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Further examples of integrable systems in two dimensions

R S KAUSHAL and S C MISHRA*

Department of Physics, Ramjas College, University of Delhi, Delhi 110007, India
*Department of Physics and Astrophysics, University of Delhi, Delhi 110007, India

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Abstract. The construction of the second constant of motion of second order for two-dimensional classical systems is carried out in terms of $z = q_1 + iq_2$ and $\bar{z} = q_1 - iq_2$. As a result a class of Toda-type potentials admitting second order invariants is explored.

Keywords. Second constant of motion; integrable systems; Toda-potential.

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In recent times, there have been several efforts in exploring both time-independent (Holt 1982; Inozemtsev 1983; Dorizzi *et al* 1983) and time-dependent (Katzin and Levine 1982, 1983; Kaushal *et al* 1984; Mishra *et al* 1984) integrable classical dynamical systems (see e.g. Whittaker 1972) in two dimensions. The invariants if constructed for such systems have utility from several points of view particularly in reducing some nonlinear dynamical problems to a quadrature, in solving several problems of plasma physics and hydrodynamics, in the study of classical analogue of Yang-Mills field equation (Chang 1984). The propagation of wave trains and solitary waves in a lattice was studied by Toda (1967) in terms of an exponential potential of the type

$$V(q_1, q_2) = \alpha_+ \exp(q_2 + \sqrt{3}q_1) + \alpha_- \exp(q_2 - \sqrt{3}q_1) + \beta \exp(-2q_2). \quad (1)$$

The integrability of this system was further studied by Ford *et al* (1973) and an exact second constant of motion (the first being the Hamiltonian) has been obtained (Berry 1978; Holt 1982; Inozemtsev 1983; Hall 1983; Ford *et al* 1973) which involves the third orders of momenta namely

$$\begin{aligned} \mathrm{I} = {} & 3\alpha_+ (p_1 - \sqrt{3}p_2)\exp(q_2 + \sqrt{3}q_1) + 3\alpha_- (p_1 + \sqrt{3}p_2)\exp(q_2 - \sqrt{3}q_1) \\ & - 6\beta p_1 \exp(-2q_2) + p_1(p_1^2 - 3p_2^2). \end{aligned} \quad (2)$$

Using Lax pair formalism Olshanetsky and Perelomov (1981, 1983) have discussed classical and quantum integrability of a large class of trigonometrical and exponential potentials in one-dimension. The unequal mass case of the free-end-Toda system has been studied recently by Dorizzi *et al* (1984). Other generalizations of the Toda system have also been done by taking recourse to Lie algebraic structures (Bogoyavlenski 1976; Mikhailov *et al* 1981; Gutkin 1985). Now the question is whether we really have to go to third order invariants (cf. equation (2)) for this system or whether there is an alternative to Toda potential which admits second order invariants? In this note we present a positive answer to this question and show that there exists a class of Toda-type potentials in two dimensions which admits second order invariants.

The method which we adopt here is the one used earlier for the study of both time-dependent (Kaushal *et al* 1984; Mishra *et al* 1984) and time-independent (Kaushal *et al* 1985) systems in two dimensions. In fact, the complexification of the two dimensions has not only led to the reproduction of known results but also provides some new integrable systems. While the earlier work (Kaushal *et al* 1985) was mainly concerned with the study of third and fourth order invariants, here we extend our analysis to rather simpler second order case.

We consider a dynamical system described by the Lagrangian

$$L = \tfrac{1}{2}\dot{z}\dot{\bar{z}} - V(z, \bar{z}), (z = q_1 + iq_2; \dot{z} = p_1 + ip_2), \tag{3}$$

with the concomitant equations of motion†

$$\ddot{z} = -2\frac{\partial V}{\partial \bar{z}}, \ddot{\bar{z}} = -2\frac{\partial V}{\partial z}. \tag{4}$$

We assume the existence of the second constant of motion (in the following called invariant), I, up to second order in momenta in a general form as*

$$I = a_0 + \tfrac{1}{2}a_{ij}\,\xi_i\,\xi_j, \tag{5}$$

where $i, j = 1, 2$, $\xi_1 = \dot{z}, \xi_2 = \dot{\bar{z}}$, and the coefficients a_0 and a_{ij} are functions of z and $\bar{z}$ only. The coefficient a_{ij} is symmetrized with respect to any interchange of its indices. Using $\mathrm{d}I/\mathrm{d}t = 0$, we find from (5),

$$a_{0,i}\,\xi_i + \tfrac{1}{2}\,a_{ij,k}\,\xi_i\xi_j\xi_k + \tfrac{1}{2}a_{ij}(\dot{\xi}_i\xi_j + \xi_i\dot{\xi}_j) = 0. \tag{6}$$

After accounting for the proper symmetrization of the coefficients and since (6) must hold identically in ξ's, we obtain the following relations:

$$a_{ij,k} + a_{jk,i} + a_{ki,j} = 0, \tag{7}$$

$$a_{0,i} + a_{ij}\,\dot{\xi}_j = 0. \tag{8}$$

Equations (7) and (8) after using (4) yield the following set of partial differential equations,

† Although from the applications point of view the Lagrangian L and hence the potential V need to be real functions of z and $\bar{z}$, it will turn out later that this reality condition is not necessary because the arbitrary constants of integration which occur in the solutions (cf. equations (16–18)) can suitably be chosen to yield real V and the invariant I.

* Here we assume that the invariant contains either only even powers or only odd powers of momenta. This is mainly because it can be seen (Holt 1982; Inozemtsev 1983 and Kaushal *et al* 1985) from a general form of I that there does not exist any coupling between the corresponding coefficients in the resultant set of partial differential equations for the case of time-independent systems. This is, however, not the case for time-dependent systems (Katzin and Levine 1982, 1983; Kaushal *et al* 1984; Mishra *et al* 1984; Kaushal 1985). Further justification to this assumption is based on the time-reversal symmetry of the corresponding Lagrangian.

$$\frac{\partial a_{11}}{\partial z}=0, \quad \frac{\partial a_{22}}{\partial \bar{z}}=0, \tag{9, 10}$$

$$\frac{\partial a_{11}}{\partial \bar{z}}+2\frac{\partial a_{12}}{\partial z}=0, \quad \frac{\partial a_{22}}{\partial z}+2\frac{\partial a_{12}}{\partial \bar{z}}=0, \tag{11, 12}$$

$$\frac{\partial a_0}{\partial z}=2\,a_{11}\frac{\partial V}{\partial \bar{z}}+2\,a_{12}\frac{\partial V}{\partial z}, \tag{13}$$

$$\frac{\partial a_0}{\partial \bar{z}}=2\,a_{22}\frac{\partial V}{\partial z}+2\,a_{12}\frac{\partial V}{\partial \bar{z}}. \tag{14}$$

To solve these equations we notice from (9) and (10) that

$$a_{11}=a_{11}(\bar{z})=\psi_1(\bar{z}) \quad \text{and} \quad a_{22}=a_{22}(z)=\phi_1(z).$$

Now differentiating (11) and (12) w.r.t. $\bar{z}$ and z, respectively and subtracting the results we obtain

$$\mathrm{d}^2\psi_1/\mathrm{d}\bar{z}^2=\mathrm{d}^2\phi_1/\mathrm{d}z^2=\text{constant } c_1 \text{ (say)}, \tag{15}$$

whose solutions are the polynomials,

$$a_{11}=\psi_1=\tfrac{1}{2}c_1\,\bar{z}^2+c_2\,\bar{z}+c_3, \tag{16}$$

$$a_{22}=\phi_1=\tfrac{1}{2}c_1 z^2+c_4\,z+c_5. \tag{17}$$

Using these expressions for a_{11} and a_{22} once again in (11) and (12) and integrating the resultant equations, we obtain an expression for a_{12} as

$$a_{12}=-\tfrac{1}{2}c_1\,z\bar{z}-\tfrac{1}{2}c_2\,z-\tfrac{1}{2}c_4\,\bar{z}+c_6. \tag{18}$$

In (16)–(18) c_i's are some arbitrary constants of integration except c_1, which is a separation constant. The solution for a_0 is not trivial unless the form of V is known. Therefore, we eliminate a_0 at this stage by differentiating (13) and (14) w.r.t. $\bar{z}$ and z, respectively and noticing that $(\partial^2 a_0/\partial\bar{z}\cdot\partial z)=(\partial^2 a_0/\partial\bar{z}\cdot\partial z)$. This would lead to a "potential" equation of the form

$$\tfrac{3}{2}(c_1 z+c_4)\frac{\partial V}{\partial z}+(\tfrac{1}{2}c_1 z^2+c_4 z+c_5)\frac{\partial^2 V}{\partial z^2}$$

$$=\tfrac{3}{2}\,(c_1\bar{z}+c_2)\frac{\partial V}{\partial \bar{z}}+(\tfrac{1}{2}c_1\bar{z}^2+c_2\bar{z}+c_3)\frac{\partial^2 V}{\partial \bar{z}^2}. \tag{19}$$

As such the solution of this potential equation is difficult; therefore, we assume $V(z,\bar{z})$ to be separable in the form

$$V(z,\bar{z})=U(z)\cdot w(\bar{z}), \tag{20}$$

which reduces (19) to a pair of equations

$$(\tfrac{1}{2}c_1 z^2+c_4 z+c_5)\frac{\mathrm{d}^2U}{dz^2}+\tfrac{3}{2}\,(c_1 z+c_4)\frac{\mathrm{d}U}{\mathrm{d}z}-c_0\,U=0, \tag{21a}$$

$$(\tfrac{1}{2}c_1\bar{z}^2+c_2\bar{z}+c_3)\frac{d^2w}{d\bar{z}^2}+\frac{3}{2}(c_1\bar{z}+c_2)\frac{dw}{d\bar{z}}-c_0w=0, \tag{21b}$$

where c_0 is another separation constant. Now, we consider the following two cases:

Case (a): Let $c_1=c_2=c_4=0$ in (21a) and (21b), then the general solutions of the resultant equations can be easily obtained as

$$U(z)=\alpha_1\exp(\varepsilon_1 z)+\beta_1\exp(-\varepsilon_1 z),\quad w(\bar{z})=\alpha_2\exp(\varepsilon_2\bar{z})+\beta_2\exp(-\varepsilon_2\bar{z}),$$

which lead to the form of V as

$$V(z,\bar{z})=v_1\exp(\varepsilon_1 z+\varepsilon_2\bar{z})+v_2\exp(\varepsilon_1 z-\varepsilon_2\bar{z})+v_3\exp(-\varepsilon_1 z+\varepsilon_2\bar{z})$$
$$+v_4\exp(-\varepsilon_1 z-\varepsilon_2\bar{z}), \tag{22}$$

where $\varepsilon_1=\sqrt{c_0/c_5}$, $\varepsilon_2=\sqrt{c_0/c_3}$ and $v_1=\alpha_1\alpha_2$, $v_2=\alpha_1\beta_2$, $v_3=\alpha_2\beta_1$, $v_4=\beta_1\beta_2$. In this case, the coefficients a_{ij} from (16)–(18) become

$$a_{11}=c_3,\quad a_{22}=c_5,\quad a_{12}=c_6,$$

and the coefficients a_0 from (13), (14) and (22), takes the form

$$a_0=\bar{k}_0+2(c_6+\sqrt{c_3c_5})[v_1\exp(\varepsilon_1 z+\varepsilon_2\bar{z})+v_4\exp(-\varepsilon_1 z-\varepsilon_2\bar{z})]$$
$$+2(c_6-\sqrt{c_3c_5})[v_2\exp(\varepsilon_1 z-\varepsilon_2\bar{z})+v_3\exp(-\varepsilon_1 z+\varepsilon_2\bar{z})],$$

where $\bar{k}_0$ is a constant of integration. With these expressions for the coefficients a_{ij} and a_0 the invariant (5) corresponding to the potential (22) can be written as (since $z=q_1+iq_2$ and $\dot{z}=p_1+ip_2$)

$$I=\bar{k}_0+2(c_6+\sqrt{c_3c_5})\{v_1\exp[(\varepsilon_1+\varepsilon_2)q_1+i(\varepsilon_1-\varepsilon_2)q_2]$$
$$+v_4\exp[-(\varepsilon_1+\varepsilon_2)q_1-i(\varepsilon_1-\varepsilon_2)q_2]\}$$
$$+2(c_6-\sqrt{c_3c_5})\{v_2\exp[(\varepsilon_1+\varepsilon_2)q_1+i(\varepsilon_1+\varepsilon_2)q_2]$$
$$+v_3\exp[-(\varepsilon_1-\varepsilon_2)q_1-i(\varepsilon_1+\varepsilon_2)q_2]\}$$
$$+\tfrac{1}{2}(2c_6+c_3+c_5)p_1^2+\tfrac{1}{2}(2c_6-c_3-c_5)p_2^2+i(c_3-c_5)p_1p_2. \tag{23}$$

In order to have an analogy with Toda potential (1) we relate the arbitrary constants ε_1 and ε_2 by

$$i(\varepsilon_1-\varepsilon_2)=1,\quad \varepsilon_1+\varepsilon_2=\sqrt{3},$$

which will imply that

$$\sqrt{c_3c_5}=c_0;\quad c_3+c_5=c_0;\quad c_3-c_5=-\sqrt{3}ic_0.$$

As a result the invariant (23), corresponding to the potential

$$V(q_1,q_2)=v_1\exp(q_2+\sqrt{3}q_1)+v_2\exp(-i(q_1-\sqrt{3}q_2))$$
$$+v_3\exp(i(q_1-\sqrt{3}q_2))+v_4\exp(-q_2-\sqrt{3}q_1), \tag{24}$$

can be written as

$$I = \bar{k}_0 + 2(c_6 + c_0)\{\tfrac{1}{2}p_1^2 + v_1 \exp(q_2 + \sqrt{3}q_1) + v_4 \exp(-q_2 - \sqrt{3}q_1\}$$
$$+ 2(c_6 - c_0)\{\tfrac{1}{2}p_2^2 + v_2 \exp(-i(q_1 - \sqrt{3}q_2))$$
$$+ v_3 \exp(i(q_1 - \sqrt{3}q_2))\}\sqrt{3}c_0 p_1 p_2 - \tfrac{1}{2}c_0(p_1^2 - p_2^2) \tag{25}$$

Note that only for $c_0 = \bar{k}_0 = 0$ and $c_6 = \frac{1}{2}$, I takes the form of Hamiltonian; otherwise the following four special cases of (24) and (25) by setting $\bar{k}_0 = 0$ are of interest. Before discussing these special cases a few remarks about the potential (24) are in order. While the potential (24) is of exponential type it consists of four terms unlike the Toda potential (1) which has only three exponential terms. In fact, the standard Toda potential has been deduced (Ford *et al* 1973) for a linear chain of three particles in a periodic lattice. Further using an appropriate canonical transformation for the change of variables, form (1) is obtained. On the other hand, the potential (24) can be visualized as corresponding to four particles in a similar lattice (Gutkin 1985) but with a choice that two relative coordinates are now made cyclic.

(1) For $v_1 = v_4 = v_0$ and $v_2 = v_3 = \bar{v}_0$, V and I are given by

$$V(q_1, q_2) = 2[v_0 \cosh(q_2 + \sqrt{3}q_1) + \bar{v}_0 \cos(q_1 - \sqrt{3}q_2)],$$
$$I = (c_6 + c_0)[p_1^2 + 4v_0 \cosh(q_2 + \sqrt{3}q_1)]$$
$$+ (c_6 - c_0)[p_2^2 + 4\bar{v}_0 \cos(q_1 - \sqrt{3}q_2)]$$
$$+ \sqrt{3}c_0 p_1 p_2 - \tfrac{1}{2}c_0(p_1^2 - p_2^2). \tag{26a}$$

(ii) For $v_1 = v_4 = v_0$ and $-v_2 = v_3 = \bar{v}_0$, V and I are given by

$$V(q_1, q_2) = 2[v_0 \cosh(q_2 + \sqrt{3}q_1) + i\bar{v}_0 \sin(q_1 - \sqrt{3}q_2)],$$
$$I = (c_6 + c_0)[p_1^2 + 4v_0 \cosh(q_2 + \sqrt{3}q_1)]$$
$$+ (c_6 - c_0)[p_2^2 + 4i\bar{v}_0 \sin(q_1 - \sqrt{3}q_2)] + \sqrt{3}c_0 p_1 p_2 - \tfrac{1}{2}c_0(p_1^2 - p_2^2). \tag{26b}$$

(iii) For $v_1 = -v_4 = v_0$ and $v_2 = v_3 = v_0$, V and I are given by

$$V = 2[v_0 \sinh(q_2 + \sqrt{3}q_1) + \bar{v}_0 \cos(q_1 - \sqrt{3}q_2)], \tag{26c}$$

$$I = (c_6 + c_0)[p_1^2 + 4v_0 \sinh(q_2 + \sqrt{3}q_1)]$$
$$+ (c_6 - c_0)[p_2^2 + 4\bar{v}_0 \cos(q_1 - \sqrt{3}q_2)] + \sqrt{3}c_0 p_1 p_2 - \tfrac{1}{2}c_0(p_1^2 - p_2^2).$$

(iv) For $v_1 = -v_4 = v_0$ and $-v_2 = v_3 = \bar{v}_0$, V and I are given by

$$V = 2[v_0 \sinh(q_2 + \sqrt{3}q_1) + i\bar{v}_0 \sin(q_1 - \sqrt{3}q_2)], \tag{26d}$$

$$I = (c_6 + c_0)[p_1^2 + 4v_0 \sinh(q_2 + \sqrt{3}q_1)]$$
$$+ (c_6 - c_0)[p_2^2 + 4i\bar{v}_0 \sin(q_1 - \sqrt{3}q_2)] + \sqrt{3}c_0 p_1 p_2 - \tfrac{1}{2}c_0(p_1^2 - p_2^2).$$

Case (b): Setting $c_2 = c_3 = c_4 = c_5 = 0$ in (21a) and (21b), the solution of the resultant equations becomes simple and is given by

$$U(z) = A_1 z^{-1\pm\eta}, \quad w(\bar{z}) = A_2 \bar{z}^{-1\pm\eta},$$

which imply the form of $V(z, \bar{z})$ as

$$V(z, \bar{z}) = A(z\bar{z})^{-1\pm\eta}, \tag{27}$$

or

$$V(|z|) = A\,r^{-2\pm 2\eta} = A(q_1^2 + q_2^2)^{-1\pm\eta},$$

where $A = A_1 A_2$, $r^2 = z\bar{z}$, and η is a real number given by $\eta = (1 + 2\,c_0/c_1)^{1/2}$. For this case, the coefficients a_{ij} from (16)–(18) reduce to

$$a_{11} = \tfrac{1}{2}c_1\bar{z}^2, a_{22} = \tfrac{1}{2}c_1 z^2, a_{12} = -\tfrac{1}{2}c_1 z\bar{z} + c_6,$$

and the coefficient a_0, after using (27) in (13) and (14) becomes

$$a_0 = 2\,A\,c_6(z\bar{z})^{-1\pm\eta} + k_0,$$

where k_0 is a constant of integration. Finally, the invariant (5) turns out to be

$$I = k_0 + 2Ac_6\,r^{-2\pm 2\eta} - c_1(q_1 p_2 - q_2 p_1)^2 + c_6(p_1^2 + p_2^2), \tag{28}$$

which is, in fact, not a new invariant but represents a linear combination of the Hamiltonian and the square of angular momentum. In this way we recover a known integrable system.

If we assume the separability of $V(z,\bar{z})$ in the form,

$$V(z, \bar{z}) = f_1(z) + f_2(\bar{z}),$$

then (19) still reduces to a pair of equations which in a compact form, can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\chi_1^{3/2}\frac{\mathrm{d}f_1}{\mathrm{d}z}\right) = \lambda\chi_1^{1/2}; \quad \frac{\mathrm{d}}{\mathrm{d}\bar{z}}\left(\chi_2^{3/2}\frac{\mathrm{d}f_2}{\mathrm{d}\bar{z}}\right) = \lambda\chi_2^{1/2},$$

where $\chi_1 = \frac{1}{2}c_1 z^2 + c_4 z + c_5$, $\chi_2 = \frac{1}{2}c_1\bar{z}^2 + c_2\bar{z} + c_3$ and λ is a separation constant. For $\lambda = 0$, the solution of these equations further yields an integrable system of the type

$$V(z, \bar{z}) = \frac{2\Lambda_1}{(2c_1 c_5 - c_4^2)}\frac{(c_1 z + c_4)}{(\frac{1}{2}c_1 z^2 + c_4 z + c_5)} + \frac{2\Lambda_2}{(2c_1 c_3 - c_2^2)}\frac{(c_1\bar{z} + c_2)}{(\frac{1}{2}c_1\bar{z}^2 + c_2\bar{z} + c_3)},$$

where Λ_1 and Λ_2 are the constants of integration. It may be noted that the structure of $V(z, \bar{z})$ turns out to be more complicated when we take $\lambda \neq 0$.

Here we have considered only a few particular solutions of the "potential" equation (19) and this could lead to various integrable systems (cf. (26a)–(26d)). To obtain other integrable systems which admit second order invariants it may be of interest to explore more general solutions of this equation in the sense of Olshanetsky and Perelomov (1981). In fact, to some extent the integrable systems obtained here are the extension of their results to two dimensions. Alternatively, if for a given system $V(q_1,q_2)$ the arbitrary constants c_i's in (19) can be determined uniquely, then also it is possible to construct the corresponding invariant (rationalization method (Kaushal *et al* 1984; Mishra *et al* 1984)). Further, it is worthwhile to study the wave propagation in a lattice through the potentials (26a)–(26d) which have Toda-type features but admit now second order invariants.

To summarize, our method based on the complexification of the two dimensions of a two-dimensional system leads to quite a few new integrable systems which to the best of our knowledge has not been discovered earlier.

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