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Microscopic theory of soliton propagation in ^{4}He films

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Abstract. A microscopic theory has been provided for propagation of solitons in superfluid ^{4}He films at temperature $T = 0°$K.

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Nonlinear excitations in ^{4}He films have received much attention during the recent years. Condat and Guyer (1982) and Nakajima *et al* (1980) treated the problem by using the two-fluid hydrodynamic equations. The validity of the phenomenological equations for films is, however, doubtful. We propose to give a microscopic treatment of the problem. Let us consider a ^{4}He film as shown in figure 1.

Let
$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i>j} V(|\mathbf{r}_i - \mathbf{r}_j|) + \sum_i W(\mathbf{r}_i) \tag{1}$$

be the Hamiltonian of the system, where $V(|\mathbf{r}_i - \mathbf{r}_j|)$ is the potential between two ^{4}He particles and $W(\mathbf{r}_i)$ is the potential felt by a ^{4}He particle due to the substrate. Then the many-body equations can be derived from the action

$$L = \int \mathrm{d}t \, \langle \psi | i\hbar \frac{\partial}{\partial t} - H | \psi \rangle, \tag{2}$$

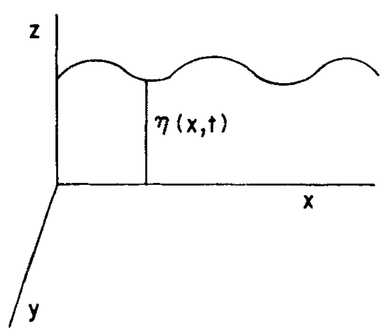

Figure 1.

where $\psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$ is the N-particle wave function. We shall use a Hartree-Fock approximation for the wave function and write

$$\psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) = \prod_{i=1}^{N} \phi(r_i, t). \tag{3}$$

Further following Condat and Guyer (1982) we shall assume that the fluid is incompressible. Let $z = \eta(x, t)$ define the surface of the film. We shall assume

$$\phi(\mathbf{r}_i, t) = \frac{[\rho_0 \Theta(z_i - \eta_i)]^{1/2}}{\sqrt{L}} \exp[i\theta(x, z, t)], \tag{4}$$

where $\eta_i = \eta(x_i, t)$ and $\phi(r_i, t)$ depend only on x, z, t and independent of the y-coordinate, $\Theta(x)$ is the Heaviside theta function. We define the film in an experimental situation where $L_x = l$ the extent in the x-direction is finite but quite large compared with the thickness of the film. The extent in the y-direction $L_y = L$ is very large. We assume that the system looks the same for all $y =$ constant planes. ρ_0 is the uniform density of the fluid. Substituting (3) and (4) in (2) and carrying out the differentiations we get the action in the form

$$\begin{aligned} L = & -i\hbar \frac{1}{2} \rho_0 \int \mathrm{d}t\, \mathrm{d}x \frac{\partial \eta_i}{\partial t} - \hbar \int \mathrm{d}t\, \mathrm{d}x\, \rho_0 \int_0^{\eta} \frac{\partial \theta}{\partial t} \mathrm{d}z \\ & - \frac{\hbar^2}{2m} \int \mathrm{d}t\, \mathrm{d}x\, \rho_0 \left[\frac{1}{4} \frac{1}{h_0} \left(-\frac{\partial \eta}{\partial x} \right)^2 + \frac{1}{2} \left(-\frac{\partial^2 \eta}{\partial x^2} \right) \right. \\ & \left. + \frac{1}{2} i \left(-\frac{\partial \eta}{\partial x} \right) \frac{\partial \theta}{\partial x} \bigg|_{z=\eta} - \int_0^{\eta} \left(\frac{\partial \theta}{\partial x} \right)^2 \mathrm{d}z + i \int_0^{\eta} \frac{\partial^2 \theta}{\partial x^2} \mathrm{d}z \right] \\ & - \frac{\hbar^2}{2m} \int \mathrm{d}t\, \mathrm{d}x\, \rho_0 \left[\frac{1}{4} \frac{1}{h_0} + i \frac{\partial \theta}{\partial z} \bigg|_{z=\eta} - \rho_0 \int_0^{\eta} \left(\frac{\partial \theta}{\partial z} \right)^2 \mathrm{d}z \right. \\ & \left. + i \int_0^{\eta} \frac{\partial^2 \theta}{\partial z^2} \mathrm{d}z - \int \mathrm{d}t \frac{\mathrm{d}x\, \mathrm{d}x'\, \mathrm{d}y\, \mathrm{d}y'}{L^2} \int_0^{\eta} \int_0^{\eta'} V(|r - r'|) \mathrm{d}z\, \mathrm{d}z' \right. \\ & - \int \mathrm{d}t \frac{\mathrm{d}x\, \mathrm{d}y}{L} \int_0^{\eta} W(r) \mathrm{d}z, \end{aligned} \tag{5}$$

where h_0 is a lower limit to the thickness of the film for which our theory is valid i.e. we have $\eta(x, t) \geqslant h_0$. This involves an assumption that there exists a regime $\xi_0 > \eta > Z \geqslant h_0$ where density variations can be neglected, ξ_0 is a coherence length. We have neglected for simplicity the effect of gravity and surface tension.

We have further that the total number of particles is conserved i.e.

$$\prod_i \iiint \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \frac{|\phi_i(x, t)|^2}{L} = \text{constant}. \tag{6}$$

From (6) it follows that

$$\int \mathrm{d}x \frac{\partial \eta}{\partial t} = 0 \tag{7}$$

so that the first term in (5) vanishes.

Let l be the extension of the system in the x-direction. We shall scale all lengths occurring in (5) w.r.t l with respect to which all lengths are small.

Then L becomes

$$L = -\hbar\rho_0 \int \mathrm{d}t\,\mathrm{d}x\, l \int_0^{\bar\eta} \frac{\partial\theta}{\partial t}\mathrm{d}\bar z - \frac{\hbar^2}{2m}\int \mathrm{d}t\,\mathrm{d}x\,\rho_0\left[\frac{1}{4}\frac{1}{h_0}\left(-\frac{\partial\bar\eta}{\partial\bar x}\right)^2\right.$$
$$+\frac{1}{2}\left(-l\frac{\partial^2\bar\eta}{\partial\bar x^2}\right)+\frac{1}{2}i\left(-\frac{\partial\bar\eta}{\partial\bar x}\frac{1}{l}\frac{\partial\theta}{\partial\bar x}\right)_{\bar z=\bar\eta} - \int_0^{\bar\eta}\left(\frac{\partial\theta}{\partial\bar x}\right)^2\frac{1}{l}\mathrm{d}\bar z$$
$$\left. + i\int_0^{\bar\eta}\frac{\partial^2\theta}{\partial\bar x^2}\frac{1}{l}\mathrm{d}\bar z\right] - \frac{\hbar 2}{2m}\int \mathrm{d}t\,\mathrm{d}x\,\rho_0\left(\frac{1}{4}\frac{1}{h_0} + i\frac{1}{l}\frac{\partial\theta}{\partial\bar z} - \frac{1}{l}\right.$$
$$\left.\int_0^{\bar\eta}\left(\frac{\partial\theta}{\partial\bar z}\right)^2\mathrm{d}\bar z + i\int_0^{\bar\eta}\frac{1}{l}\frac{\partial^2\theta}{\partial\bar z^2}\mathrm{d}\bar z\right) - \int\frac{\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}x'\,\mathrm{d}y'}{L^2}$$
$$\int_0^{\bar\eta}\int_0^{\bar\eta'} V(|r-r'|)\,\mathrm{d}\bar z\,\mathrm{d}\bar z' - \int\frac{\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y}{L}\,l\int_0^{\bar\eta} W(r)\,\mathrm{d}\bar z. \tag{8}$$

By varying L w.r.t $\bar\eta$ and θ we get the following set of equations

$$-\frac{\partial\bar\eta}{\partial\bar x}\frac{\partial\theta}{\partial\bar x} + \frac{\partial^2\theta}{\partial\bar x^2} + \frac{\partial^2\theta}{\partial\bar z^2} - \frac{\partial\theta}{\partial z} = 0 \tag{9}$$

$$-\hbar\frac{\partial\theta}{\partial t}\bigg|_{z=\bar\eta} - \frac{\hbar^2}{2m}\frac{1}{4lh_0}\frac{\partial^2\bar\eta}{\partial\bar x^2} - \frac{\hbar^2}{2m}\left(\frac{\partial\theta}{\partial z}\right)^2\bigg|_{z=\bar\eta}$$
$$-\frac{\hbar^2}{2m}\left(\frac{\partial\theta}{\partial x}\right)^2\bigg|_{z=\bar\eta} - \int V(|r-r'|)\bigg|_{\substack{z=\eta\\ z'=\eta'}}\mathrm{d}x'\,\mathrm{d}y' - W(r)\bigg|_{z=\eta} = 0. \tag{10}$$

Let us expand $\theta(x,z,t)$ as a power series in z i.e. we write

$$\theta = \sum_{n=0}^{\infty}\theta_n(\bar x,t)\,\bar z^n \tag{11}$$

$\theta_{\bar z} = 0$ at $\bar z = 0$, $\theta_1 \equiv 0$.

Substituting in (9)

$$\frac{\mathrm{d}^2\theta_n}{\mathrm{d}\bar x^2} + (n+2)(n+1)\,\theta_{n+2} - \eta_x\frac{\mathrm{d}\theta_n}{\mathrm{d}\bar x} - (n+1)\theta_{n+1} = 0.$$

This gives

$$\theta_2 = \frac{1}{2!}\left(-\theta_{0\bar x\bar x} + \eta_x\theta_{0\bar x}\right) \tag{12}$$

$$\theta_3 = \frac{1}{3!}\left(-\theta_{0\bar x\bar x} + \eta_{\bar x}\theta_{0\bar x}\right). \tag{13}$$

Assuming $\theta(\bar{x}, \bar{z}, t)$ to be a very slowly varying function of $\bar{z}$ we have

$$\theta_{0\bar{x}\bar{x}} \simeq \bar{\eta}_{\bar{x}}\,\theta_{0\bar{x}}$$

or

$$\theta_{0\bar{x}} \simeq C\,e^{\bar{\eta}}.$$

The constant C can be evaluated to be $C = mVl/h$ where V is the uniform velocity of the fluid at infinitesimal thickness.

We have

$$\theta_{0\bar{x}} = \frac{mVl}{h}\,e^{\bar{\eta}}. \tag{14}$$

From (10) neglecting smaller terms we have

$$-\hbar\frac{\partial\theta_0}{\partial t} - \frac{\hbar^2}{8\,mlh_0}\frac{\partial^2\bar{\eta}}{\partial\bar{x}^2} - \alpha - (\beta+\beta')\bar{\eta}^2 - 2\beta d\bar{\eta} + \beta d^2 = 0, \tag{15}$$

where α, β, β', d are constants. Differentiating (15) w.r.t $\bar{x}$ and substituting from (14)

$$-\hbar\frac{\partial}{\partial t}\frac{mVl}{h}\left(1+\bar{\eta}+\frac{\bar{\eta}^2}{2}\right) - \frac{\hbar^2}{8\,mlh_0}\frac{\partial^3\bar{\eta}}{\partial\bar{x}^3} - 2(\beta+\beta')\bar{\eta}\frac{\partial\bar{\eta}}{\partial\bar{x}} - 2\beta d\frac{\partial\bar{\eta}}{\partial\bar{x}} = 0 \tag{16}$$

or

$$-mVl\left[\frac{\partial\bar{\eta}}{\partial t} + \bar{\eta}\frac{\partial\bar{\eta}}{\partial t}\right] - 2(\beta+\beta')\bar{\eta}\frac{\partial\bar{\eta}}{\partial\bar{x}} + 2\beta d\frac{\partial\bar{\eta}}{\partial\bar{x}} - \frac{\hbar^2}{8\,m\,h_0}\frac{3-}{\bar{x}^3} = 0 \tag{17}$$

or

$$\gamma\frac{\partial\bar{\eta}}{\partial t} + \gamma\bar{\eta}\frac{\partial\bar{\eta}}{\partial t} + 2(\beta+\beta')\,l\bar{\eta}\frac{\partial\bar{\eta}}{\partial\bar{x}} - 2\beta d\frac{\partial\bar{\eta}}{\partial\bar{x}} + \frac{\hbar^2}{8mlh_0}\frac{\partial^3\bar{\eta}}{\partial\bar{x}^3} = 0. \tag{18}$$

This is a variation of the $K-\mathrm{d}V$ equation obeyed by the film surface $\eta(x, t)$, γ, β, β' are constants. $\gamma = mVl$ and

$$\beta = \left\langle \varepsilon\left(\frac{\sigma}{[(x-x')^2+(y-y')^2]^{1/2}}\right)^{12}\frac{6}{(x-x')^2+(y-y')^2}\right\rangle$$
$$-\left\langle \varepsilon\left(\frac{\sigma}{[(x-x')^2+(y-y')^2]^{1/2}}\right)^{6}\frac{3}{(x-x')^2+(y-y')^2}\right\rangle, \tag{19}$$

$$\beta' = \frac{3}{2}\varepsilon'\left\langle\frac{\sigma^{3/2}}{(x^2+y^2)^{3/2}\,(x^2+y^2)}\right\rangle, \tag{20}$$

where we have taken $V(|\mathbf{r}-\mathbf{r}'|)$ as the Lennard-Jones potential and $W(\mathbf{r})$ as the van der Waal's potential due to the substrate. The brackets indicate averages over the whole x-y plane with suitable distribution functions so that they are finite. d is the equilibrium thickness of the film.

Linearization of the nonlinear equation (18) gives the dispersion relation

$$\omega_K = \frac{2\beta d}{\gamma}k - \frac{\hbar^2}{8mlh_0\gamma}k^3$$
$$= C_3 k\left(1 - \frac{\hbar^2}{16mlh_0\beta d}k^2\right) \tag{21}$$

where $C_3 = 2\beta d/\gamma$ is the third sound velocity. [Here C_3 has the dimension of $(\text{time})^{-1}$ since K is dimensionless]. Let us change time t to a dimensionless time τ by $\tau = C_3 t$. The equation becomes

$$\frac{\partial \bar{\eta}}{\partial t} + \bar{\eta}\frac{\partial \bar{\eta}}{\partial \tau} + \alpha_1 \bar{\eta}\frac{\partial \bar{\eta}}{\partial \bar{x}} - \frac{\partial \bar{\eta}}{\partial \bar{x}} + \alpha_2 \frac{\partial^3 \bar{\eta}}{\partial \bar{x}^3} = 0, \quad (22)$$

where

$$\alpha_1 = \frac{2(\beta + \beta')l}{C_3 \gamma}, \quad \alpha_2 = \frac{\hbar^2}{8mh_0 \gamma C_3 l}.$$

We put $\bar{\eta} = f(\bar{x} - C\tau) = f(\xi)$. Then $f(\xi)$ satisfies the equation

$$-(C+1)f' + (\alpha_1 - C)ff' + \alpha_2 f''' = 0,$$

which can be integrated twice to give

$$f = A \operatorname{sech}^2 \frac{1}{2}\left(\frac{1+C}{\alpha_2}\right)^{\frac{1}{2}} \xi, \quad (23)$$

where $A = \{[3(1+C)]/(\alpha_1 - C)\}$. Note that α_1, α_2, C are all dimensionless quantities. Thus we get the soliton solution given by

$$f = A \operatorname{sech}^2 \frac{1}{2}\left(\frac{1+C}{\alpha_2}\right)^{\frac{1}{2}} (\bar{x} - C\tau). \quad (24)$$

The amplitude A of the soliton is given by $A = \{[3(1+C)]/(\alpha_1 - C)\}$ and is a function of the velocity C of the soliton. The width Δ of the soliton is given by $\Delta = 2[\alpha_2/(1+C)]^{\frac{1}{2}}$ which also depends on the velocity C of the soliton.

References

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