\theta-1)\frac{x^2}{2} - \theta x_0 x\right]\right\}, \tag{3}$$

where Z is the normalization constant and

$$x_0 = m_1(\infty) = Z^{-1} \int_{-\infty}^{\infty} \mathrm{d}x \exp\left\{-\frac{1}{D}\left[\frac{x^4}{4} + (\theta-1)\frac{x^2}{2} - \theta x_0 x\right]\right\}. \tag{4}$$

It is clear that $x_0 = 0$ is always a solution of (4). Expressing the integral in (4) in terms of parabolic cylinder functions $D_\nu(Z)$ (Erdelyi 1953) and using the rule of signs (Korn and Korn 1961) for the roots of the algebraic equation, it is possible to look for the existence of non-trivial solutions to (4). For a given θ, there exists a critical value D_c of the diffusion constant below which this occurs. D_c is given by

$$D_c^{1/2} = \frac{\theta}{\sqrt{2}} \frac{D_{-3/2}(z)}{D_{-1/2}(z)}, \quad z = (\theta - 1)(2D_c)^{-1/2}. \tag{5}$$

A plot of D_c as a function of θ is given in figure 1, for $0{\cdot}001 \leqslant \theta \leqslant 10$. An outline of the numerical procedure used to obtain this is given in Appendix A. It may be noted that D_c is of the order of θ when $\theta \ll 1$.

2.2 *The Gaussian decoupling*

We have seen that the steady state solution bifurcates at $D = D_c$. In order to prove the existence of the phase transition, we have to show that the trivial steady state becomes unstable and the nontrivial one becomes stable at D_c. We must also find the value of x_0 (in the nontrivial branch) at D_c in order to determine the nature of the transition. One of the ways of doing this is to perform a linear stability analysis of the equations for the moments, namely,

$$\begin{aligned} \dot{m}_n(t) = n[&(1-\theta)m_n(t) - m_{n+2}(t) + \theta m_1(t) m_{n-1}(t) \\ &+ (n-1)D m_{n-2}(t)]. \; n = 1, \ldots \infty, \end{aligned} \tag{6}$$

around the steady state solutions of (6). However, these equations are open-ended. Some decoupling procedure must therefore be used to truncate the set. The simplest approximation is to treat the stochastic process $x(t)$ as Gaussian and decouple the third and fourth moments appearing in the equations for $m_1(t)$ and $m_2(t)$ accordingly. This

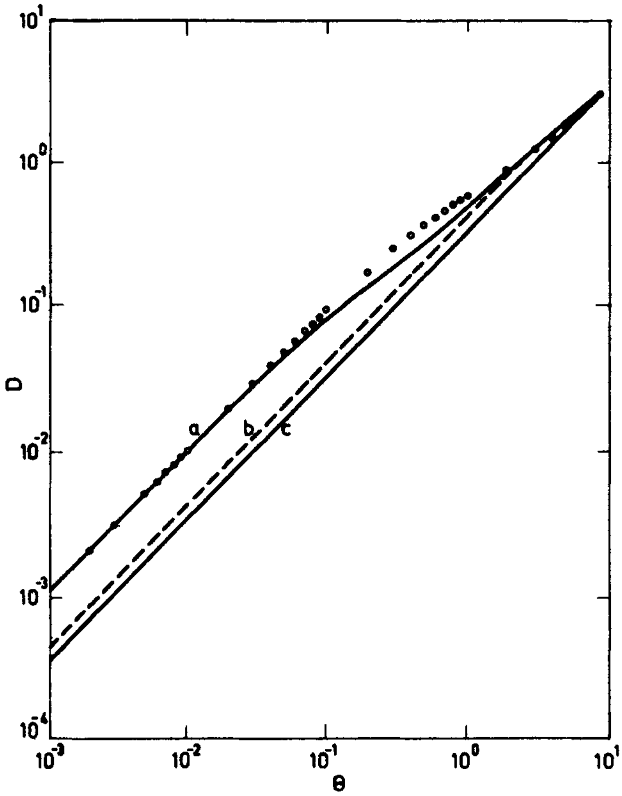

Figure 1. The critical value D_c of the diffusion constant D at which the trivial steady state becomes unstable is plotted as a function of θ. The curves a, b, c correspond to the exact result, the cumulant expansion to fourth order and the Gaussian decoupling, respectively. The open circles are obtained using self-consistent linearization of the present work.

yields

$$\dot{m}_1(t) = m_1(t)[1 - 3m_2(t) + 2m_1^2(t)],$$
$$\dot{m}_2(t) = 2[(1-\theta)m_2(t) - 3m_2^2(t) + 2m_1^4(t) + \theta m_1^2(t) + D]. \tag{7}$$

The trivial steady state of (7) is

$$m_1(\infty) = 0, \quad m_2(\infty) = \{1 - \theta + [(1-\theta)^2 + 12D]^{1/2}\}/6, \tag{8a}$$

for all θ and D. The nontrivial steady states are given by

$$m_1^2(\infty) = \tfrac{1}{4}\{2 - \theta + [(2+\theta)^2 - 24D]^{1/2}\}, m_2(\infty) = \tfrac{1}{3}\{1 + 2m_1^2(\infty)\},$$

for $\quad \theta < 2, 0 \leqslant D \leqslant \dfrac{(2+\theta)^2}{24}$ and $\theta \geqslant 2, 0 \leqslant D \leqslant \theta/3;$ (8b)

and

$$m_1^2(\infty) = \tfrac{1}{4}\{2 - \theta - [(2+\theta)^2 - 24D]^{1/2}\}, m_2(\infty) = \tfrac{1}{3}\{1 + 2m_1^2(\infty)\},$$

for $\quad \theta < 2 \quad$ and $\quad \dfrac{\theta}{3} \leqslant D \leqslant \dfrac{(2+\theta)^2}{24}.$ (8c)

Linear stability analysis of (7) around the steady states shows that the trivial branch given by (8a) becomes unstable at $D = D_c^{(G)} = \theta/3$; the nontrivial branch given by (8b) becomes stable at $D_c^{(G)} = \theta/3$; and the other non-trivial branch is always unstable.

It thus becomes clear that the bifurcation of the asymptotic solution and the dynamical instability occur at the same value of D when $\theta \geqslant 2$. However, this is not so for $\theta < 2$. This can also be seen from the value of the non-trivial order parameter $m_{1c}(\infty)$ at $D_c^{(G)} = \theta/3$. The result is

$$m_{1c}(\infty) = \pm\tfrac{1}{2}\{[2-\theta+|2-\theta|]^{1/2}\} = \begin{cases} 0, \text{ for } \theta \geqslant 2 \\ \pm(1-\theta/2)^{1/2}, \text{ for } \theta < 2. \end{cases} \tag{9}$$

We have presented the value of $D_c^{(G)}$, obtained by this approximation, as a function of θ in figure 1. It is clear that the agreement with the exact (numerical) result is not at all good for small θ. It is interesting to observe that the agreement is reasonably good for $\theta \gg 1$. We will come back to this point in §4.

3. The cumulant expansion

Having seen how the Gaussian decoupling fares, we would like to investigate whether higher order cumulant expansions yield better results or not. It is well known that the cumulant expansion is not a systematic expansion in any small parameter. But the hope is that the higher order expansions will be better than Gaussian decoupling (which is a cumulant expansion to second order). Like the Gaussian decoupling, this is also easy to implement.

We now proceed to analyze (6) for the moments m_n with the fourth order cumulant expansion. This amounts to neglecting all the cumulants k_n for $n \geqslant 5$. This in turn leads to the approximate relations

$$m_5 = 5m_4m_1 + 10m_3m_2 - 20m_3m_1^2 - 30m_2^2m_1 + 60m_2m_1^3 - 24m_1^5, \tag{10a}$$

and

$$m_6 = 15m_4m_2 + 10m_3^2 - 60m_3m_2m_1 - 30m_2^3 + 90m_2^2m_1^2 - 24m_1^6. \tag{10b}$$

The evolution equations to be analyzed are

$$\dot{m}_1 = m_1 - m_3,$$

$$\dot{m}_2 = 2[(1-\theta)m_2 - m_4 + \theta m_1^2 + D],$$

$$\dot{m}_3 = 3[(1-\theta)m_3 - m_5 + \theta m_1m_2 + 2Dm_1],$$

and

$$\dot{m}_4 = 4[(1-\theta)m_4 - m_6 + \theta m_1m_3 + 3Dm_2]. \tag{11}$$

The trivial steady states of (11) are given by

$$m_1(\infty) = m_3(\infty) = 0, \tag{12a}$$

$$m_2^3(\infty) - \frac{1-\theta}{2}m_2^2(\infty) + \frac{(1-\theta)^2 - 12D}{30}m_2(\infty) + \frac{D(1-\theta)}{30} = 0, \tag{12b}$$

and

$$m_4(\infty) = (1-\theta)m_2(\infty) + D. \tag{12c}$$

Applying the rule of signs, (12b) allows for one positive root for $\theta > 1$ and two for $\theta < 1$.

The nontrivial steady states are given by the coupled equations

$$m_1(\infty) = m_3(\infty), \tag{13a}$$

$$m_4(\infty) = (1-\theta)m_2(\infty) + \theta m_1^2(\infty) + D, \tag{13b}$$

$$30m_2^2(\infty) - [15 - 6\theta + 60m_1^2(\infty)]m_2(\infty) + 24m_1^4(\infty) + (20 - 5\theta)m_1^2(\infty) + 1 - \theta - 3D = 0, \tag{13c}$$

and

$$24m_1^6(\infty) - 90m_2^2(\infty)m_1^2(\infty) + (60 - 15\theta)m_2(\infty)m_1^2(\infty) + [\theta(2-\theta) - 10]m_1^2(\infty) + 30m_2^3(\infty) - 15(1-\theta)m_2^2(\infty) + [(1-\theta)^2 - 12D]m_2(\infty) + D(1-\theta) = 0. \tag{13d}$$

The method solution of (13c) and (13d) is discussed briefly in Appendix B. The number of nontrivial positive steady states is seen to be six for $\theta \ll 1$ and three for $\theta \sim 1$.

We have performed the stability analysis of (11) around all the permissible steady states. The details of the eigenvalue equation associated with the stability matrix are given in Appendix C. The results are as follows:

One of the trivial steady states, which is stable for large D, becomes unstable at a critical value $D_c^{(K)}$ of the diffusion coefficient. The second trivial branch, which exists for $\theta < 1$, is always unstable. A plot of $D_c^{(K)}$ as a function of θ is given in figure 1. It can be seen that $D_c^{(K)} \sim 0{\cdot}4\theta$ for $\theta < 1$ and that it tends to the Gaussian value for $\theta \gg 1$. Thus the agreement is not good for $\theta < 1$.

Our numerical study shows that the phase transition, in this approximation, is continuous for $\theta \geqslant 1{\cdot}295$. However, the transition is discontinuous for $\theta < 1{\cdot}295$. Figure 2 shows a plot of the permissible steady states for $\theta = 1$, as a function of D. It is obvious that the value of the order parameter at $D_c^{(K)} (= 0{\cdot}4)$ is nonzero. Further, for $0{\cdot}4 \leqslant D \leqslant 0{\cdot}4044$, both the trivial branch and the nontrivial branch b are stable.

The situation becomes still more complicated when $\theta \ll 1$. There exists a wide range of D (for any $\theta \ll 1$) where two non-trivial steady states and the trivial steady state are simultaneously stable. Figure 3 is a plot of the permissible steady states, as a function of D, for a typical value of $\theta = 10^{-2}$. Note that at $D_c^{(K)} = 0{\cdot}00992$, the trivial branch becomes unstable and the nontrivial branch a alone is stable. However, for $0{\cdot}0435 < D < 0{\cdot}061$, the nontrivial branches a and c as well as the trivial branch are all stable.*

We have integrated (11) to crosscheck this observation. The computation was performed using the standard Runge-Kutta-Gill algorithm (Korn and Korn 1961). Figure 4 shows a plot of $[m_1(\infty) - m_1(t)]/(m_1(\infty) - m_1(0))$ as a function of t for three different initial conditions belonging respectively to the neighbourhoods of the above-

* In a parameter regime in which the dynamical system is confronted with more than one stable steady state, the 'phase space' of the system breaks up into sectors.

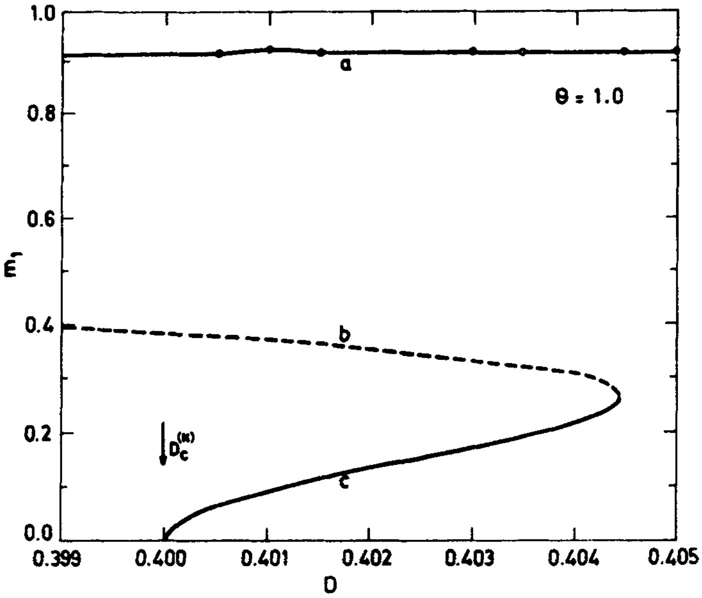

Figure 2. The positive nontrivial steady states m_1 obtained with the cumulant expansion to fourth order are plotted as a function of D for $\theta = 1$. The branches a and c are always unstable. The remaining branch b is stable. Notice that the value of m_1 along this branch is non-zero when $D = D_c^{(K)}$.

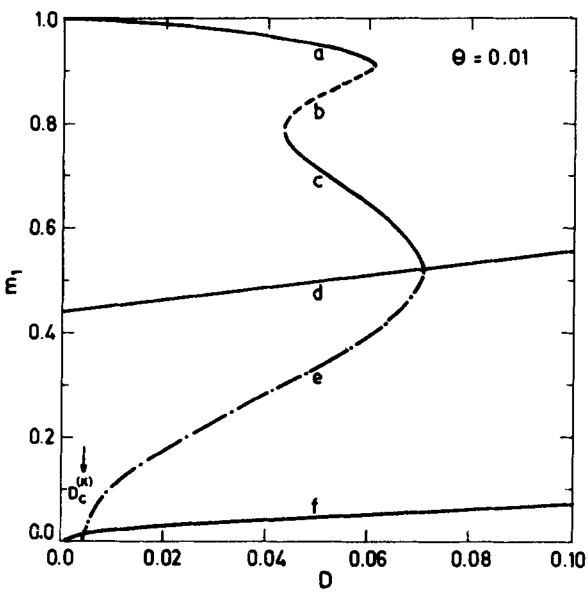

Figure 3. The positive nontrivial steady state values m_1 obtained with the cumulant expansion to fourth order are plotted as functions of D for $\theta = 0{\cdot}01$. The branches a and c are always stable. Other branches are unstable.

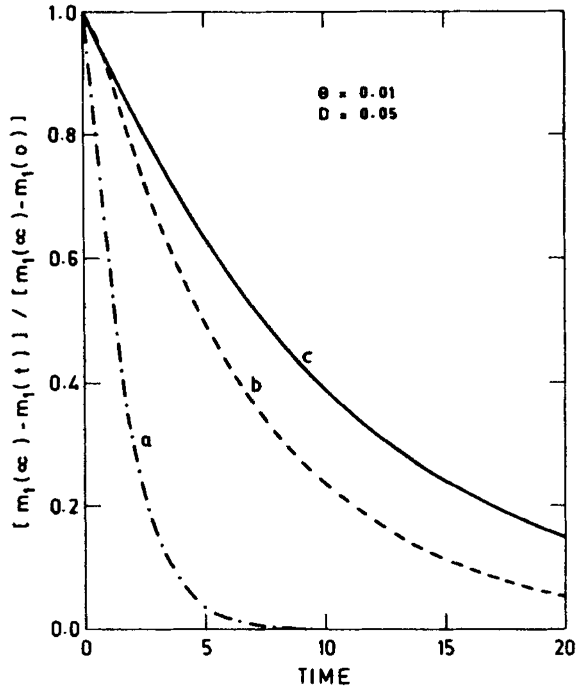

Figure 4. The time evolution of $m_1(t)$ obtained with the cumulant expansion to fourth order for three different initial conditions. Computations were done for $\theta = 0{\cdot}01$ and $D = 0{\cdot}05$ with the initial conditions $(m_1(0), m_2(0), m_3(0), m_4(0) = (0{\cdot}9525, 0{\cdot}9401, 0{\cdot}9540, 0{\cdot}9902), (0{\cdot}7114, 0{\cdot}7280, 0{\cdot}7113, 0{\cdot}7758)$ and $(0{\cdot}001, 0{\cdot}4597, 0{\cdot}001, 0{\cdot}5050))$ for curves a, b and c, respectively.

mentioned steady states. It can be seen that all these relax to the corresponding steady states, in accordance with the above observation.

To summarize, we find that the cumulant expansion to fourth order converges to Gaussian decoupling when $\theta \gg 1$. In the other limit of $\theta \ll 1$, the results of the cumulant expansion are very different from those of the Gaussian decoupling. Agreement with the exact results is not satisfactory, either, in this limit.

4. Self-consistent linearization

Linearization of nonlinear Langevin equations has been discussed extensively in literature (Budgor 1976; Eaves and Reinhardt 1981; Valsakumar *et al* 1983; West *et al* 1983; Indira *et al* 1983; Ito 1984). In this scheme, the original nonlinear Langevin equation is replaced by a linear Langevin equation with time-dependent coefficients. The probability distribution of this equivalent linear equation is Gaussian (since the driving noise is Gaussian and white). Hence self-consistency demands the higher order moments (appearing as coefficients in the effective equation) to be decoupled using the solution of the latter equation, which is Gaussian. Among all possible linearizations, this Gaussian decoupling is seen to minimize the error (or Kullback-Leibler entropy (Friedman 1975)) in the probability distribution. Gaussian decoupling performs extremely well when the asymptotic distribution is going to be unimodal (Valsakumar *et al* 1983; West *et al* 1983; Indira *et al* 1983).

The above result has to be accepted with some caution. There could be situations where the non-Gaussian character of the process is important. A simple example of this is diffusion in a bistable potential. Clearly one cannot get any satisfactory approximation to the probability distribution using a linear Langevin equation, which always leads to a Gaussian distribution. One could, however, try to obtain a reasonably good description of the first few moments. This is possible if one decouples the higher order moments appearing in their evolution equations using an ad hoc probability distribution which respects the *symmetry* of the potential. This idea has been applied to the problem of diffusion in a bistable potential, with remarkable success (Suzuki 1981; Valsakumar *et al* 1983).

We now proceed to analyze (1) in this perspective. We note that (1), in the limit $\theta \to 0$, becomes

$$\dot{x} = x - x^3 + \eta(t). \tag{14}$$

This equation for diffusion in a bistable potential has been studied extensively (Suzuki 1977, 1981; Haake 1978; de Pasquale and Tombesi 1979; Dekker 1983; Valsakumar 1983, 1985). The scaling theory gives the best account of fluctuations in this system. On the other hand, if one chooses to work in the framework of linearization, then the Suzuki Ansatz (Suzuki 1981)

$$x^3 = \langle x^2 \rangle x = m_2 x, \tag{15}$$

gives the best description of fluctuations. This ad hoc decoupling has been justified (Valsakumar *et al* 1983) using a bimodal Gaussian decoupling, which respects the symmetry of the potential. Thus any decoupling of x^3 we propose must also tend to (15) in the limit $\theta \to 0$.

We have also noted (see §2) that the Gaussian decoupling is satisfactory when $\theta \gg 1$. We therefore feel that a 'smooth' interpolation between these two extremes should be able to reproduce all the essential physical features embodied in (1). With this premise we suggest the replacement of the nonlinear term x^3 in (1) by the expression

$$x^3 = (1+\gamma) m_2 x - \gamma m_1^3, \tag{16a}$$

where γ is a function of θ (for a general nonlinear Langevin equation, we have to express *all* the nonlinear terms in it by linear combinations of the corresponding Suzuki and Gaussian decoupling expressions). Using the exact asymptotic distribution (given by (3)) we find that

$$\gamma \sim \theta, \text{ for } \theta \ll 1 \text{ and } \gamma \sim 2, \text{ for } \theta \gtrsim 2. \tag{16b}$$

Assuming θ dependence of γ to be exponential, we obtain the approximate relation

$$\gamma = 2[1 - \exp(-\theta/2)]. \tag{16c}$$

We observe that $\gamma - \theta$ is negative. Equation (6) in conjunction with (15) gives the trivial steady state

$$m_1(\infty) = 0 \text{ and } m_2(\infty) = \frac{1 - \theta + [(1-\theta)^2 + 4D(1+\gamma)]^{1/2}}{2(1+\gamma)} \tag{17a}$$

for all θ and D and the nontrivial steady state is given by

$$m_1(\infty) = \pm\left[\frac{[(\theta+\gamma)^2 - 4\gamma(1+\gamma)D]^{1/2} - (\theta-\gamma)}{2\gamma}\right]^{1/2}$$

for all θ, $0 \leqslant D \leqslant \dfrac{\theta}{1+\gamma}$,

and
$$m_2(\infty) = \frac{1+\gamma m_1^2(\infty)}{1+\gamma}. \tag{17b}$$

The asymptotic distribution bifurcates at $D = \theta/(1+\gamma)$.

Stability analysis shows that the trivial branch becomes unstable (and the nontrivial one stable) at a critical value $D_c^{(\mathrm{SCL})} = \theta/(1+\gamma)$ of the diffusion coefficient. Thus the bifurcation of the asymptotic solution and the dynamical instability of the trivial branch occur at the same point in this approximation. From (17b), it can be seen that the nontrivial branch goes to zero continuously at $D_c^{(\mathrm{SCL})}$. Thus we see that the phase transition, as described by this approximation, is continuous for all θ. The critical exponent β associated with the order parameter m_1 also can be obtained from (17b). We get

$$m_1(\infty) = \pm\left[\frac{1+\gamma}{(\theta\gamma)}\right]^{1/2}(D_c^{(\mathrm{SCL})} - D)^{1/2},\ D_c^{(\mathrm{SCL})} - D \to 0_+. \tag{18}$$

Thus the mean field exponent $\beta = 1/2$ is also reproduced accurately. The critical boundary is plotted in figure 1, along with the exact result and other approximations. The agreement with the exact result is very good for both small and large values of θ. The maximum deviation occurs near $\theta \sim 1$, where the nature of the potential changes.

Figure 5 shows a plot of the steady state values of the order parameter as a function of D for $\theta = 10^{-2}$. The agreement of the approximate result with the exact one is good. The critical point is reproduced to an accuracy of 1 %. As is clear from (17b), $m_1(\infty)$ tends to unity as D tends to zero.

We now examine the dynamics in this approximation. For comparison, Monte Carlo simulation of (1) was carried out for a few values of θ. The simulation was performed using the finite-difference algorithm

$$x(t+\Delta t) = x(t) + \Delta t[(1-\theta)x(t) - x^3(t) + \theta m_1(t)] + g(t), \tag{19a}$$

where $g(t)$ is Gaussian with the normalization

$$\langle g(t)\rangle = 0 \quad \text{and} \quad \langle g^2(t)\rangle = 2D\,\Delta t. \tag{19b}$$

In contrast with the usual Monte Carlo simulations of Langevin equations (Valsakumar *et al* 1983; West *et al* 1983; Indira *et al* 1983; Valsakumar 1985), the number of realizations of the stochastic process (trajectories) has to be fixed from the beginning in the present case owing to the presence of the term $m_1(t)$ in (19). Given $x(0)$, $m_1(0)$ is computed. Using these in (19), $x(\Delta t)$ and $m_1(\Delta t)$ are computed. The process is continued till the final time of interest. Choosing 5000 as the number of trajectories was found to be sufficient to ensure good statistics. The value $\Delta t = 2 \times 10^{-3}$ was sufficient

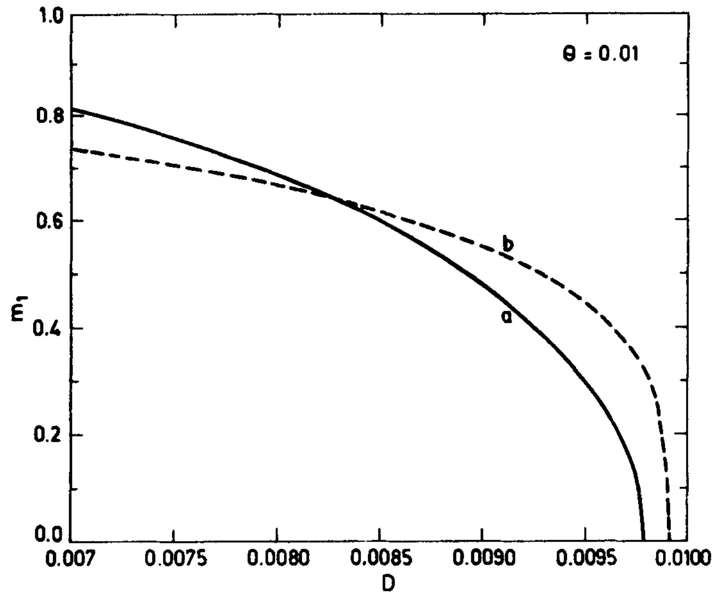

Figure 5. The value of the order parameter m_1 is plotted as a function of D for $\theta = 10^{-2}$. The curves a and b correspond respectively to the exact result and the present self-consistent linearization. We note that the critical points agree with each other to an accuracy of 1%.

to ensure a cumulative error of less than 0·2% up to $t = 5$ sec. The approximate solutions were also obtained with the finite-difference algorithm and the same Δt, to eliminate the error due to the choice of the algorithm. These results are shown in figure 6, for $\theta = 10^{-2}$ and $D = 10^{-4}$. The agreement for this particular choice of the parameters is excellent. The Monte Carlo simulation very near the critical point is prohibitively time consuming and so we have not attempted it.

At this stage we would like to comment on the work by Brey *et al* (1984). They studied the time evolution of the same system after placing it very near the central maximum of the bistable potential. Scaling theory, with some modifications, was used to obtain a good description of the fluctuations. As mentioned in the beginning of this section, this special case also is encompassed by the self-consistent linearization of the present work.

To summarize, the present decoupling scheme reproduces all the basic features of the model in addition to giving a reasonably good account of the dynamics.

5. Conclusions

This investigation clearly demonstrates that an approximation based on the cumulant expansion is not suitable for the analysis of the critical phenomena associated with the model considered. In marked contrast to the continuous phase transition possessed by the model, this approach predicts a discontinuous phase transition for a wide range of parameters.

The smallness of the diffusion constant forms the basis for most of the approximation methods for the solution of nonlinear Langevin equations. However, retention of the symmetry properties of the potential should form the guiding principle for

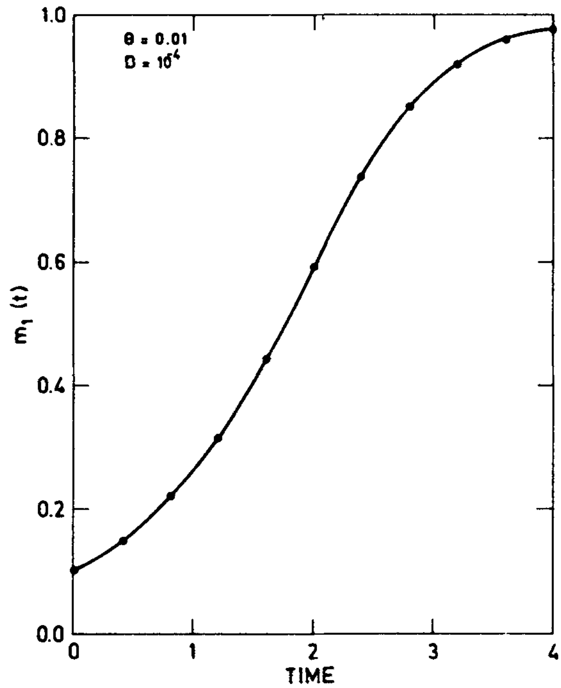

Figure 6. The time evolution of $m_1(t)$ obtained using Monte Carlo simulation for $\theta = 10^{-2}$, $D = 10^{-4}$ and $x(0) = m_1(0) = 10^{-1}$. The full circles correspond to values obtained by our self-consistent linearization.

approximations in the analysis of systems near phase transitions. A simple linearization scheme based on this principle has been shown to reproduce all the essential features of the exact solution in the model we have considered. The simplicity of implementation and the availability of closed form expressions for the quantities of interest makes this approximation scheme worthwhile.

Admittedly, the arguments used to arrive at our decoupling scheme are not rigorous. In fact, there could be other interpolations between the Suzuki and Gaussian limits; but the success of the scheme in the entire parameter space reinforces our faith in its applicability. Further work is required to see how this decoupling emerges rigorously.

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Appendix A. Evaluation of the critical boundary

The root of equation (5) for $\theta = 1$ is found to be

$$D_c = 4\Gamma^2(3/4)/\Gamma^2(1/4) = 0{\cdot}4569.$$

When $\theta \neq 1$, (5) can be rewritten as

$$G(z) = z + (1-\theta)I_2/I_1 = 0, \tag{A.1}$$

where

$$I_n = \int_0^\infty \mathrm{d}t\, t^{n-1/2} \exp[-(zt + t^2/2)]. \tag{A.2}$$

The solution to (A.1) is obtained using the Newton-Raphson method. If z_0 is the trial solution, then the correction $\Delta z (z = z_0 + \Delta z)$ is given by

$$\Delta z = -G(z_0) \Big/ \frac{\mathrm{d}}{\mathrm{d}z_0} G(z_0), \tag{A.3}$$

where

$$\frac{\mathrm{d}G}{\mathrm{d}z} = 1 + (1-\theta)[-I_3/I_1 + I_2^2/I_1^2]. \tag{A.4}$$

The integrals I_n are computed using Simpson rule with the step-length adjusted to give an accuracy of one part in 10^5. Convergence under iteration is found to be rapid.

Appendix B. Determination of the nontrivial steady states

We have to find the positive roots of the coupled algebraic equations

$$30v^2 - [15 - 6\theta + 60u]v + 24u^2 + (20 - 5\theta)u + 1 - \theta - 3D = 0, \quad \text{(B.1)}$$

and

$$a_1 v + a_2 u + a_3 v^2 + a_4 uv + 30v^3 - 90v^2 u + 24u^3 + a_5 = 0, \quad \text{(B.2)}$$

where

$$u = m_1^2(\infty),\ v = m_2(\infty),\ a_1 = (1-\theta)^2 - 12D,\ a_2 = \theta(2-\theta) - 10,$$
$$a_3 = -15(1-\theta),\ a_4 = 60 - 15\theta \text{ and } a_5 = D(1-\theta). \quad \text{(B.3)}$$

From (B.1) we get

$$v = u + d_1 \pm [c_1 u^2 + c_2 u + c_3]^{1/2}, \quad \text{(B.4)}$$

where

$$d_1 = \frac{5 - 2\theta}{20},\quad c_1 = \frac{1}{5},\ c_2 = \frac{-(5+\theta)}{30},$$

and

$$c_3 = \frac{35 + 12\theta^2 - 20\theta + 120D}{1200}. \quad \text{(B.5)}$$

Substituting (B.4) in (B.2) and rationalizing the surd gives

$$Au^6 + Bu^5 + Cu^4 + Du^3 + Eu^2 + Fu + G = 0, \quad \text{(B.6)}$$

where $A, \ldots, F$ are polynomials in a_i, c_i and d_1. Note that (B.6) and (B.4) have exactly twice the number of roots as (B.1) and (B.2). The correct roots were selected by substituting them in (B.1) and (B.2) and verifying that these equations are also satisfied.

Equation (B.6) was solved using the standard programme ZORP available in the subroutine package of the Honeywell Bull DPS-8 computer system. The whole computation was done in double precision to give an accuracy of one part in 10^{14} for the roots.

Appendix C. Stability analysis

Expressing the moments as $m_1 = m_1(\infty) + \delta m_i$ in (11), using (10) and retaining only terms upto first order in δm_i, we get

$$\begin{pmatrix} \delta\dot{m}_1 \\ \delta\dot{m}_2 \\ \delta\dot{m}_3 \\ \delta\dot{m}_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 4\theta m_1(\infty) & 2(1-\theta) & 0 & -2 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix} \begin{pmatrix} \delta m_1 \\ \delta m_2 \\ \delta m_3 \\ \delta m_4 \end{pmatrix} \quad \text{(C.1)}$$

where

$$\alpha_1 = 3[2D - 5m_4(\infty) + 40m_1(\infty)m_3(\infty) + 30m_2^2(\infty) - 180m_2(\infty)m_1^2(\infty) + 120m_1^4(\infty) + \theta m_2(\infty)],$$

$$\alpha_2 = 3[\theta m_1(\infty) - 10m_3(\infty) + 60m_2(\infty)m_1(\infty) - 60m_1^3(\infty)],$$

$$\alpha_3 = 3[(1-\theta) - 10m_2(\infty) + 20m_1^2(\infty)], \quad \alpha_4 = -15m_1(\infty),$$

$$\beta_1 = 4[\theta m_3(\infty) + 60m_3(\infty)m_2(\infty) - 180m_2^2(\infty)m_1(\infty) + 144m_1^5(\infty)],$$

$$\beta_2 = 4[3D - 15m_4(\infty) + 60m_3(\infty)m_1(\infty) + 90m_2^2(\infty) - 180m_2(\infty)m_1^2(\infty)],$$

$$\beta_3 = 4[\theta m_1(\infty) - 20m_3(\infty) + 60m_2(\infty)m_1(\infty)],$$

and

$$\beta_4 = 4[1 - \theta - 15m_2(\infty)]. \tag{C.2}$$

The fourth order algebraic equation for the eigenvalues of the stability matrix in (C.1) is solved using the subroutine discussed in Appendix B.