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A radiating Kerr-Newman solution

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Abstract. A non-static exact solution of Einstein's equations corresponding to a field of flowing null radiation plus an electromagnetic field is presented. The geometry of the solution is described by the Kerr-Schild metric. The solution admits a shear-free, geodetic null congruence. It has the symmetry of the Kerr-Newman solution and when a certain parameter is put equal to zero the solution becomes static and reduces to the Kerr-Newman solution.

Keywords. General relativity; Kerr-Newman metric; pure radiation field.

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1. Introduction

Many investigators have shown keen interest in obtaining generalizations of different static or stationary metrics to the non-stationary situation, especially, in relation to certain astrophysical problems, namely when the energy density of the radiation emitted by the source cannot be neglected (e.g. supernovae, quasistellar radio sources etc).

The nonstatic generalizations of the Schwarzschild exterior solution and the Nordstrom solution are obtained by Vaidya (1951) and Bonnor and Vaidya (1970). The Vaidya solution describes the exterior gravitational field of a radiating star. The Bonnor and Vaidya solution describes the field of a moving charge particle emitting null fluid and pervaded by a null current field.

Kerr (1963) derived a stationary generalization of Schwarzschild exterior solution which describes the exterior gravitational field of a rotating body. Newman *et al* (1965) obtained the electromagnetic generalization of Kerr solution. Their solution is known as Kerr-Newman solution in the literature. The Kerr-Newman solution is believed to represent the ultimate state of a collapsing body with rotation, mass and electric charge. Therefore considerable significance is attached to the Kerr-Newman solution.

Vaidya and Patel (1973) obtained a non-static generalization of the Kerr solution for the shining Kerr particle. Many radiating Kerr-like solutions are also discussed by Vaidya (1974) and Herlt (1980). It would be interesting to find out if there exists a radiating Kerr-Newman solution.

In the present paper we give such a generalization of Kerr-Newman solution in the form of an exact solution of Einstein-Maxwell equations

$$R_{ik} = -8\pi[E_{ik} + \sigma\xi_i\xi_k], \xi_i\xi^i = 0 \quad (1)$$

$$F^{ik}_{;k} = 4\pi J^i \quad (2)$$

in terms of the Kerr-Schild (1965) metric

$$g_{ik} = \eta_{ik} + H\,\xi_i\xi_k, \tag{3}$$

where ξ_i is a null geodetic shear-free congruence with non-zero twist. Here, the semicolon indicates covariant derivative and η_{ik} is the metric tensor of the Minkowskian space-time. H is a function of co-ordinates, F_{ik} is the electromagnetic field tensor, J^i is 4-current vector, E_{ik} is the electromagnetic energy tensor and $\sigma\xi_i\xi_k$ is the tensor arising out of flowing null radiation.

We shall be freely using the method of real tetrads introduced by Vaidya (1972, 1974). The next section is devoted to a very brief description of this real tetrad method.

2. The real tetrad method

Consider a Minkowskian space-time with signature -2 and assume that it is pervaded by a null geodetic and shear-free congruence ξ_i so that

$$\xi_i\xi^i = 0, \quad \xi^i_{,k}\xi^k = 0, \quad (\xi^i_{,k} + \xi_{k,l}\eta^{li})\xi^k_{,i} - (\xi^i_{,i})^2 = 0, \tag{4}$$

η_{ik} being the metric of the Minkowskian space-time and a comma indicating ordinary derivative.

We use the geometrical framework developed by Vaidya (1972) to obtain a real tetrad system in the Minkowskian space-time appropriate to the congruence ξ_i. In such a space-time we can always find four uniform vector fields such that (i) any two of them are mutually orthogonal and (ii) one of them is time-like and the other three space-like. Let λ^i be the unit tangent to the time-like vector field through a point P (co-ordinates x^i) and A^i, B^i and C^i be the unit tangents to the space-like vector fields. We raise and lower the indices with the help of η^{ik} or η_{ik}. These four uniform vector fields give rise to a Euclidean reference frame with co-ordinates x,y,z,t for P such that

$$x_{,i} = A_i \quad y_{,i} = B_i \quad z_{,i} = c_i \quad \text{and} \quad t_{,i} = \lambda_i.$$

Let us denote by S the 3-flat at right angles to λ^i at the point P. If l_i is the projection of ξ_i on S at P, we shall take $\xi_i = \lambda_i + l_i$. In 3-flat S let l_i have the spherical angles α and β with respect to the triad A_i, B_i, C_i.

We can now define an orthonormal triad l_i, $\bar{l}_i$, $\bar{m}_i$ as follows

$$\begin{aligned} l_i &= \sin\alpha\, m_i + \cos\alpha\, C_i, \\ \bar{l}_i &= \cos\alpha\, m_i - \sin\alpha\, C_i, \\ m_i &= \cos\beta\, A_i + \sin\beta\, B_i, \\ \bar{m}_i &= -\sin\beta\, A_i + \cos\beta\, B_i. \end{aligned}$$

The derivatives of these vectors are listed in Appendix III. We take α and β as functions of the co-ordinates x^i.

The conditions (4) after some simplification will lead to

$$\alpha_{,i}\xi^i = 0, \quad \beta_{,i}\xi^i = 0, \tag{5}$$

and

$$\bar{m}^i\alpha_{,i} + \bar{l}^i \sin\alpha\beta_{,i} = 0, \tag{6}$$
$$\bar{l}^i\alpha_{,i} - \bar{m}^i \sin\alpha,\beta_{,i} = 0.$$

It can be verified that if we define

$$u = x\sin\alpha\cos\beta + y\sin\alpha\sin\beta + z\cos\alpha + t,$$
$$V = x\cos\alpha\cos\beta + y\cos\alpha\sin\beta - z\sin\alpha,$$
$$W = x\sin\beta - y\cos\beta,$$

then $u_{,i}\xi^i = 0$, $V_{,i}\xi^i = 0$ and $W_{,i}\xi^i = 0$. Therefore the conditions (5) can be integrated and exhibited in the form

$$W = W(u,\alpha,\beta), \quad V = V(u,\alpha,\beta).$$

The conditions (6) will now lead to two partial differential equations for V and W. For our purpose we note that if

$$V = -a\cos\alpha W, \quad W = (au+k)\sin\alpha(1+a^2\cos^2\alpha)^{-1}, \tag{7}$$

the conditions (6) will be satisfied, a and k being constants of integration.

The principal results of this real tetrad method which are used in the present investigation are reproduced in Appendix III for ready reference.

3. The Maxwell field

We now consider the Maxwell equations in a Riemannian space-time described by the metric

$$g_{ik} = \eta_{ik} + H\xi_i\xi_k, \tag{8}$$

where ξ_i is the null congruence discussed in §2. It can be easily seen that

$$g^{ik} = \eta^{ik} - H\xi^i\xi^k, \quad g = |g_{ik}| = -1. \tag{9}$$

We choose the electromagnetic 4-potential φ_i as

$$\varphi_i = P\theta\xi_i, \quad P_{,i}\xi^i = 0. \tag{10}$$

Here $\theta = \xi^i_{,i}$ is the expansion of the congruence ξ_i.
Here it should be noted that ξ_i continues to be null, geodetic and shear-free in the Riemannian space-time defined by (8) (see Vaidya 1974).

For the choice (10) of φ_i the electromagnetic field tensor $F_{ik} = \varphi_{i,k} - \varphi_{k,i}$ is given by

$$F_{ik} = (P\theta)_{,k}\xi_i - (P\theta)_{,i}\xi_k + P\theta(\xi_{i,k} - \xi_{k,i}).$$

Using the results given in Appendix III, $F^{ik}_{;k}$ can be simplified to the form

$$F^{ik}_{;k} = \eta^{ab}(P\theta)_{,ab}\xi^i - \tfrac{1}{2}[P(\theta^2+\Omega^2)]_{,k}\eta^{ik}$$
$$+ P\theta\eta^{ab}\xi^i_{,ab} + \tfrac{1}{2}\eta^{ab}(P\theta)_{,a}\xi^i_{,b}.$$

Using the above result and the results of Appendix III we have verified that

$$F^{ik}_{;k}\xi_i = 0$$

and

$$-2F^{ik}_{;k}\bar{l}_i = [P(\theta^2+\Omega^2)_{,a}\bar{l}^a + \eta^{ab}(P\theta)_{,a}\alpha_{,b},$$

$$-2F^{ik}_{;k}\bar{m}_i = [P(\theta^2+\Omega^2)]_{,a}\bar{m}^a + \eta^{ab}(P\theta)_{,a}\sin\alpha\beta_{,b}.$$

As $P_{,i}\xi^i = 0$, we can take P as a function of u, α, β.
Using the result $P_{,i} = P_\alpha \alpha_{,i} + P_u u_{,i} + P_\beta \beta_{,i}$ the equations $J^i\bar{l}_i = 0$ and $J^i\bar{m}_i = 0$ imply the following two differential equations:

$$P_\alpha + P_u V + 2PV_u = 0,$$

and

$$P_u W + 2PW_u - P_\beta \operatorname{cosec}\alpha = 0.$$

Here $P_u = \partial P/\partial u$ etc. If we assume that P is a function of u only, then we have

$$P_u/P = -2V_u/V = -2W_u/W.$$

On using the forms of V and W given by (7) we find that

$$P_u/P = -2a/(au+k). \tag{11}$$

Equation (11) can be easily integrated to have

$$P = e(au+k)^{-2}, \tag{12}$$

where e is a constant of integration.

The 4-current vector J^i can be expressed as

$$4\pi J^i = \frac{ea\xi^i}{(au+k)^2}\left\{\frac{\cos\alpha(1+a^2)(\theta^2+\Omega^2)}{(1+a^2\cos^2\alpha)^2} - \frac{\Omega^2-\theta^2}{au+k}\right\}, \tag{13}$$

where

$$\Omega^2 = (\xi_{i,k} - \xi_{k,i})\xi^i_{,l}\eta^{lk},$$

so that Ω measures the rotation of the null congruence ξ_i.

The electromagnetic energy tensor E_{ik} for the choice (10) of φ_i is recorded in Appendix II for reference. In the next section we shall direct our attention to the Einstein-Maxwell field.

4. The Einstein-Maxwell field

Vaidya (1974) computed the components R_{ik} of the Ricci tensor for the metric (8). They are given in Appendix I for reference. From these expressions for R_{ik} it is easy to see that

$$R = g^{ik}R_{ik} = (H\theta + h)_{,i}\xi^i + \theta(H\theta + h), \tag{14}$$

where $h = H_{,a}\xi^a$. For the field equations (1) $R = 0$.

This equation can be solved to get

$$H = M\theta + N\Omega, \quad M_{,i}\xi^i = 0, \quad N_{,i}\xi^i = 0. \tag{15}$$

Now we shall obtain an exact solution of the field equations (1) corresponding to the Maxwell potential φ_i given by (10) and the gravitational potentials g_{ik} given by (8).

From the field equations (1) it is easy to see that the vectors ξ_i, $\overline{l}_i$ and $\overline{m}_i$ are eigenvectors of $(R_{ik} + 8\pi E_{ik})$, the corresponding eigen values being zero. The equation $(R_{ik} + 8\pi E_{ik})\xi^i = 0$, after substituting R_{ik} and E_{ik}, will lead to

$$N\Omega = 4\pi e^2 (au + k)^{-4} (\theta^2 + \Omega^2). \tag{16}$$

Here it should be noted that

$$\left[\frac{\theta^2 + \Omega^2}{\Omega}\right]_{,i} \xi^i = 0.$$

Equation (16) determines the function N. Now using the expressions for R_{ik} and E_{ik} and equation (16) in the equations

$$(R_{ik} + 8\pi E_{ik})\overline{l}^i = 0, \quad (R_{ik} + 8\pi E_{ik})\overline{m}^i = 0,$$

and simplifying with the aid of the results given in Appendix III we get

$$M_u / M = -3a/(au + k). \tag{17}$$

Here we have assumed that M is a function of u only. Equation (17) can be easily solved to get

$$M = m(au + k)^{-3}, \tag{18}$$

m being a constant of integration.

Using the relevant results of this section and those given in Appendix III in the equation

$$(R_{ik} + 8\pi E_{ik})\lambda^i \lambda^k = -8\pi\sigma$$

we can find the radiation density σ. It is given by

$$\begin{aligned} 8\pi\sigma(au + k)^4 = a[&(\theta^2 + \Omega^2)\{-\tfrac{3}{2}m - 8\pi e^2 \theta (au + k)^{-1}\} \\ &+ 8\pi a e^2 \sin^2\alpha\, \theta^2 (\Omega^2 - \theta^2)(1 + a^2\cos^2\alpha)^{-2} \\ &+ 8\pi a^3 e^2 \sin^2\alpha \cos^2\alpha\, \theta^2 (\theta^2 + 3\Omega^2)(1 + a^2\cos^2\alpha)^{-2} \\ &- 4\pi a^2 e^2 \cos\alpha \sin^2\alpha\, \Omega\theta^3 (1 + a^2\cos^2\alpha)^{-2}], \end{aligned} \tag{19}$$

where θ and Ω are given by

$$\begin{aligned} \tfrac{1}{2}p &= \frac{\Omega}{\theta^2 + \Omega^2} = \frac{(1 + a^2)(au + k)\cos\alpha}{2(1 + a^2\cos^2\alpha)^2} \\ \tfrac{1}{2}q &= -\frac{\theta}{\theta^2 + \Omega^2} = \frac{1}{2}\left[r - \frac{a(au + k)(1 - 2\cos^2\alpha - a^2\cos^4\alpha)}{(1 + a^2\cos^2\alpha)^2}\right] \end{aligned} \tag{20}$$

Here r is defined by $r = t - u$. This completes the task of solving the field equations (1).

5. Discussion of the solution

One can use u, α, β, t as the co-ordinates. The metric of our solution can be written down in an explicit form as

$$\begin{aligned} ds^2 = & -(p^2+q^2)(d\alpha^2+\sin^2\alpha\, d\beta^2) + 2(\xi_i dx^i)dt \\ & -\frac{2(\xi_i dx^i)a\sin\alpha}{1+a^2\cos^2\alpha}[(p-aq\cos\alpha)d\alpha \\ & -(pa\cos\alpha+q)\sin\alpha\, d\beta] \\ & -(\xi_i dx^i)^2\left[1+\frac{a^2\sin^2\alpha}{1+a^2\cos^2\alpha}+\left\{\frac{2mq-16\pi e^2(au+k)^{-1}}{(p^2+q^2)(au+k)^3}\right\}\right], \end{aligned} \tag{21}$$

where p and q are given by (20) and

$$\xi_i dx^i = du + \frac{(au+k)\sin\alpha}{1+a^2\cos^2\alpha}(a\cos\alpha\, d\alpha + \sin\alpha\, d\beta). \tag{22}$$

The function M and P for our solution are functions of u only. This is a noteworthy feature of our solution. When we put $e = 0$, the results (13) and (19) give

$$J^i = 0, \quad 8\pi\sigma = -\frac{3ma}{2}(\theta^2+\Omega^2)(au+k)^{-4}.$$

In this case we recover a particular case of the radiating Kerr solution discussed by Herlt (1980) in connection with Kerr-Schild pure radiation fields with axial symmetry.

When $a = 0$, the results (13) and (19) imply that $J^i = 0$ and $\sigma = 0$. In this particular case $q = r$ and $p = k\cos\alpha$. Let us put $\bar{m} = m/k^3$ and $\bar{e}^2 = 4e^2/k^4$. The metric (21) now reduces to the Kerr-Newman metric in the form

$$\begin{aligned} ds^2 = & \, 2(du + k\sin^2\alpha\, d\beta)dt - (r^2+k^2\cos^2\alpha)(d\alpha^2+\sin^2\alpha\, d\beta^2) \\ & -\left[1+\frac{2\bar{m}r-4\pi\bar{e}^2}{r^2+k^2\cos^2\alpha}\right](du+k\sin^2\alpha\, d\beta)^2. \end{aligned} \tag{24}$$

From the above discussion one can arrive at the conclusion that our solution is a radiating Kerr-Newman solution.

Also when $N = 0$, we obtain $e = 0$ and consequently the electromagnetic field disappears. Thus, according to the scheme developed in the present paper, the radiation in the solution of Herlt (1980) is not electromagnetic.

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Appendix I

$$
\begin{aligned}
-R_{ik} = {} & (H\theta+h)_{,}(i^{\xi}k)+(H\theta+h)\xi_{(i,k)} \\
& -H\xi_{i,a}\xi_{k,b}\eta^{ab} \\
& -\eta^{ab}[H\xi_{(i}\xi_{k),ab}+2H_{,b}\xi_{(i}\xi_{k),a}] \\
& +\xi_i\xi_k[\tfrac{1}{2}Hh\theta+\tfrac{1}{2}Hh_{,a}\xi^a-\tfrac{1}{2}\eta^{ab}H_{,ab}+\tfrac{1}{2}H^2\Omega^2],
\end{aligned}
$$

where $h = H_{,a}\xi^a$, $\theta = \xi^i_{,i}$ and round brackets imply symmetrization.

Appendix II

$$E_{ik} = K\,\xi_i\xi_k + L_i\xi_k + L_k\xi_i + L_{ik},$$

where

$$K = -\eta^{lm}(P\theta)_{,l}(P\theta)_{,m} + \frac{H}{8}P^2(\theta^2+\Omega^2)^2,$$

$$L_i = \frac{P}{2}(\Omega^2-\theta^2)(P\theta)_{,i} - P\theta\eta^{lm}(P\theta)_{,l}(\xi_{i,m}-\xi_{m,i}),$$

$$
\begin{aligned}
L_{ik} = {} & -P^2\theta^2\eta^{lm}(\xi_{i,l}-\xi_{l,i})(\xi_{k,m}-\xi_{m,k}) \\
& +\frac{P^2}{8}(\Omega^2-\theta^2+2\theta\Omega)(\theta^2-\Omega^2+2\theta\Omega)\eta_{ik},
\end{aligned}
$$

where P is given by (12).

Appendix III

$$
\begin{aligned}
& \xi_{i,k} = l_{i,k} = \alpha_{,k}\bar{l}_i + \sin\alpha\beta_{,k}\,\bar{m}_i, \\
& \bar{l}_{i,k} = -l_i\alpha_{,k} + \cot\alpha\sin\alpha\beta_{,k}\,\bar{m}_i, \\
& m_{i,k} = \beta_{,k}\,\bar{m}_i, \quad \bar{m}_{i,k} = -\beta_{,k}\,m_i \\
& \alpha_{,i} = -\tfrac{1}{2}\theta(\bar{l}_i - V_u\xi_i) - \tfrac{1}{2}\Omega(\bar{m}_i + W_u\xi_i), \\
& \sin\alpha\beta_{,i} = \tfrac{1}{2}\Omega(l_i - V_u\xi_i) - \tfrac{1}{2}\theta(\bar{m}_i + W_u\xi_i), \\
& \theta(\theta^2+\Omega^2)^{-1} = -\tfrac{1}{2}(V_\alpha + u - t), \quad \Omega(\theta^2+\Omega^2)^{-1} = \tfrac{1}{2}(VW_u + W_\alpha), \\
& \theta_{,i}\xi^i = \tfrac{1}{2}(\Omega^2-\theta^2), \quad \Omega_{,i}\xi^i = -\theta\Omega \\
& \theta_{,k}\bar{l}^k + \Omega_{,k}\bar{m}^k = \Omega(\Omega V_u + \theta W_u), \\
& \theta_{,k}\bar{m}^k - \Omega_{,k}\bar{l}^k = \Omega(\theta V_u - \Omega W_u) \\
& \eta^{ik}\alpha_{,ik} + \tfrac{1}{4}(\theta^2+\Omega^2)\cot\alpha = 0, \\
& \eta^{ik}(\sin\alpha\beta_{,i})_{,k} = 0, \quad \eta^{ik}\xi^a_{,ik}\tfrac{1}{2} = (\theta^2+\Omega^2)l^a,
\end{aligned}
$$

$$\eta^{ik}\theta_{,ik} = -\tfrac{1}{2}(\theta^2+\Omega^2)_{,i}\lambda^i,$$
$$(\theta^2+\Omega^2)_{,i}\xi^i + \theta(\theta^2+\Omega^2) = 0.$$

The functions V and W are given by (7).

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