Pramāṇa – J. Phys., Vol. 26, No. 2, February 1986, pp. 93–108. © Printed in India.

Special Stäckel electrovac spacetimes

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MS received 25 January 1985; revised 21 November 1985

Abstract. Classification of all electrovac spacetimes permitting the separation of variables in the Hamilton-Jacobi equation for a charged test particle is carried out. This separation requires the existence of a complete set consisting of Killing's vectors and tensors of a special kind. Every complete set defines its own type of metric and electromagnetic potential in the separable coordinate system. There exist seven types of separation of variables for electromagnetic spaces. For every type an additional classification is carried out by transformation of coordinates without any disturbance of the separation conditions, the gradient transformation of electromagnetic potential and the conformal-constant transformation of metric.

The key step in solving the problem is the extraction of an autonomous subsystem which determines the metric from only the Einstein-Maxwell equations for every type of separation of variables.

Representatives of all classes of metrics and electromagnetic potential are given for every type of separation of variables with the exception of the spaces found in the well-known work by Carter.

The problem is solved in terms of metric formalism. The classes of electrovac spacetimes obtained are found to be related to Petrov's classification.

Keywords. General relativity; electrovac spacetimes; separation of variables.

PACS No. 3·50; 4·40

1. Introduction

We shall call the Riemannian space of a signature $(+, -, -, -)$ as a special Stäckel electrovac spacetime in which the Hamilton-Jacobi equation for a massive charge in an external electromagnetic field defined by an electromagnetic potential $A_i(x)$, $i, j = 0, 1, 2, 3$.

$$g^{ij}(x)(S_{,i} + A_i)(S_{,j} + A_j) - m^2 = 0, \tag{1}$$

can be integrated by the complete separation of variables and the metric tensor g^{ij} together with the electromagnetic potential A_i satisfy the vacuum (with λ-term) Einstein-Maxwell equations

$$R_{ij} - g_{ij}R/2 + \lambda g_{ij} = 4\pi æ T_{ij}, \quad \nabla_i F^{ij} = 0, \tag{2}$$

(we use for a single term the rule of summation with respect to the superscript and the subscript appearing twice, unless a reservation is made). Here R_{ij} is the Ricci tensor; R is the scalar curvature; ∇_i is the covariant derivative with respect to x^i; $F_{ij} = A_{j,i} - A_{i,j}$, $T_{ij} = (g_{ij}F_{kl}F^{kl} - F_{il}F_j^{\cdot l})/4\pi$; $g_{il}g^{lj} = \delta_i^j$, δ_i^j is the Kronecker symbol; æ is an arbitrary constant; $k, l = 0, 1, 2, 3$. The term "special" is introduced to distinguish the above determined spaces from the Stäckel spaces in which the eikonal equation, $g^{ij}S_{,i}S_{,j} = 0$, can be integrated by the method of complete separation of variables.

Interest in special Stäckel spaces of electrovacuum is due to the possibility existing in this case to investigate their global properties on the basis of exact solutions of the equations of motion for test particles.

The problem of classification of such spaces repeatedly attracted researcher's attention (Carter 1968; Walker and Penrose 1970; Boyer *et al* 1981; Demiansky and Francaviglia 1981; Bagrov and Obukhov 1982, 1983; Bagrov *et al* 1983). A detailed bibliographical review has been given by Bagrov *et al* (1983). In the present paper this problem is completely solved in terms of metric formalism.

2. Separation of variables

In this section we give some statements of the theory of separation of variables which will be needed in the subsequent discussion. A more detailed consideration of the theory is given by Bagrov *et al* (1973), Shapovalov (1978, 1979), Benenty (1980), Benenty and Francaviglia (1979).

The conditions necessary and sufficient for complete separations of variables in a linear second-order non-parabolic differential equation were first given by Bagrov *et al* (1973) in covariant terms of symmetry operator complete sets. Stäckel spaces of a general form are studied by Shapovalov (1978, 1979). Benenty (1980) considered the theory in terms of structure properties of Riemannian manifold. The metric tensor and the electromagnetic potential are also found in a general form in separable coordinates by Bagrov *et al* (1973), Shapovalov (1978, 1979) and assumed as a basis for our work.

By definition, equation (1) is solvable by complete separation of variables if in any privileged coordinate system its complete integral can be presented in the form

$$S = \sum_{i=0}^{3} S_i(x^i) + Q(x), \tag{3}$$

where $Q(x)$ is independent of separation constants. According to general theory (see, for example, Shapovalov 1978, 1979; Benenty 1980) the necessary condition of variables complete separation in (1) is the existence of a complete set consisting of pairwise commuting $n(0 \leqslant n \leqslant 3)$ Killing's vector fields $Y^i_p(x)$ (indices of p and q run from 0 to $n-1$ through the text) and $(3-n)$ Killing's tensor fields $Y^{ij}_\nu(x)$ independent of each other and of g^{ij} (indices denoted by small Greek letters run from n to 3). We denote $n' = n\text{-rank}\,(Y^i_p Y_{qi})$. In the Riemannian space of a signature $(+,-,-,-)$ n' can take a value of 0 or 1 (Shapovalov 1978, theorem 4; 1979, theorem 2). Numbers n and n' characterize the complete set. We shall take a complete set defined by integers n and n' as (n,n')-type set. Accordingly, a special Stäckel electrovac space with an (n,n')-type complete set will be taken as a (n,n')-type space. The case is possible when a space would have sets of both (n,n')- and (n_0,n'_0)-types with $n > n_0$. Then we shall relate it to the (n,n')-type. There are only seven different combinations of integers (n,n'), they are (0·0), (1·0), (1·1), (2·0), (2·1), (3·0), (3·1). That leads to the seven correspondent types of Stäckel spaces with signature $(+, -, -, -)$.

The following notation is adopted here. Separable (privileged) coordinate system is picked up so that $Y^i_p = \delta^i_p$ and will be further designated as (u_i). Functions of one variable will be denoted by small Greek letters with an obligatory singular right subscript indicating the number of the variable. For example: $\varphi^{ij}_{\nu} = \varphi^{ij}_{\nu}(u_\nu)$. In this case a derivative with respect to the corresponding variable is denoted by a dot. The quantities squared denoted by small Greek letters ε, ξ, ζ without indices are equal to 1. Constants are denoted by small Roman letters only (indices are possible), for example: a^{ij}, a^i_j, a_{ij} are constants. Exceptions are metric tensors denoted by g_{ij}, g^{ij}, Kronecker's symbol $\delta^{ij} \equiv \delta^i_j \equiv \delta_{ij}$, x^i, u_i, $z = (u_2 + iu_3)/2$. Other functions are denoted by capital letters. In order that one may not confuse the value of a single superscript with the power we shall put the numerical value of the index in parenthesis, so that $a^{(2)}_p = a^i_p$, when $i = 2$; $a^2_p = (a_p)^2$.

Let there be two special Stäckel spaces of electrovacuum V and V' being denoted by the following sets of quantities

$$g_{ij}(u),\ A_i(u),\ \lambda,\ æ, \tag{4}$$

and

$$g'_{ij}(u'),\ A'_i(u'),\ \lambda',\ æ', \tag{5}$$

respectively, ((u), (u') are privileged coordinate systems). V and V' are considered to be equivalent if quantities (4) and (5) are related by relationships of the form

$$g'_{ij}(u') = a \sum_{k,l} (\partial u_k/\partial u'_i)(\partial u_l/\partial u'_j) g_{kl}(u), A'_i(u') = \xi \sum_k (\partial u_k/\partial u'_i) A_k(u)$$
$$+ \partial F(u')/\partial u'_i,\ \lambda' = \lambda/a,\ æ' = a æ,\ u' = u'(u),\ \det(\partial u'/\partial u) \neq 0. \tag{6}$$

This definition is used only once for establishing the equivalence of (1·0)-type to (2·0)-type spaces. In all other cases we employ a narrower concept of equivalence when using in formulae (6) so-called permissible transformations of coordinates

$$u'_p = c^q_p u_q + c^\nu_{pq} \alpha^q_\nu, \quad u'_\nu = \alpha_{S(\nu)}(u_{S(\nu)}), \tag{7}$$

where $\det c^p_q \neq 0$, $S(\nu)$ is a certain permutation of the set $(n, \ldots, 3)$. Using this definition, let us write down the conditions of complete separation of variables in the privileged coordinate system (see, for example, Bagrov *et al* 1973)

$$g^{ij} = \phi^\nu_3 \pi^{ij}_\nu, \tag{8}$$

$$A_i = \phi^\nu_3 g_{ij} \pi^j_\nu, \tag{9}$$

$$g^{ij} A_i A_j = \phi^\nu_3 \pi_\nu, \tag{10}$$

where $\phi^\mu_\nu = (\varphi^{-1})^\mu_\nu$, φ^μ_ν is the Stäckel matrix, π^{ij}_ν, π^i_ν are determined by the expressions:

type $(n{\cdot}0)$: $\pi^{ij}_\nu = \delta^i_\nu \delta^j_\nu + \gamma^{pq}_\nu \delta^i_p \delta^j_q, \quad \pi^i_\nu = \delta^i_p \pi^p_\nu,$

type $(n{\cdot}1)$: $\pi^{ij}_\nu = (1 - \delta^n_\nu)\delta^i_\nu \delta^j_\nu + \delta^n_\nu \gamma^p_\nu (\delta^i_n \delta^j_p + \delta^i_p \delta^j_n) +$ (11)

$$+ \gamma^{pq}_\nu \delta^i_p \delta^j_q, \quad \pi^i_\nu = \delta^i_n \delta^n_\nu \pi^n_n + \delta^i_p \pi^p_\nu.$$

The metrics and electromagnetic potentials in the canonical form are represented here for all possible seven types of Stäckel spaces mentioned above.

Type (0·0): $ds^2 = \delta^{ki}\delta^{kj}du_i du_j/\phi_3^k,\ A_i = 0.$

Type (1.0): $$ds^2 = (du_0^2/\Omega + \sum_\nu du_\nu^2/\varepsilon_\nu W^\nu)\Delta,\ \Omega A_0 = \gamma_\nu w^\nu,\ A_\nu = 0, \tag{12}$$

Type (1·1): $$ds^2 = (2du_0 du_1/\varepsilon_1 W^{(1)} - \Omega du_1^2/(\varepsilon_1 W^{(1)})^2 + du_2^2/\varepsilon_2 W^{(2)} + du_3^2/\varepsilon_3 W^{(3)})\Delta,$$
$$A_0 = \gamma_1,\quad A_1 = (\theta_\nu W^\nu - \Omega A_0)/\varepsilon_1 W^{(1)},\quad A_2 = A_3 = 0. \tag{13}$$

For $(1,n')$-types $\Delta = \delta_\nu W^\nu, \Omega = \omega_\nu W^\nu, W^{(1)} = \tau_2 - \tau_3,$

$$W^{(2)} = \tau_3 - \tau_1,\quad W^{(3)} = \tau_1 - \tau_2.$$

Type (2·0): $$ds^2 = \left(\sum_{p,q} G_{pq} du_p du_q + \sum_\nu (-1)^\nu du_\nu^2/\tau_\nu\right)\Delta,\ A_\nu = 0,$$
$$A_p = G^{pq}(\gamma_2^q + \gamma_3^q),\ \Delta = \delta_2 + \delta_3,\ G_{pp'}G^{p'q} = \delta_p^q, G^{pq} = \alpha\xi_2^q + \alpha\xi_3^q. \tag{14}$$

Type (2·1): $$ds^2 = -[(\varphi_2 du_0 + du_1 - \Omega du_2)^2/G + 2du_0 du_2 - \gamma_3 du_2^2 + du_3^2]\Delta,$$
$$-G = \varphi_2^2\gamma_3 + 2\varphi_2\chi_3 + \xi_2 + \xi_3,\ \Omega = \varphi_2\gamma_3 + \chi_3,\ \Delta = \delta_2 + \delta_3,$$
$$A_0 = \sigma_2 + \varphi_2 A_1,\quad A_1 = (\varphi_2\theta_3 - \sigma_2\Omega + \zeta_2 + \zeta_3)/G, \tag{15}$$
$$A_2 = \theta_3 - \sigma_2\gamma_3 - \Omega A_1,\quad A_3 = 0.$$

Type (3.n'): $$g_{ij} = g_{ij}(u_3),\quad A_p = \alpha\xi_3,\quad A_3 = 0. \tag{16}$$

$$(1)\ n' = 0,\quad g^{p3} = 0.\quad (2)\ n' = 1,\quad g^{p2} = g^{q3} = 0. \tag{17}$$

All the functions in the above relations can be apparently expressed in terms of the π_ν^{ij}, π_ν^i, φ_ν^μ in (8)–(10).

Let us briefly describe the general scheme.

Expressions comprising complementary information (to (8) and (9)) on the metrics and potential structure specific for every Stäckel space type are extracted from (10).

Functional equations autonomous subsystem comprising only functions φ_ν^μ and π_ν^{ij} is derived from equation (2) by the expressions mentioned above. The autonomous subsystem integration is reduced to the algebraic equations system solution. The rest system equations (2) and (8)–(10) are thus integrated.

3. (1·0)-type spaces

The electromagnetic potential, correct to equivalence, has the form

$$A_0 = \alpha_1,\quad A_\nu = 0. \tag{18}$$

To prove this let us write down (10) in the form

$$(\gamma_\nu W^\nu)^2 = (\omega_\nu W^\nu)(\zeta_\mu W^\mu),\quad \nu, \mu = 1, 2, 3. \tag{19}$$

Let the variable u_1 be fixed and introduce the functions

$$\Omega_{\nu_0} \equiv (\omega_{\nu_0} - \omega_1)/T_{\nu_0} + \mu_1,\quad \Gamma_{\nu_0} \equiv (\gamma_{\nu_0} - \gamma_1)/T_{\nu_0} + \nu_1,\quad \Xi_{\nu_0} \equiv (\zeta_{\nu_0} - \zeta_1)/T_{\nu_0} + \lambda_1,$$

$T_{\nu_0} = (-1)^{\nu_0}(\tau_{\nu_0} - \tau_1)$, here $\nu_0 = 2, 3$. Then one can represent (19) in the form

$$(\Gamma_2 - \Gamma_3)^2 = (\Omega_2 - \Omega_3)(\Xi_2 - \Xi_3). \tag{20}$$

Linear combinations (with constant coefficients) of the functions $(\Gamma_{\nu_0,\nu_0}; \Omega_{\nu_0,\nu_0}; \Xi_{\nu_0,\nu_0})$ forms a linear space M_{ν_0}; moreover $\dim M_2 + \dim M_3 \leqslant 3$.
(1). $\dim M_2 > 0$, $\dim M_3 > 0$. Because of the symmetry of (20) with respect to the permutation of indices 2 and 3, one can take $\dim M_3 = 1$. It allows us to represent functional equation (20) as a following system:

$$\Gamma_{\nu_0}^2 = \Omega_{\nu_0}\Xi_{\nu_0}, \quad 2\Gamma_2\Gamma_3 = \Omega_2\Xi_3 + \Omega_3\Xi_2. \tag{21}$$

As $\dim M_2 \neq 0$, we have

$$A_0 = \Gamma_2/\Omega_2 = \Gamma_3/\Omega_3 = \alpha_1. \tag{22}$$

(2). Let now $\dim M_2 = 0$, then $\Gamma_2 = \Omega_2 = \Xi_2 = 0$.
Because of $\tau_2 \neq 0$ (otherwise we have (2·0)-type spaces), from here it follows:

$$\gamma_1 = \gamma_2 = \omega_1 = \omega_2 = \zeta_1 = \zeta_2 = 0, \tag{23}$$

which is equivalent to (18). When condition (18) is satisfied, $T_{\nu\mu} = 0\,(\nu \neq \mu)$, then

$$R_{\nu\mu} = 0 \quad (\nu \neq \mu). \tag{24}$$

All solutions of system (24) were found by Obukhov (1977). This together with condition (18) makes it possible to completely integrate system (2). We can write the solutions in the following manner:

$$\mathrm{d}s^2 = \mathscr{P}[\mathrm{d}u_0^2/\sigma_1 + \mathrm{d}u_1^2 + (u_2 - u_3)(\eta_2\mathrm{d}u_2^2 - \eta_3\mathrm{d}u_3^2)],$$

$$A_0 = \alpha_1, \; A_\nu = 0, \quad \eta_\nu = 4(ru_\nu^3 + bu_\nu^2 + ku_\nu + q).$$

$$(1) \quad \mathscr{P} = 1/u_2u_3, \quad \sigma_1 = \varepsilon\beta_1^2, \quad \dot{\beta}_1^2 = \beta_1(p\beta_1^2 + 3r),$$

$$\dot{\alpha}_1 = 2(\varepsilon\lambda/æ)^{\frac{1}{2}}/\beta_1, \quad k = \lambda, \quad q = 0. \tag{25}$$

$$(2) \quad \mathscr{P} = \varepsilon, \quad \dot{\sigma}_1^2 = -4\varepsilon\sigma_1^2(p\sigma_1 + \varepsilon\lambda + \varepsilon\xi æ a^2/2),$$

$$\alpha_1 = a\int(-\xi\dot{\sigma}_1^2\sigma_1)^{-\frac{1}{2}}\mathrm{d}\sigma_1, \quad 2\varepsilon r = \xi æ a^2 - 2\lambda. \tag{26}$$

$$(3) \quad \mathscr{P} = \varphi_1, \quad 3\dot{\varphi}_1^2 = -2(2\lambda\varphi_1^2 + 6r\varphi_1^2 + 3\xi æ a^2\varphi_1 + p\varphi_1^{3/2}),$$

$$\sigma_1 = \varepsilon\varphi_1^3/\dot{\varphi}_1^2, \quad \alpha_1 = 2a\xi(-\xi\varphi_1)^{\frac{1}{2}}. \tag{27}$$

It is shown that the above mentioned special Stäckel electrovac spacetimes are equivalent, in terms of relations (6), to certain spaces of the (n·0)-type, where $n > 1$. For this purpose we shall prove that, in addition to Killing's vector field $Y_0^i = (1,0,0,0)$ belonging to the complete set, these spaces permit the existence of another Killing's vector field, i.e. $Y_1^i = (0,0,A(u_2,u_3),B(u_2,u_3))$ commuting with Y_0^i and A^i. For solving (25) the field Y_1^i is defined by the formulae: $A = u_2(\eta_2\eta_3u_2u_3)^{-\frac{1}{2}}(u_2 - u_3)^{-1}$, $B = -u_3(\eta_2\eta_3u_2u_3)^{-\frac{1}{2}}(u_2 - u_3)^{-1}$. In cases (26) and (27) the existence of Y_1^i follows from the following considerations. The Killing's system is written as

$$2A_{,3} = D/\eta_2, \quad 2B_{,2} = -D/\eta_3, \quad D = D(u_2, u_3), \tag{28}$$

$$2A_{,2} = -A\dot{\eta}_2/\eta_2 + (B-A)/(u_2-u_3),$$

$$2B_{,3} = -B\dot{\eta}_3/\eta_3 + (B-A)/(u_2-u_3).$$

Compatibility conditions of this sytem

$$2D_{,2}\eta_2 = 3(B-A)/(u_2-u_3)^2 - B\dot{\eta}_3/\eta_3(u_2-u_3)$$
$$- D[1/(u_2-u_3) - \dot{\eta}_2/\eta_2]/\eta_2,$$
$$2D_{,3}\eta_3 = 3(B-A)/(u_2-u_3)^2 - A\dot{\eta}_2/\eta_2(u_2-u_3)$$
$$- D[1/(u_2-u_3) + \dot{\eta}_3/\eta_3]/\eta_3,$$

with (28) form a completely integrated system of equations which has nontrivial solutions A, B, D and this is a proof of the existence of the vector field Y_1^i. Making transformation of variables u_2, u_3 which diagonalize Y_1^i, we obtain metrics and potentials of the special Stäckel spaces of electrovacuum of (2·0)- and (3·0)-types in the privileged coordinate system. Thus, we have constructed the proof of Theorem 1. There are no special Stäckel spaces of electrovacuum of (1·0)-type.

Note: the same is also true for spaces of (0·0)-type (Iwata 1969).

4. (1·1)-type spaces

Let us consider the functional equation (10). Using expression (13), (10) reduces to the form

$$2\theta_\nu W^\nu \gamma_1 - \omega_\nu W^\nu \gamma_1^2 - æ_\nu W^\nu = 0, \tag{29}$$

where $æ_\nu \equiv \pi_\nu/\varepsilon_\nu$. It should be noted that (29) is symmetric with respect to permutation of variables u_2, u_3 and that the functions which depend only on one of these variables enter into (29) linearly. The latter means that to obtain a general solution of (29) it is necessary to exhaust all cases of linear dependence between the functions $\theta_a, æ_a, \omega_a, \tau_a$, $a = 2, 3$. The existence of the group of equation (29):

$$\theta'_\nu = \theta_\nu + a\tau_\nu + b\omega_\nu + c, \quad æ'_\nu = æ_\nu + a_1\tau_\nu + 2b\theta_\nu + b^2\omega_\nu + c_1,$$

$$\omega'_\nu = \omega_\nu + a_2\tau_\nu + c_2, \quad \gamma'_1 = \gamma_1 + b, \quad \tau'_\nu = \tau_\nu + t \tag{30}$$

($a, b, c, a_1, c_1, a_2, c_2, t$–group parameters)
enables one, without loss of generality, to confine himself to the non-equivalent with respect to (30) solutions of (29):

(1) $\quad \gamma_1 = æ_\nu = 0 \Rightarrow A_0 = 0, \quad A_1 = \theta_\nu W^\nu/\varepsilon_1 W^{(1)}, \quad A_2 = A_3 = 0.$

(2) $\quad æ_\nu = \theta_\nu = \omega_\nu = 0 \Rightarrow A_0 = \gamma_1, \quad A_\nu = \Omega = 0. \qquad (31)$

The conditions $T_{\nu\mu} = 0, \nu \neq \mu$ are valid in the both cases, hence it follows $R_{\nu\mu} = 0$, $\nu \neq \mu$. A solution of this system was obtained by Obukhov (1977) and defined the space linear element in the form

$$ds^2 = \phi[2du_0du_1 + 2du_1^2\Omega/W + W(du_2^2/\alpha_2^{(2)} + du_3^2/\alpha_3^{(3)})], \tag{32}$$

where $W = \tau_2 - \tau_3$, $\Omega = \omega_2 - \omega_3$. Substituting (32) and (31) into Einstein-Maxwell equations (2), we obtain a system of functional equations for functions $\theta_\nu, \gamma_1, \omega_\nu, \tau_\nu, \phi$. The solutions of this system define all spaces of (1·1)-type. They are:

I. $A_\nu = 0,\quad A_0 = u_1(2\lambda/æ)^{\frac{1}{2}},\quad \phi = 1,\quad \dot{\tau}_\nu^2 = (-1)^\nu[a + 2n\tau_\nu + l\tau_\nu^2 - 8\lambda\tau_\nu^3],$

$\Omega = 0,\quad \alpha_1^\nu = 1.$

II. $A_0 = A_2 = A_3 = 0.$

(1). $A_1 = 12au_2,\quad \phi = -3/\lambda u_2^2,\quad W = 1,\quad \Omega = (ru_3^2 + su_3 + q)/\tau_1^2$

$-12\lambda æ a^2 u_2^4 + k_0u_2^3 - ku_2^2,\ \dot{\tau}_1^2 = 2(k\tau_1^2 - r/\tau_1^2),\ \alpha_1^{(2)} = 1,\ \alpha_1^{(3)} = \tau_1^2.$

(2). $A_1 = (au_2 + r)/\sigma_1^2 + (a_0u_3 + r_0)/\tau_1^2,\quad \phi = W = 1, \lambda = 0,\quad \Omega = (bu_2^2 + cu_2$

$+ l)/\sigma_1^2 + (b_0u_3^2 + c_0u_3 + l_0)/\tau_1^2,\quad \ddot{\sigma}_1/\sigma_1 + \ddot{\tau}_1/\tau_1 - 2b/\sigma_1^4 - 2b_0/\tau_1^4$

$+ æ(a^2/\sigma_1^6 + a_0^2/\tau_1^6) = 0,\quad \alpha_1^{(2)} = \sigma_1^2,\quad \alpha_1^{(3)} = \tau_1^2.$

(3). $A_1 = \mathscr{F} + \bar{\mathscr{F}},\quad \phi = \varphi_1,\quad W = \partial\bar{\partial}\mathscr{F}_0\bar{\mathscr{F}}_0,\quad \dot{\varphi}_1^2 = -(2r_0\varphi_1^2 + æn_0^2\varphi_1$

$+ s_0\varphi_1^3),\quad \Omega = r_0\partial\bar{\partial}(\mathscr{F}_0\bar{\mathscr{F}}_0)^2/4 + \mathscr{P}\partial\bar{\mathscr{F}}_0 + \bar{\mathscr{P}}\partial\mathscr{F}_0,\quad \bar{l} = l,$

$\bar{p} = p,\quad \bar{k} = k,\quad \partial = \partial/\partial z,\quad \alpha_1^\nu = 1.$

(a) $\mathscr{F}_0 = (kz^2 + 2qz + r)\exp(-il),\quad \mathscr{F} = (n_0\mathscr{F}_0 + k_0)\exp(il),$

$\mathscr{P} = \int[3kr_0\mathscr{F}_0^2\exp(il) + p\mathscr{F}_0 + t]dz,$

(b) $\mathscr{F}_0 = c\exp(l^{\frac{1}{2}}z) + t\exp(-l^{\frac{1}{2}}z) + k_0,\quad \mathscr{F} = 0,$

$\mathscr{P} = -2l^{\frac{1}{2}}r_0\bar{k}_0(c^2\exp(2l^{\frac{1}{2}}z))$

$-t^2\exp(-2l^{\frac{1}{2}}z) - (p + 3lr_0k_0\bar{k}_0)z(\mathscr{F}_0 - k_0)$

$+ c_0\exp(l^{\frac{1}{2}}z) + t_0\exp(-l^{\frac{1}{2}}z).$

5. (2·0)-type spaces

The spaces of this type were systematically studied in the well-known work by Carter (1968) under some additional restrictions imposed on the metric. We shall make a classification without these restrictions.

Let us write relationship (10) using formulae (14) and differentiate it with respect to u_2, u_3. The equation obtained, after simple transformations can be represented in the form $T_{23} = 0$. As $g_{23} = 0$ it follows that $R_{23} = 0$. The latter equality can be written as $[\ln(\Delta^2 \det G_{pq})]_{,23} = 0$. By integrating this we obtain

$$^2 \det(G_{pq}) = \tau_2\tau_3. \tag{33}$$

Thus, equation (33) follows from the separability conditions of Hamilton-Jacobi equation (1) and is not an additional restriction as it was supposed by Carter (1968). The only essential restriction accepted by Carter (1968) is the existence of constants a^{pq} such that the condition

$$\det(\alpha_\nu^{pq} + (-1)^\nu a^{pq}) = 0 \tag{34}$$

is valid for the functions α_ν^{pq} entering in metric (14). Having considered the metrics which do not satisfy condition (34) we establish the validity of the following.

Theorem 2. The set of special Stäckel spaces of electrovacuum of (2·0)-type can be divided into the following non equivalent, nonintersecting classes of spaces:

(a) class of Carter's spaces. Functions α_ν^{pq} in metric (14) can obey the condition (34);

(b) class of plane wave spaces. By means of permissible coordinate transformations (7) the component g_{00} of a metric tensor can be made equal to zero. All solutions of this class enter as special cases into the solutions of section 4;

(c) class of spaces of constant curvature (with no electromagnetic field).

Let us outline the proof. Formulae (14) enable one to extract from Einstein-Maxwell system (2) an autonomous subsystem (containing no functions of electromagnetic field). Indeed, as it is not difficult to see that

$$g^{\nu\mu} T_{\nu\mu} = g^{pq} T_{pq} = 0, \tag{35}$$

hence

$$g^{\nu\mu} R_{\nu\mu} = g^{pq} R_{pq} = 2\lambda. \tag{36}$$

Equations (36) together with (33) form the autonomous subsystem to be found. All nonequivalent solutions of this subsystem not satisfying condition (34) and the equality $g_{00} = 0$ are represented below:

(1) $$\mathrm{d}s^2 = \varepsilon(u_2 - b)(u_3 + b)\mathrm{d}u_0^2 + \varepsilon(u_2 + b)(u_3 - b)\mathrm{d}u_1^2 - [\mathrm{d}u_2^2/(4\lambda u_2 - a)(u_2^2 - b^2) + \mathrm{d}u_3^2/(4\lambda u_3 + a)(u_3^2 - b^2)]\, 3(u_2 + u_3).$$

(2) $$\mathrm{d}s^2 = u_3[(\alpha_2 \mathrm{d}u_0 + \mathrm{d}u_1)^2 \exp\varphi_2 + \tau_2 \exp(-\varphi_2)\mathrm{d}u_0^2 - \varepsilon \mathrm{d}u_2^2/\tau_2 + \mathrm{d}u_3^2/(-4\lambda u_3^3/3 + \varepsilon n u_3^2)], \quad \tau_2 = n u_2^2 + 2q u_2 + t, \quad \alpha_2 = \int \frac{[\dot\varphi_2(\dot\tau_2 - \dot\varphi_2 \tau_2) - n]^{\frac{1}{2}} \mathrm{d}u_2}{\exp\varphi_2}$$

(3) $$\mathrm{d}s^2 = \theta_3[(\alpha_3 - ct\rho_2)\mathrm{d}u_0^2/(\alpha_3^2 - \xi) + 2st\rho_2 \mathrm{d}u_0 \mathrm{d}u_1/(\alpha_3^2 - \xi) + \zeta(\alpha_3 + ct\rho_2)\mathrm{d}u_1^2/(\alpha_3^2 - \xi) + \varepsilon \mathrm{d}u_2^2 - \varepsilon \mathrm{d}u_3^2/\tau_3],$$

$$\rho_2 = p u_2, \quad \alpha_3^2 = \xi + \zeta\theta_3^2/\tau_3$$

$$(ct^2 x + \zeta st^2 x \equiv \xi, \mathrm{d}(ctx)/\mathrm{d}x \equiv -\zeta stx; \mathrm{d}(stx)/\mathrm{d}x \equiv ctx),$$

$$4\zeta\tau_3^2(\xi\tau_3 + \zeta\theta_3^2)(2\theta_3\ddot\theta_3 - \dot\theta_3^2 - \xi p^2) = (2\theta_3\dot\theta_3\tau_3 - \theta_3^2\dot\tau_3)^2.$$

(4) $$\mathrm{d}s^2 = \xi[(\Delta + 4a - \gamma_3)\mathrm{d}u_0^2 + 2\gamma_2 \mathrm{d}u_0 \mathrm{d}u_1 + \varepsilon(\Delta + 4a + \gamma_3)\mathrm{d}u_1^2]/\Delta + \Delta[\mathrm{d}u_2^2/(u_2 + a)(u_2 + b) - \mathrm{d}u_3^2/(u_3 + a)(u_3 - b)],$$

$$\Delta = u_2 + u_3, \quad \gamma_2^2 = 8\varepsilon a(u_2 + a), \quad \gamma_3^2 = 8a(u_3 + a), \quad \lambda = 0.$$

(5) $$\mathrm{d}s^2 = \varepsilon\zeta u_3[(1/u_3 + \varepsilon\zeta(u_2^2 - 3)/2)(\mathrm{d}u_0^2 + \varepsilon \mathrm{d}u_1^2) - \gamma_2(\mathrm{d}u_0^2 - \varepsilon \mathrm{d}u_1^2)]/\Delta + \xi\Delta(4\mathrm{d}u_2^2/(u_2 - 3\zeta\varepsilon) + \mathrm{d}u_3^2/u_3) - 2\zeta\gamma_3 \mathrm{d}u_0 \mathrm{d}u_1/\Delta, \quad \Delta = u_2 + u_3,$$

$4\gamma_2^2 = (u_2 - 3\varepsilon\zeta)(u_2 + \varepsilon\zeta)^3, \quad \gamma_3^2 = \zeta(\varepsilon\zeta - u_3)^3, \quad \lambda = 0.$

(6) $ds^2 = [\varepsilon\zeta u_2 u_3^2 du_0^2 + 2(\zeta u_2)^{\frac{1}{2}} u_2 \, du_0 \, du_1 - \varepsilon u_2 (2/u_3 + 1/u_2) du_1^2]/\Delta$
$+ \Delta(du_2^2/4u_2 + du_3^2/u_3), \quad \lambda = 0, \quad \Delta = u_2 + u_3.$

(7) $ds^2 = \sum_{i,j} g_{ij}(u_3) du_i \, du_j.$

The first solution is a space of a constant curvature. The second one is self-consistent on substitution into system (2) only if $A_i = 0$ and $\varphi_2 = \ln u_2$. Under these conditions the above solution also describes a space of a constant curvature. The seventh solution results in spaces of (3·0)-type. The rest of the solutions, when substituted in sytem (2), lead to some contradictions. Then the theorem is proved.

Plane-wave solutions are represented below for completeness. They follow from (31) and (32) in which it should be assumed that $A_0 = A_2 = A_3 = 0$, $\alpha_1^{\nu} = 1$.

(1) $A_1 = 2a \exp(ru_3) \cos(ru_2), \quad \phi = 1/W\tau_2, \quad \dot{\tau}_2^2 = -\tau_2(r^2\tau_2 + 4\lambda/3),$
$W = \exp(-ru_3), \quad \Omega = (q \exp(-ru_3) + q_0) \exp(-ru_3) + 3r^2(b - æa^2)\tau_2^2\lambda^{-1}$
$+ 2b\tau_2 - 2b\lambda/3r^2 + f\chi_2 \int \tau_2 \chi_2^{-1} du_2, \quad \chi_2 = r^2\tau_2^2 + \frac{4}{3}\lambda\tau_2 - \frac{4}{9}\lambda^2 r^{-2}.$

(2) $A_1 = au_2 + a_0 u_3, \quad \phi = -3/\lambda u_2^2, \quad W = 1, \quad \Omega = k(u_2^2 + u_3^2) + s_0 u_3$
$- æ\lambda(a^2 + a_0^2)u_2^4/12 + f u_2^3 + t.$

(3) $A_1 = \mathscr{F}/\partial\mathscr{F}_0 + \overline{\mathscr{F}/\partial\mathscr{F}_0}, \quad \lambda = 0, \quad W = \partial\bar{\partial}\mathscr{F}_0\bar{\mathscr{F}}_0, \quad \Omega = æ\mathscr{F}\bar{\mathscr{F}} + \mathscr{P}\overline{\partial\mathscr{F}_0}$
$+ \bar{\mathscr{P}}\partial\mathscr{F}_0, \quad \partial = \partial/\partial z, \quad \bar{l} = l, \quad \bar{p} = p, \quad \bar{n} = n.$

(*a*) $\mathscr{F}_0 = 3kz^2 + 2qz + r, \quad \mathscr{F} = 2n(kz^3 + qz^2) + 3fz + c,$
$\mathscr{P} = æn[n(kz^5/5 + qz^4/3) + f z^3 + cz^2] + 2p(kz^3 + qz^2) + s_0 z + t.$

(b) $\mathscr{F}_0 = h \exp(l^{\frac{1}{2}}z) + t \exp(-l^{\frac{1}{2}}z) + k, \quad \mathscr{F} = nz(\mathscr{F}_0 - k) + h_0 \exp(l^{\frac{1}{2}}z)$
$+ t_0 \exp(-l^{\frac{1}{2}}z), \; 2l^{\frac{1}{2}}\mathscr{P} = [n^2 æhz^2 + (-n^2 æ h/l^{\frac{1}{2}} + 2h_0 næ + 2l^{\frac{1}{2}}p_0 h)z$
$+ s_0]\exp(l^{\frac{1}{2}}z) + [-n^2 æ tz^2 + (-n^2 æ t/l^{\frac{1}{2}} - 2t_0 n æ + 2l^{\frac{1}{2}}p_0 t)z$
$+ s_1] \exp(-l^{\frac{1}{2}}z).$

(4) $A_1 = a_\nu \dot{\tau}_\nu / W, \quad \phi = 1/\tau_2\tau_3, \quad \dot{\tau}_\nu^2 = (-1)^\nu (r\tau_\nu^2 - 4\lambda\tau_\nu/3),$
$\omega_\nu = (-1)^\nu \tau_\nu \dot{\tau}_\nu \int [p\tau_\nu^2 + k\tau_\nu + l + (-1)^\nu \lambda^2 æ (a_2^2 + a_3^2)\tau_\nu^2/9]/\tau_\nu \dot{\tau}_\nu^2 du_\nu.$

6. (2.1)-type spaces

Similar to the space types considered above we shall write (10), using formulae (15), in the form $A_1^2 = -G(2\sigma_2\theta_3 - \sigma_2^2\gamma_3 + \eta_2 + \eta_3) +$. From this it follows that $[(GA_1)^2]_{,23} = -2\dot{\sigma}_2(F_{23} + \Omega_{,3}A_1 + \Omega F_{13})$, which, in its turn, can be represented in the form

$$(\dot{\sigma}_2 + \varphi_2 A_1)(F_{23} + \Omega F_{13}) + GF_{12}F_{13} = 0, \tag{37}$$

or $T_{23} = 0$. The latter equation, subject to (2), leads to the condition for the metric R_{23}

$= 0 \Leftrightarrow (G_{,2}/G - 2\mathscr{P}_2)_{,3} = 0$, where $\mathscr{P}_\nu \equiv \Delta_{,\nu}/\Delta$. From this equation we obtain by integrating

$$G = -(\Delta/\beta_2\beta_3)^2. \tag{38}$$

Formulae (15) together with (38) permits one to write system (2) of Einstein-Maxwell equations in the following form

$$\ddot{\beta}_3 = 2\lambda\beta_3\Delta, \tag{39}$$

$$2\mathscr{P}_{3,3} + \mathscr{P}_3^2 - 2\dot{\beta}_3\mathscr{P}_3/\beta_3 - (\dot{\varphi}_2\beta_2\beta_3/\Delta)^2 = 0, \tag{40}$$

$$\mathscr{P}_3^2 - 4\dot{\beta}_3\mathscr{P}_3/\beta_3 + (\dot{\varphi}_2\beta_2\beta_3/\Delta)^2 = -2æ[(F_{13}\Delta/\beta_2\beta_3)^2 + (\dot{\gamma}_2 + \dot{\varphi}_2 A_1)^2/\Delta - 4\lambda\Delta, \tag{41}$$

$$\ddot{\varphi}_2 - \dot{\varphi}_2(2\mathscr{P}_2 - 3\dot{\beta}_2/\beta_2) + \Omega_{,33} + 3\dot{\beta}_3\Omega_{,3}/\beta_3 - 2\mathscr{P}_3\Omega_{,3} = -2æ\Delta[F_{12}(\dot{\sigma}_2 + \dot{\varphi}_2 A_1) - F_{13}(F_{23} + \Omega F_{13})]/\beta_2^2\beta_3^2, \tag{42}$$

$$\ddot{\gamma}_3 + \dot{\gamma}_3\dot{\beta}_3/\beta_3 - (\beta_2\beta_3\Omega_{,3}/\Delta)^2 - 2\ddot{\beta}_2/\beta_2 - \mathscr{P}_2^2 + 4\dot{\beta}_2\mathscr{P}_2/\beta_2 = 2æ[(\Delta F_{12}/\beta_2\beta_3)^2 + (F_{23} + \Omega F_{13})^2]/\Delta, \tag{43}$$

$$(\Delta F_{13}/\beta_2\beta_3)_{,3} + \dot{\varphi}_2\beta_2\beta_3(\dot{\sigma}_2 + \dot{\varphi}_2 A_1)/\Delta = 0, \tag{44}$$

$$[\beta_2\beta_3(F_{23} + \Omega F_{13})/\Delta]_{,3} - [\beta_2\beta_3(\dot{\sigma}_2 + \dot{\varphi}_2 A_1)/\Delta]_{,2} = 0. \tag{45}$$

Let us put $\mathscr{P}_3 = 0$. From (40) it follows that $\dot{\varphi}_2 = 0$, equivalent to $\varphi_2 = 0$, while from (39) we have $\dot{\beta}_3^2 = 2\lambda\beta_3 + p$. Equation (38) gives in addition one more of the conditions (a) $\beta_2 = \delta_2$, (b) $\beta_3 = 1$. Integrating (44) together with (41) makes it possible to obtain an expression for potential A_1:

$$A_1 = \beta_2[(-\dot{\sigma}_2^2 - 2\lambda\delta_2^2/æ)^{\frac{1}{2}}\int\beta_3 du_3 + \rho_2]/\delta_2$$

while (37) enables one to express $F_{23} + \Omega F_{13}$ through A_1 if $\dot{\sigma}_2 \neq 0$. If $\dot{\sigma}_2 = 0$, we can use this for (45). The rest of the equations can be integrated in quadratures.

Let now $\mathscr{P}_3 \neq 0$. From (39) it follows that $\lambda\dot{\delta}_2 = 0$, and from (40) and (38) we have $(\dot{\varphi}_2\beta_2)^2 = 4\delta_2$. Let us integrate (39) and (40) taking into consideration the above expressions; then we obtain $\beta_3^2 = \dot{æ}_3^2 = (\lambda/3)[æ_3^4 + 6(\delta_2 æ_3)^2] + 2kæ_3 + q$, $\delta_3 = æ_3^2$. Since $\dot{æ}_3 \neq 0$ (in the opposite case $\mathscr{P}_3 = 0$), one can make the substitution of variables $u'_3 = æ_3(u_3)$, then $g_{33} = -\Delta/\beta_3^2$ and the above formula takes the form:

$$\beta_3^2 = (\lambda/3)[u_3^4 + 6(\delta_2 u_3)^2] + 2ku_3 + q, \quad \delta_3 = u_3^2. \tag{46}$$

It is supposed below that the substitution of variables is made. Let us consider two cases

(1) $\varphi_2 = 0$. Then (41), (44), (45) can be integrated:

(a) $\beta_2 = 1, \quad 2q = -æ\rho^2, \quad \dot{\sigma}_2 = \rho\cos\alpha_2,$

$$A_1 = \rho(\sin\alpha_2 + \rho_2 u_3)/u_3,$$

$$A_{2,3} + \Omega A_{1,3} = p\Delta[\dot{\alpha}_2\sin\alpha_2 + u_3\dot{\rho}_2 tg\alpha_2]/u_3\beta_3^2. \tag{47}$$

(b) $\sigma_2 = q = 0, \quad A_1 = \alpha_2, \quad A_{2,3} = \rho_2\Delta/\beta_3^2.$ (48)

The unknown functions entering into (47), (48) should be chosen so that (41), (42) would be self-consistent. Then these equations can be integrated in quadratures.

(2) $\dot{\varphi}_2 \neq 0$. Let us integrate (44) and (45); we obtain

$$A_1 = \rho_2 \sin 2[\operatorname{arctg}(u_3/\omega_2) + æ_2] - \dot{\sigma}_2 \dot{\varphi}_2,$$

$$\beta_3^2(A_{2,3} + \Omega A_{1,3})/\Delta = p\{(\dot{\omega}_2/\omega_2 - \dot{\beta}_2/\beta_2)\cos 2[\operatorname{arctg}(u_3/\omega_2) + æ_2] + 2\dot{æ}_2 \sin 2[\operatorname{arctg}(u_3/\omega_2) + æ_2] + (\dot{\omega}_2 \cos 2[\operatorname{arctg}(u_3/\omega_2) + æ_2])/2\omega_2\}/\omega_2 + \theta_2, \quad \delta_2 = \omega_2^2. \tag{49}$$

Substituting (49) into (41) enables us to give concrete expressions to the quantities entering into (48)

$$\rho_2 = p\beta_2/\omega_2, \quad q = -(\lambda\omega_2^4 + 2æp^2), \quad æ_2 = b,$$

$$\theta_2 = p\dot{\omega}_2\omega_2^{-2}\cos b, \quad (\dot{\sigma}_2/\dot{\varphi}_2)_{,2} = p\beta_2\dot{\omega}_2\omega_2^{-2}\sin 2b. \tag{50}$$

From (38), by fixing in turn variables u_2, u_3 and taking into account the linear independence of the functions φ_2 and φ_2^2 we obtain the following expressions:

$$\beta_3^2\gamma_3 = c_0 + 2c_1\delta_3 + c_2\delta_3^2, \quad \beta_3^2\chi_3 = b_0 + 2b_1\delta_3 + b_2\delta_3^2,$$

$$\beta_3^2\xi_3 = a_0 + 2a_1\delta_3 + a_2\delta_3^2. \tag{51}$$

Substituting again (51) into (38), collecting coefficients of the linearly independent functions δ_3, δ_3^2 and equating them to zero we obtain

$$\beta_2^2(c_0\varphi_2^2 + 2b_0\varphi_2 + a_0) = \delta_2^2 - r\xi_2\beta_2^2,$$

$$\beta_2^2(c_1\varphi_2^2 + 2b_1\varphi_2 + a_1) = \delta_2 - l\xi_2\beta_2^2,$$

$$\beta_2^2(c_2\varphi_2^2 + 2b_2\varphi_2 + a_2) = 1 - n\xi_2\beta_2^2,$$

$$\xi_2[n\delta_3^2 + 2l\delta_3 + r - \beta_3^2] = 0. \tag{52}$$

The latter equation of system (52) together with the relation between β_3^2 and δ_3 defined by (46) enables us to come to the conclusion that either $\xi_2 = 0$ or $\beta_3 = \text{const}$. Solutions of the rest of equations of system (52) together with the conditions $\lambda\dot{\delta}_2 = 0$, $(\dot{\varphi}_2\beta_2)^2 = 4\delta_2$ gives

(a) $\xi_2 = 0, \quad \beta_2 = 1, \quad \varphi_2 = \delta_2 = u_2^2, \quad \beta_3\gamma_3 = 1, \quad \chi_3 = u_3^2/\beta_3^2, \quad \xi_3 = u_3^4/\beta_3^2,$

(b) $\xi_2 = 0, \quad \beta_2 = [a(\xi\varphi_2^2 + 1)]^{-\frac{1}{2}}, \quad \dot{\varphi}_2^2 = 4c^2a(\xi\varphi_2^2 + 1), \quad \delta_2 = c,$

$\gamma_3 = a\xi(u_3^2 + c^2)/\beta_2^2, \quad \chi_3 = 0, \quad \xi_3 = a(u_3^2 + c^2)/\beta_3.$

(c) $\xi_2 = \delta_2^2(c_2\varphi_2^2 + 2b_2\varphi_2 + a_2), \quad \beta_2^2 = 1/(c_2\varphi_2^2 + 2b_2\varphi_2 + a_2),$

$\delta_2 = \beta_2^2(c_1\varphi_2^2 + 2b_1\varphi_2 + a_1), \quad \dot{\varphi}_2^2 = 4\delta_2/\beta_2^2.$

Thus, all the functions entering into g_{ij}, A_i are determined. System (2) and conditions of separability (8)–(10) are satisfiable identically. In conclusion, all the solutions obtained are listed below.

$$ds^2 = -\beta_2^2\beta_3^2(\varphi_2 du_0 + du_1 - \Omega du_2)^2/\Delta$$
$$-\Delta(2du_0du_2 - \gamma_3 du_2^2 + du_3^2/\alpha_3^2),$$
$$A_0 = \sigma_2 + \varphi_2 A_1, \quad A_3 = 0.$$

I. $48\alpha_3^2 = 48\beta_3^2 = \lambda[16u_3^4 + 24(\dot\varphi_2\beta_2 u_3)^2 - 3(\dot\varphi_2\beta_2)^4] + 48ku_3 - 96æp^2$

$\Delta = u_3^2 + (\dot\varphi_2\beta_2/2)^2$.

(1) $\varphi_2 = 2cu_2, \quad \beta_2 = 1, \quad \sigma_2 = \gamma_3 = \Omega = 0, \quad A_1 = p[2cu_3\cos a + (c^2 - u_3^2)\sin a]/c(u_3^2 + c^2), \quad A_2 = 0.$

(2) $\lambda = 0, \quad \varphi_2 = u_2^2, \quad \beta_2 = 1, \quad \sigma_2 = -2pu_2\sin a, \quad \Omega = (u_2^2 + u_3^2)/\beta_3^2,$
$\beta_3^2\gamma_3 = 1, \quad A_1 = 2p(u_3\cos a + u_2\sin a)/(u_2^2 + u_3^2), \quad A_2 = 0.$

(3) $\lambda = 0, \quad \dot\varphi_2^2 = 4(r\varphi_2^2 + \varepsilon\varphi_2), \quad k = 0, \quad 2æp^2 = -1, \quad \varphi_2\beta_2 = 1, \quad \varepsilon = 0, 1,$
$\sigma_2 = 2\varepsilon p\varphi_2\theta_2\sin a, \quad \Omega = \varphi_2 u_3^4 + (2r\varphi_2 + \varepsilon)u_3^2, \quad \gamma_3 = u_3^4 + 2ru_3^2,$
$A_1 = p[2\theta_2 u_3\cos a + (\theta_2^2 - u_3^2 - 2\varepsilon r\varphi_2\Delta - \varepsilon\Delta)\sin a]/\varphi_2\theta_2\Delta,$
$A_2 = \theta_2 pr[2\theta_2 u_3\cos a + (\theta_2^2 - u_3^2 - \varepsilon\Delta)\sin a]/\Delta, \quad \theta_2^2 = r + \varepsilon/\varphi_2.$

(4) $\varphi_2 = 0, \quad \beta_2 = 1, \quad \sigma_2 = 2pu_2\cos a, \quad \gamma_3 = -c\Omega, \quad \Omega = -cu_3^4/\beta_3^2,$
$A_1 = 2pu_3^{-1}\sin a + c\sigma_2, \quad A_2 = 0.$

(5) $\varphi_2 = 0, \quad \beta_2 = 1, \quad \sigma_2 = 0, \quad \Omega = 4æ\int_0^{u_3} x^3(rx + 2cp^2)\beta_3^{-4}(x)dx,$

$$\gamma_3 = 4æ\int_0^{u_3} dx\,\beta_3^{-2}(x)\left\{\int_0^x dt\,\beta_3^{-4}(t)t^2[4æ(rt + 2cp^2)^2 + 2p^2c^2\beta_3^2 t)]^3 + h\right\},$$

$$A_1 = 2p/u_3, \quad A_2 = 2p\int_0^{u_3}[cx^2\beta_3^{-2}(x) + \Omega(x)x^{-2}]dx.$$

(6) $\varphi_2 = 0, \quad \dot\beta_2^2 = r\beta_2^2 + b, \quad \sigma_2 = k = p = \Omega = 0, \quad \gamma_3 = -3(2\lambda ru_3^2 + 2\lambda cu_3 - 3æl^2)/2\lambda^2u_3^4, \quad A_1 = \alpha_2, \quad A_2 = 3(l^2 - \dot\alpha_2^2/\beta_2^2)^{\frac{1}{2}}/\lambda u_3,$

(a) $\alpha_2 = (a/r)(r\beta_2^2 + b)^{\frac{1}{2}}$, (b) $\alpha_2 = au_2^2, \quad \beta_2 = u_2.$

(7) $\varphi_2 = 0, \quad \beta_2 = 1, \quad p = \sigma_2 = 0, \quad \Omega = -c\int_0^{u_3}\frac{x^4}{\beta_3^4(x)}dx,$

$$\gamma_3 = \int_0^{u_3}\frac{dx}{\beta_3^2(x)}\left\{\int_0^x \frac{t^2[c^2t^2 + 2l^2æ\beta_3^2(t)]d(t)}{\beta_3^4(t)} + h\right\}, \quad A_1 = \alpha_2,$$

$$A_2 = (l^2 - \dot\alpha_2^2)^{\frac{1}{2}}\int_0^{u_3}\frac{x^2dx}{\beta_3^2(x)}$$

(a) $\alpha_2 = 0$,

(b) $k = c = 0, \quad \gamma_3 = -9(l^2 \ae + n u_3)/2\lambda^2 u_3^4$,

$\alpha_2 = a u_2, \quad A_2 = 3(l^2 - a^2)^{1/2}/\lambda u_3.$

II. $\varphi_2 = 0, \quad \alpha_3^2 = p + 2\lambda\beta_3^2, \quad \Delta = \delta_2.$

(8) $\beta_2 = \delta_2 = 1, \quad \sigma_2 = (-2\lambda\ae)^{1/2} u_2 \cos a, \quad \beta_3 = u_3, \quad \Omega = -c/\lambda u_3^2,$

$\gamma_3 = (c/\lambda u_3)^2, \quad A_1 = (\alpha_3 \sin a + 2 c u_2 \cos a)/(-2\lambda\ae)^{1/2}, \quad A_2 = 0.$

(9) $\beta_2 = \delta_2 = 1, \quad \beta_3 = u_3, \quad \sigma_2 = 0, \quad \Omega = \int_0^{u_3^{1/2}} \left(\frac{n}{(p+2\lambda x)^{1/2}} - r\left(\frac{-2\ae}{\lambda}\right)^{1/2} \right) \frac{dx}{x^2},$

$$\gamma_3 = \int_0^{u_3^{1/2}} \frac{dx}{x(p+2\lambda x)^{1/2}} \left\{ \int_0^x \frac{[n - r(-2\ae(p/\lambda + 2t))^{1/2}]^2 + 2\ae r^2 t}{t^2 (p+2\lambda t)^{1/2}} dt + h \right\},$$

$$A_1 = \alpha_3/(-2\lambda\ae)^{1/2}, \quad A_2 = \int_0^{u_3^{1/2}} \left[\frac{r}{x} - \left(-\frac{\lambda}{2\ae}\right)^{1/2} \Omega \right] \frac{dx}{(p+2\lambda x)^{1/2}}.$$

(10) $\lambda = \sigma_2 = \ddot\beta_2 = 0, \quad \delta_2 = \beta_2, \quad \beta_3 = u_3, \quad \Omega = c/u_3^2, \quad \gamma_3 = c^2/u_3^2 + l\ae \ln^2 u_3$

$+ b \ln u_3, \quad A_1 = 0, \quad A_2 = (l\beta_2)^{1/2} \ln u_3.$

(11) $\lambda = \sigma_2 = 0, \quad \beta_3 = 1, \quad \Omega = c u_3, \quad \gamma_3 = r u_3^2 + n u_3, \quad A_1 = \alpha_2,$

$A_2 = -c\alpha_2 u_3, \quad \delta_2 [2\ddot\beta_2/\beta_2 + (\dot\delta_2/\delta_2)^2 + 4(\dot\beta_2 \dot\delta_2)/\beta_2\delta_2 + (q\beta_2/\delta_2)^2 - 2r]/2\ae$

$+ (\delta_2 \dot\alpha_2/\beta_2)^2 + c^2 \alpha_2^2 = 0.$

7. (3·n′)-type spaces

The spaces of this type admit a three-parametric Abelian group of motions. In separable coordinates, a metric tensor and an electromagnetic potential depend on one variable u_3. In the present section, as earlier, constants are denoted by small Roman letters and by small Greek letters without right subscript. All other letters are used to denote functions of u_3. As before, the exceptions concerning the symbols δ^i_j, g_{ij}, g^{ij} are valid.

7.1. (3.0)-*type spaces*

The metric, in correspondence with (16), has the form

$$ds^2 = \Delta \sum_{p,q} G_{pq} du_p du_q - \Delta^3 du_3^2/\Omega^2 G, \tag{53}$$

where

$$G = 1/\det(G_{pq}), \quad G^{pp'} G_{qp'} = \delta^p_q, \quad p,q = 0,1,2.$$

Integrating Maxwell equations we have

$$\dot A_p = G_{pq} c^q/\Omega \tag{54}$$

Using permissible coordinate transformations

$$u'_p = a_{p.}^{\cdot q} u_q, \quad u_p = a_{\cdot p}^{q\cdot} u'_q, \quad (a_{q'.}^{\cdot p} a_{\cdot q}^{q'\cdot} = \delta_q^p), \tag{55}$$

we turn $c^{(1)}$ and $c^{(2)}$ to zero. This condition is invariant about (55) if we demand for

$$a_{a.}^{\cdot(0)} = a_{\cdot a}^{(0)\cdot} = 0, \quad a = 1, 2. \tag{56}$$

Let us denote $J_q^p \equiv \dot{G}^{pp'} G_{p'q}$, $\mathscr{P} \equiv \dot{\Delta}/\Delta$ then system (12) can be represented in the form

$$(\Omega J_q^p)^{\cdot} = \{\delta_q^p [\text{æ} c^2 \Delta G G_{00} - 2\lambda\Delta^3 + (\Omega\mathscr{P})^{\cdot}\Omega G - 2\text{æ} c^2 \Delta G G_{0q} \delta_o^p\}/\Omega G, \tag{57}$$

$$3(\Omega\mathscr{P})^{\cdot} - (\dot{G}\Omega/G)^{\cdot} - \Omega[3\mathscr{P}^2 - 2\mathscr{P}\dot{G}/G + \dot{G}^{pq}\dot{G}_{pq}/2] = (\text{æ} c^2 \Delta G G_{00} + 2\lambda\Delta^3)/\Omega G. \tag{58}$$

Integrating (57) under with $p = a = 1, 2$, we obtain

$$\Omega J_q^a = \gamma_3 \delta_q^a + k_q^a, \tag{59}$$

where γ_3 is a certain function. In addition, from (57) it also follows that

$$\Delta = \Omega[3(\Omega J_1^{(1)})^{\cdot} - (\Omega\dot{G}/G)^{\cdot}]/2c^2 \text{æ} G_{00}. \tag{60}$$

The rest of the equations can be reduced to the form

$$(\Omega J_1^{(0)})^{\cdot} G_{00} + [(\Omega J_1^{(1)})^{\cdot} - (\Omega J_0^{(0)})^{\cdot}] G_{01} = 0,$$

$$(\Omega J_1^{(0)})^{\cdot} G_{02} - (\Omega J_2^{(0)})^{\cdot} G_{01} = 0,$$

$$(\Omega^2 \dot{G}^{pq} \dot{G}_{pq})^{\cdot} - 2(\Omega\dot{G}/G)^{\cdot}\Omega\dot{G}_{00}/G_{00} + 2(\Omega J_1^{(1)})^{\cdot}\Omega(\dot{G}/G + 3\dot{G}_{00}/G_{00}) = 0, \tag{61}$$

$$2(3\Omega\mathscr{P} - \Omega\dot{G}/G)^2 + (\Omega\dot{G}/G)^2 + 3\Omega^2 \dot{G}^{pq}\dot{G}_{pq} + 12(\text{æ} c^2 \Delta G G_{00} - 2\lambda\Delta^3)/G = 0. \tag{62}$$

System (61) is a differential-algebraic consequence system (59), (60), (62). Multiplying (59) by G^{pq} and summing over p, we obtain the system

$$\Omega\dot{G}^{pa} = \gamma_3 G^{pa} + k_q^a G^{qp}. \tag{63}$$

Let us eliminate γ_3 from (63) and integrate the subsystem obtained. By transformations (55), (56) we turn matrix k_q^a to one of the two possible canonical forms

$$\text{I. } k_p^a = \begin{Vmatrix} 0 & 0 & k \\ 0 & n & 0 \end{Vmatrix}, \qquad \text{II. } k_p^a = \begin{Vmatrix} k & 0 & 0 \\ 0 & n & 0 \end{Vmatrix}$$

Discarding the solutions leading to special cases of (2.0)- and (2.1)-type spaces we obtain three nonequivalent classes of the solutions G_{pq}, which are represented as quadric quantics $G(x,x) = \sum_{p,q} G_{pq} x_p x_q$, x_p are parameters:

(1) $G(x,x) = \{x_0^2 - 2\theta_3 x_0 x_1 + (\ddot{\theta}_3\Omega + \dot{\theta}_3^2)x_1^2 + 2(u_3\dot{\theta}_3 - \theta_3)x_0 x_2$

$$-2[u_3\Omega\ddot{\theta}_3+\dot{\theta}_3(u_3\dot{\theta}_3-\theta_3)]x_1x_2+[\Omega\theta_3\ddot{\theta}_3+(u_3\dot{\theta}_3$$
$$-\theta_3)^2]x_2^2\}/\ddot{\theta}_3\Omega^2,$$

where $\theta_3=\Omega+u_3^2$.

(2) $G(x,x)=\{\Omega x_0^2+a\theta_3x_1^2-2u_3\theta_3x_1x_2+\theta_3x_2^2\}/\theta_3\Omega,\quad \Omega=a-u_3^2.$

(3) $G(x,x)=\{(a-b^2u_3)x_0^2+2a\theta_3x_0x_1+2b\theta_3x_0x_2+a\theta_3(\theta_3-u_3\dot{\theta}_3)x_1^2$
$$+2b\theta_3(\theta_3-u_3\dot{\theta}_3)x_1x_2-\theta_3\dot{\theta}_3x_2^2\}/u_3\theta_3[(b^2u_3-a)\dot{\theta}_3-b^2\theta_3],$$
$\theta_3=\Omega/u_3$.

Every of these equations includes an arbitrary function θ_3, defined from (62). All the rest of the equations of system (57), (58) are satisfied identically.

7.2. (3.1)-*type spaces*

According to (17) the metric of the space has the form

$$ds^2=\sum_{i,j}g_{ij}\,du_idu_j,\ g^{33}=g^{p2}=0,\ g^{p3}=\beta_3^p,\ \beta_3^{(2)}=1.$$

From system (2) it follows that $\beta_3^{(0)}=\beta_3^{(1)}=0$. Indeed,

$$R_{pq}=-\dot{\beta}_3^{p'}\dot{\beta}_3^{q'}g_{pp'}g_{qq'}=g_{pq}(T+\lambda),\ \text{where } T=2\pi æ(\beta_3^pF_{p3})^2,$$

hence $(\dot{\beta}_3^{(0)}g^{11}-\dot{\beta}_3^{(1)}g^{01})^2=-gg^{11}(T+\lambda),$

$$(\dot{\beta}_3^{(0)}g^{01}-\dot{\beta}_3^{(1)}g^{00})^2=-gg^{00}(T+\lambda),$$
$$(\dot{\beta}_3^{(0)}g^{11}-\dot{\beta}_3^{(1)}g^{01})(\dot{\beta}_3^{(0)}g^{01}-\dot{\beta}_3^{(1)}g^{00})=-gg^{01}(T+\lambda),\ \text{where } g=\det(g_{ij}).$$

Since $g\neq 0$, we have $T+\lambda=0$, $\dot{\beta}_3^{(0)}g^{11}-\dot{\beta}_3^{(1)}g^{01}=0$, $\dot{\beta}_3^{(0)}g^{01}-\dot{\beta}_3^{(1)}g^{00}=0$. There is, up to equivalence, the only solution of this system:

$$\beta_3^{(0)}=\beta_3^{(1)}=\lambda=T=0,\ 2(\dot{g}/g)^{\cdot}+\dot{g}^{pq}\dot{g}_{pq}+2æg^{pq}\dot{A}_p\dot{A}_q=0,\quad A_2=0,$$

which is solvable by a quadrature.

8. Conclusion

In conclusion, it should be noted since the problem is solved in terms of metric formalism it is desirable to obtain the Petrov classification of the solutions obtained. Additional analysis has shown that (2.0)- and (3.0)-type spaces belong to Petrov type I, (2.1)-type spaces are mainly of Petrov type II; and that (3.1), (1.1)-types belong to the confluent Petrov type II.

References

Bagrov V G, Meshkov A G, Shapovalov V N, Shapovalov A V 1973 *Izv. vuzov SSR Fiz.* **11** 66
Bagrov V G and Obukhov V V 1982 *Izv. vuzov SSSR Fiz.* **4** 13

Bagrov V G and Obukhov V V 1983 *Ann. Phys.* **40** 181
Bagrov V G, Obukhov V V and Shapovalov A V 1983 *Izv. vuzov SSSR Fiz.* **1** 6
Benenti S 1980 *Lect. Notes Math.* **836** 512
Benenti S and Francaviglia M 1979 *Gen. Relat. Gravit.* **10** 79
Boyer C P, Kalnins E G and Miller W Jr 1981 *J. Phys.* **A14** 1675
Carter B 1968 *Commun. Math. Phys.* **10** 280
Demiansky M and Francaviglia M 1981 *J. Phys.* **A14** 173
Iwata G 1969 *Nat. Sci. Rept. Ochanomizu Univ.* **20** 92
Obukhov V V 1977 Deposited in VINITI, N 2640-77 Dept.
Shapovalov V N 1978 *Izv. vuzov SSSR Fiz.* **9** 18, 25
Shapovalov V N 1979 *Sib. Mat. Z.* **20** 1117
Walker M and Penrose R 1970 *Commun. Math. Phys.* **18** 26