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Polarization parameters in systems with spin-spin interactions

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Abstract. The spin-spin interaction of two arbitrary spin systems is considered in some detail. The temporal evolution of the polarization parameters and the correlation parameters has been worked out. Applications of the formalism and the interpretation of the results to processes such as heavy-ion interactions, muon and nuclear repolarization and depolarization in muonic atoms and interactions of multilevel systems are outlined.

Keywords. Polarization; density matrix; coupled spin systems; heavy-ions; multilevel systems; correlations.

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1. Introduction

The study of polarization phenomena has assumed increasing importance in recent years (Mukherjee 1984) although it has been given considerable attention in earlier years in various contexts such as fine and hyperfine interactions, interactions of spin systems with external magnetic field, the discovery of V-A form of the weak interaction from asymmetry experiments in β-decay, μ-decay, μ-capture and recently the neutral current processes, the emission and absorption of polarized light and various angular correlations. A recent addition to this impressive list is the study of heavy-ion physics. In this context the observation of Mukherjee may be quoted here, "surprisingly the scattering of light heavy-ions provides us with a fair number of puzzling problems that may not be solved without polarization." He goes on to say, "the polarized heavy ion scattering is expected to provide a rich source of information about the spin-dependence of heavy ion interaction about which very little is known." From a theoretical point of view also the subject is very interesting since the projectile as well as the target could in general be characterized by high spin values j_1 and j_2 respectively. In discussing the polarization of a spin 1/2 system it is customary to characterize the system by specifying the numbers $N(\uparrow)$ and $N(\downarrow)$ of the spin 1/2 particles with the spin pointing up or down respectively with respect to an axis in the three-dimensional configuration space which specifies the direction of polarization. If the same type of description is extended to spin 1 assemblies by expressing the vector polarization $t_0^1 = \sqrt{\frac{3}{2}}\,(N_1 - N_{-1})$ and the tensor polarization $t_0^2 = (1 - 3N_0)/\sqrt{2}$ where N_1, N_0 and N_{-1} denote the populations with spin projections $+1$, 0 and -1 respectively (see for e.g., Mukherjee 1984), [The parameters t_0^1 and t_0^2 can be made identical to the celebrated

Fano statistical tensors G_1 and G_2 by choosing a slightly different normalization as is done in this paper.] the situation is one which is particularly simple, viz the axially symmetric or oriented one. In general in the case of spin systems, with spin $j > \frac{1}{2}$ the assembly is characterized by specifying the populations $N_1, N_2 \ldots N_n$ where $n = 2j+1$ corresponding to the occupation states $\psi_1, \psi_2 \ldots \psi_n$ which cannot always be identified with the states $|jm\rangle, m = -j, \ldots, +j$ with respect to a suitable z-axis which is usually called the axis of orientation. This is so since the unitary group in n-dimensions is homomorphic to the rotation group in 3-dimensions only in the particular case of $n = 2$. Therefore for spins $j > \frac{1}{2}$ a spin assembly may or may not be characterized by an axis of orientation and we say that the system is non-oriented* when an inherent orientation axis does not exist for the assembly. In a heavy-ion interaction involving particles with spin greater than $\frac{1}{2}$ the reaction products would in general be non-oriented even if one takes care to prepare the colliding spin systems in an oriented manner initially.

The standard description of the density matrix in terms of the spherical tensor parameters t^k_q is sufficiently general to describe non-oriented systems as well although the Fano statistical tensors are inadequate. An alternative description of the density matrix in terms of the generators of the group SU(n) is available (Ramachandran and Murthy 1979). However we shall adopt the language of t^k_q's here as they are well known to experimentalists as well.

The purpose of this paper is to develop a detailed formalism to describe the temporal evolution of two interacting spin systems where the spins j_1 and j_2 are arbitrary. The basic density matrix description for the combined system is outlined in § 2 and the time development of the parameters is discussed in § 3. We demonstrate in § 4 some of the interesting features of the results obtained in § 3 by considering few special cases and also compare our method with other methods used in the literature.

2. Definitions and formalism

The density matrix $\rho(j)$ for an assembly of spin j particles is given by

$$\rho(j) = \sum_{k=0}^{2j} \sum_{q=-k}^{k} t^k_q \tau^{k^+}_q, \tag{1}$$

where τ^k_q denotes the spherical tensor operators of rank k which are normalized such that

$$\langle jm' | \tau^k_q | jm \rangle = C(j k j; m q m') \frac{[k]}{[j]}, \tag{2}$$

*The word 'non-oriented' has been used occasionally (Bernardini, 1966) to denote a random distribution of spin and Blum (1981) uses the word 'oriented' to denote systems with non-zero vector polarization and 'polarized' to denote a non-random distribution although a majority of authors use the word 'polarised' to denote systems with non-zero vector polarization, 'aligned' to denote systems with non-zero tensor polarization and 'oriented' to denote systems characterised by an axis of orientation i.e., axially symmetric (Ramachandran and Mallesh 1984; Simonius 1973).

with $[x] = (2x+1)^{1/2}$. Here we choose a normalization* which is slightly different from the Madison convention (Darden 1970) in view of future convenience. The spherical tensor parameters t^k_q satisfy

$$t^{k*}_q = (-1)^q t^k_{-q} \tag{3}$$

and constitute a total number n^2-1 of independent real parameters where $n = 2j+1$. If the system is oriented the number of independent parameters is reduced to $n+1$ given by $n-1$ independent Fano statistical tensors and two angles specifying the axis of orientation (Ramachandran and Mallesh 1984). For composite systems of spins j_1 and j_2 the density matrix ρ is the direct product of the density matrices $\rho(j_1)$ and $\rho(j_2)$ so long as the systems are non-interacting. To discuss the time evolution when the spins are interacting it is convenient to express the density matrix in the form

$$\rho = \sum_{k_1=0}^{2j_1} \sum_{k_2=0}^{2j_2} \sum_{k=|k_1-k_2|}^{k_1+k_2} \sum_{q=-k}^{k} f^k_q(k_1k_2;t)(\tau^{k_1}(1)\otimes\tau^{k_2}(2))^{k^+}_q, \tag{4}$$

where the parameters $f^k_q(k_1k_2;t)$ denote the average expectation values

$$f^k_q(k_1k_2;t) = \mathrm{Tr}\,\{\rho(\tau^{k_1}(1)\otimes\tau^{k_2}(2))^k_q\}. \tag{5}$$

It may be noted that while the direct product of $\rho(j_1)$ and $\rho(j_2)$ is completely specified by $n_1^2+n_2^2$ independent real parameters, the density matrix given by (4) needs $n_1^2n_2^2-1$ independent real parameters apart from $\mathrm{Tr}\,\rho$. The form (4) is slightly different from the form given by equation (B1) of the appendix B of Blum (1981) in that the $f^k_q(k_1k_2;t)$ transform under rotations like spherical tensors of rank k. The tensor operators

$$(\tau^{k_1}(1)\otimes\tau^{k_2}(2))^k_q$$

have the matrix elements

$$\langle j_1j_2j'm'|(\tau^{k_1}(1)\otimes\tau^{k_2}(2))^k_q|j_1j_2jm\rangle = \frac{[k_1][k_2]}{[j_1][j_2]}C(jkj';mqm') \begin{bmatrix} j_1 & j_2 & j \\ k_1 & k_2 & k \\ j_1 & j_2 & j' \end{bmatrix}^{**}, \tag{6}$$

so that

$$\mathrm{Tr}\,\{(\tau^{k_1}(1)\otimes\tau^{k_2}(2))^k_q(\tau^{k'_1}(1)\otimes\tau^{k'_2}(2))^{k'^+}_{q'}\} = \delta_{k_1k'_1}\delta_{k_2k'_2}\delta_{kk'}\delta_{qq'}. \tag{7}$$

One can also write the density matrix ρ in the form

$$\rho = \sum_{j'=|j_1-j_2|}^{j_1+j_2} \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{k=|j'-j|}^{j'+j} \sum_{q=-k}^{k} t^k_q(j'j;t)\tau^k_q(j'j)^+$$

which is essentially the equation (4.3.4) of Blum (1981). Here the operators $\tau^k_q(j'j)$ are

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* With this normalization, t^1_0 and t^2_0, referred to in the introduction will be given by $t^1_0 = (1/\sqrt{2})(N_1 - N_{-1})$, $t^2_0 = (1/\sqrt{6})(1-2N_0)$.

**The $9-j$ symbol used here is related to the Wigner $9-j$ symbol through

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = [g][h][f][c]\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Bmatrix}$$

simply related to the earlier operators through

$$\tau^k_q(j'j) = \sum_{k_1k_2} \begin{bmatrix} j_1 & j_2 & j' \\ j_1 & j_2 & j \\ k_1 & k_2 & k \end{bmatrix} (\tau^{k_1}(1) \otimes \tau^{k_2}(2))^k_q \tag{9}$$

and satisfy

$$\mathrm{Tr}\,\{\tau^k_q(j'j)\tau^{k'^\dagger}_{q'}(\lambda'\lambda)\} = \delta_{j'\lambda'}\delta_{j\lambda}\delta_{kk'}\delta_{qq'}. \tag{10}$$

From (9) it follows that the parameters in (4) and (8) are related through

$$f^k_q(k_1k_2;t) = \sum_{j'j} \begin{bmatrix} j_1 & j_2 & j' \\ j_1 & j_2 & j \\ k_1 & k_2 & k \end{bmatrix} t^k_q(j'j;t). \tag{11}$$

To proceed further we evaluate the commutation rules satisfied by the operators occurring in (4) and (8). We have

$$[(\tau^{k_1}(1)\otimes\tau^{k_2}(2))^k_q,\ (\tau^{k'_1}(1)\otimes\tau^{k'_2}(2))^{k'}_{q'}] = \sum_{\lambda_1\lambda_2\Lambda} A^x_{\lambda_1\lambda_2\Lambda}(\tau^{\lambda_1}(1)\otimes\tau^{\lambda_2}(2))^\Lambda_\mu, \tag{12}$$

where

$$A^x_{\lambda_1\lambda_2\Lambda} = [k_1][k_2][k'_1][k'_2]C(kk'\Lambda;qq'\mu)W(j_1k_1j_1k'_1;j_1\lambda_1)$$
$$W(j_2k_2j_2k'_2;j_2\lambda_2)\{(-1)^{k_1+k'_1-\lambda_1}(-1)^{k_2+k'_2-\lambda_2}-1\} \begin{bmatrix} k_1 & k_2 & k \\ k'_1 & k'_2 & k' \\ \lambda_1 & \lambda_2 & \Lambda \end{bmatrix} \tag{13}$$

with $x = \{k_1k_2kq,\ k'_1k'_2k'q'\}$, and

$$[\tau^k_q(j'j),\ \tau^{k'}_{q'}(\lambda'\lambda)] = \sum_{\alpha'\alpha\Lambda} B^y_{\alpha'\alpha\Lambda}\tau^\Lambda_\mu(\alpha'\alpha), \tag{14}$$

where

$$B^y_{\alpha'\alpha\Lambda} = [k][k']C(kk'\Lambda;qq'\mu)\{(-1)^{k+k'-\Lambda}$$
$$W(kj'k'\lambda;j\Lambda)\delta_{j\lambda'}\delta_{\alpha\lambda}\delta_{\alpha'j'} - W(kjk'\lambda';\lambda\Lambda)\delta_{j'\lambda}\delta_{\alpha j}\delta_{\alpha'\lambda'}\}, \tag{15}$$

with $y = \{j'jkq,\ \lambda'\lambda k'q'\}$,

which define the Lie algebra associated with these operators. Moreover these operators enable us to write down the most general form of the spin-spin interaction hamiltonian $\mathscr{H}$ for two interacting spin systems with spins j_1 and j_2. We have

$$\mathscr{H} = \sum_{k_1k_2kq} \varepsilon^k_q(k_1k_2)(\tau^{k_1}(1)\otimes\tau^{k_2}(2))^{k^+}_q, \tag{16}$$

in terms of the first set of operators or

$$\mathscr{H} = \sum_{j'jkq} h^k_q(j'j)\tau^k_q(j'j)^+ \tag{17}$$

in terms of the second set of operators. It is clear that the spin-spin energy parameters $\varepsilon^k_q(k_1k_2)$ and $h^k_q(j'j)$ are related to each other through

$$h^k_q(j'j) = \sum_{k_1k_2} \begin{bmatrix} j_1 & j_2 & j' \\ j_1 & j_2 & j \\ k_1 & k_2 & k \end{bmatrix} \varepsilon^k_q(k_1k_2). \tag{18}$$

The basic problem considered here is to discuss the evolution in time of the parameters $f^k_q(k_1k_2;t)$ or equivalently $t^k_q(j'j;t)$ by using the appropriate form of $\mathscr{H}$ namely (16) and (17) respectively. Observing that the time evolution of ρ is given by the Neumann-Liouville equation

$$i\hbar\, \mathrm{d}\rho/\mathrm{d}t = [\mathscr{H}, \rho] \tag{19}$$

we are now in a position to deduce the equations of motion for the parameters $f^k_q(k_1k_2;t)$ and $t^k_q(j'j;t)$ by using the commutation relationships derived above. They can be written in the form

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t} f^k_q(k_1k_2;t) = -\sum_{\lambda_1\lambda_2\Lambda\mu}\ \sum_{k_1'k_2'k'q'} A^x_{k_1k_2k}\,\varepsilon^\Lambda_\mu(\lambda_1\lambda_2) f^{k'}_{q'}(k_1'k_2';t), \tag{20}$$

and

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t} t^k_q(j'j;t) = -\sum_{\lambda'\lambda\Lambda\mu}\ \sum_{\alpha'\alpha k'q'} B^y_{j'jk}\, h^\Lambda_\mu(\lambda'\lambda)\, t^{k'}_{q'}(\alpha'\alpha). \tag{21}$$

Obviously a solution of the above equation in all its generality is forbidding. We would therefore consider a simple form of the spin-spin interaction and show that it leads to a solvable problem.

3. Temporal evolution of the parameters

We consider in this section a specific kind of spin hamiltonian which is sufficiently general to embrace the usual spin-spin interaction familiar from atomic physics as a particular case. For this interaction we evaluate the polarization parameters of the final system. The hamiltonian considered is of the form

$$\mathscr{H} = \sum_\alpha a_\alpha \tau^\alpha(1)\cdot\tau^\alpha(2), \quad \alpha = 1\ldots x \quad \text{where } x = \begin{cases} 2j_1, & j_1 < j_2 \\ 2j_2, & j_2 < j_1. \end{cases} \tag{22}$$

It represents a sum of various multipole scalar interactions that may exist between the two spin systems. $\alpha = 1$ corresponds to the usual hyperfine interaction between the two spin systems. We wish to discuss this interaction in some detail in the hope that it is capable of describing the physics in the context of heavy-ion collisions as well. Further the results obtained with the above formalism are quite general so that they can be applied to other areas such as muon repolarization, correlations in multilevel systems etc., where similar interactions as above are considered.

The hamiltonian in (22) when expressed in terms of the $\tau^k_q(j'j)$ takes the form

$$\mathscr{H} = \sum_j [j]\,\omega_j \tau^0_0(jj), \tag{23}$$

where

$$\omega_j = \sum_\alpha (-1)^\alpha \frac{[\alpha]}{[j]} a_\alpha \begin{bmatrix} j_1 & j_2 & j \\ j_1 & j_2 & j \\ \alpha & \alpha & 0 \end{bmatrix}, \tag{24}$$

which implies that it is diagonal in the total angular momentum basis. We therefore have

$$\rho_{jm,j'm'}(t) = \rho_{jm,j'm'}(0)\exp\left\{\frac{it}{\hbar}(\omega_{j'} - \omega_j)\right\}, \tag{25}$$

from which we obtain

$$t_q^k(j'j;t) = t_q^k(j'j;0)\exp\left\{\frac{it}{\hbar}(\omega_{j'}-\omega_j)\right\}, \tag{26}$$

and using now equation (11) and its inverse relation we obtain

$$f_q^k(k_1k_2;t) = \sum_{j'jk_1'k_2'} \begin{bmatrix} j_1 & j_2 & j' \\ j_1 & j_2 & j \\ k_1 & k_2 & k \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j' \\ j_1 & j_2 & j \\ k_1' & k_2' & k \end{bmatrix} \begin{matrix} (t^{k_1'}(1)\otimes t^{k_2'}(2))_q^k \\ \exp\{(it/\hbar)(\omega_{j'}-\omega_j)\}, \end{matrix} \tag{27}$$

where we have replaced $f_q^k(k_1'k_2';0)$ by $(t^{k_1'}(1)\otimes t^{k_2'}(2))_q^k$ corresponding to

$$\rho(0) = \rho(j_1)\times\rho(j_2) = \sum_{k_1k_2kq} (t^{k_1}(1)\otimes t^{k_2}(2))_q^k(\tau^{k_1}(1)\otimes\tau^{k_2}(2))_q^{k^+}. \tag{28}$$

Equations (26) and (27) give the most general solutions for the time evolution of the two sets of parameters. It is interesting to note that the rank k and the projection q are preserved in the time evolution which follows from the fact that the hamiltonian contains only scalars in terms of the spin operators. Apart from being quite general, (26) and (27) contain interesting features which are important from the experimental point of view. In fact one can prepare the initial system in an appropriate way to get the final system with a desired kind of polarization. Further the number of independent parameters needed to describe a coupled system of spins j_1 and j_2 becomes very large for high spin values. Examination of (26) and (27) suggests that if the initial parameters

$$(t^{k_1'}(1)\otimes t^{k_2'}(2))_q^k$$

are so chosen that all those with $q\neq 0$ are zero then all the final parameters $f_q^k(k_1k_2)$ with $q\neq 0$ also vanish which leads to a considerable reduction in the number of parameters. Such a situation can be realised by taking two spin systems which are initially oriented with respect to a common axis of quantization. It is easy to see that in such a case for $j_1=j_2=\frac{1}{2}$, out of the 15 parameters 10 will vanish while for $j_1=j_2=1$, out of the 80 parameters, 62 of them will vanish. Fano (1983) recently discussed the particular case of two spin $\frac{1}{2}$ particles interacting via the hyperfine coupling where he highlights the reduction of parameters by symmetry considerations employing however the cartesian operators. In this context the above formalism is a generalisation of his approach in terms of spherical tensor operators since it can embrace not only arbitrary spins j_1 and j_2 (or multilevel systems) but also more general interactions as given by (22).

Still further reduction in the number of parameters occurs when one is interested only in the time-averaged parameters. Such parameters are considered in certain cases such as polarization transfer between muon and nucleus in muonic atoms due to hyperfine interactions (Mukhopadhyay and Hintermann 1979). On averaging (26) and (27) over a time much larger compared to the precessional period we get the time-averaged parameters

$$T_q^k(j'j) = \delta_{j'j}t_q^k(jj;0), \tag{29}$$

and

$$F_q^k(k_1k_2) = \sum_{k_1'k_2'j} \begin{bmatrix} j_1 & j_2 & j \\ j_1 & j_2 & j \\ k_1 & k_2 & k \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j \\ j_1 & j_2 & j \\ k_1' & k_2' & k \end{bmatrix} (t^{k_1'}(1)\otimes t^{k_2'}(2))_q^k, \tag{30}$$

provided $\omega_{j'}\neq\omega_j$ if $j'\neq j$. The fortuitous circumstance when $\omega_{j'}=\omega_j$ even if $j'\neq j$

implies that a_α should satisfy some conditions. We are not discussing such a special case here. It is obvious that all those parameters in (29) and (30) with $j' \neq j$ and k_1+k_2+k odd respectively vanish and further $F^0_0(\lambda\lambda)$ remains unchanged while $F^\lambda_q(\lambda 0)$ and $F^\lambda_q(0\lambda)$ do change in the interaction.

The correlations between two spin $\frac{1}{2}$ systems is usually expressed in terms of the expectation values $\Sigma_{xy} = \langle \sigma_x \sigma_y \rangle - \langle \sigma_x \rangle \langle \sigma_y \rangle$ as can be found in Fano (1983). In our formalism such correlations for multilevel systems can be generalised to

$$D^{k_1k_2}_{q_1q_2} = \langle t^{k_1}_{q_1}(1) t^{k_2}_{q_2}(2) \rangle - \langle t^{k_1}_{q_1}(1) \rangle \langle t^{k_2}_{q_2}(2) \rangle \tag{31}$$

$$= \sum_k C(k_1k_2k;\, q_1q_2q)\, C^k_q(k_1k_2),$$

where

$$C^k_q(k_1k_2) = f^k_q(k_1k_2) - (f^{k_1}(k_10) \otimes f^{k_2}(0k_2))^k_q, \tag{32}$$

which of course reduce to spherical tensor analogues of Σ_{xy} for $j_1 = j_2 = \frac{1}{2}$.

4. Special cases

We consider in this section some of the special cases of interest and discuss in some detail about the polarization parameters of the two systems.

4.1 $j_1 = j_2 = \frac{1}{2}$

The parameter α defined in (22) takes the value 1 only for this case. The various parameters are given by

$$f^1_q(10; t) = \tfrac{1}{2}\{[10, 1q] - [01, 1q]\} \cos(2a_1t) + \frac{i}{\sqrt{2}}[11, 1q] \sin(2a_1t) + \tfrac{1}{2}\{[10, 1q] + [01, 1q]\}, \tag{33}$$

$$f^1_q(01; t) = -\tfrac{1}{2}\{[10, 1q] - [01, 1q]\} \cos(2a_1t) - \frac{i}{\sqrt{2}}[11, 1q] \sin(2a_1t) + \tfrac{1}{2}[10, 1q] + [01, 1q]\}, \tag{34}$$

$$f^0_0(11; t) = [11, 00], \tag{35}$$

$$f^1_q(11; t) = \frac{i}{\sqrt{2}}\{[10, 1q] - [01, 1q]\} \sin(2a_1t) + [11, 1q] \cos(2a_1t), \tag{36}$$

$$f^2_q(11; t) = [11, 2q], \tag{37}$$

where we have replaced

$$(t^{k_1}(1) \otimes t^{k_2}(2))^k_q$$

by the symbol $[k_1k_2, kq]$. The vector parameters oscillate with a frequency $2a_1$ while the scalar and alignment parameters remain constant. When (33) to (37) are averaged over time we get

$$F^1_q(10) = \tfrac{1}{2}\{[10, 1q] + [01, 1q]\} = F^1_q(01), \tag{38}$$

$$F_0^0(11) = [11, 00], \tag{39}$$

$$F_q^1(11) = 0, \tag{40}$$

$$F_q^2(11) = [11, 2q]. \tag{41}$$

It is interesting to note that the initial vector polarizations of the two systems are equally shared by the two systems after the interaction which is of course a general result for two equal spin systems. This is easily proved by utilising the following identity satisfied by the $9-j$ symbol

$$\sum_k \begin{bmatrix} j & j & k \\ j & j & k \\ 1 & 0 & 1 \end{bmatrix}^2 = \tfrac{1}{2}. \tag{42}$$

The constancy of the sum of the two vector polarizations of course follows from the fact that the hamiltonian commutes with the operator $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$.

If the two spin systems are prepared initially such that only $q = 0$ terms are nonvanishing we see that there remain only 4 parameters for the final system which are nonvanishing.

4.2 $j_1 = \frac{1}{2}, j_2$ *arbitrary*

Here again α takes the value 1 only. For this case instead of evaluating the parameters using our general result (27) we outline an alternative procedure by making use of (20) in order to bring out the similarity with the procedure given by Mukhopadhyay and Hintermann (1979) who however use the cartesian tensors. The hamiltonian takes the form

$$\mathscr{H} = -(3a_1)^{1/2}(\tau^1(1) \otimes \tau^1(2))_0^0, \tag{43}$$

which implies that $\varepsilon_0^0(11) = -(3a_1)^{1/2}$ in (16) is the only nonvanishing parameter. Substituting this in (20) we get

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} f_q^k(k_1 k_2; t) = -\sum_{k'_1 k'_2} f_q^k(k'_1 k'_2; t)\, \varepsilon_0^0(11) A_{k_1 k_2 k}^{x}, \tag{44}$$

from which we obtain the coupled equations

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} f_q^\lambda(0\lambda; t) = -\alpha f_q^\lambda(1\lambda; t), \tag{45}$$

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} f_q^\lambda(1\lambda; t) = -\alpha f_q^\lambda(0\lambda; t) + \beta f_q^\lambda(1\lambda+1; t) + \gamma f_q^\lambda(1\lambda-1; t) \tag{46}$$

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} f_q^\lambda(1\lambda+1; t) = \beta f_q^\lambda(1\lambda; t), \tag{47}$$

and

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} f_q^\lambda(1\lambda-1; t) = \gamma f_q^\lambda(1\lambda; t), \tag{48}$$

where

$$\alpha = (\lambda(\lambda+1)/p)^{1/2} \varepsilon_0^0(11),$$

$$\beta = ((2j_2+\lambda+2)(2j_2-\lambda)\lambda/p(2\lambda+1))^{1/2}\varepsilon_0^0(11),$$

$$\gamma = ((2j_2+\lambda+1)(2j_2-\lambda+1)(\lambda+1)/p(2\lambda+1))^{1/2}\varepsilon_0^0(11)$$

and

$$p = 2j_2(j_2+1)(2j_2+1).$$

These equations can be easily solved subjected to the initial condition given by (28) to get the following solutions

$$f_q^\lambda(1\lambda; t) = [1\lambda, \lambda q]\cos(\nu t) - \frac{i}{\hbar\nu}\{-\alpha[0\lambda, \lambda q] + \beta[1\lambda+1, \lambda q] + \gamma[1\lambda-1, \lambda q]\}\sin(\nu t), \tag{49}$$

$$f_q^\lambda(0\lambda; t) = \frac{i\alpha}{\hbar\nu}([1\lambda, \lambda q]\sin\nu t) + \frac{\alpha}{\hbar^2\nu^2}\{-\alpha[0\lambda, \lambda q] + \beta[1\lambda+1, \lambda q] + \gamma[1\lambda-1, \lambda q]\}(1-\cos\nu t) + [0\lambda, \lambda q], \tag{50}$$

$$f_q^\lambda(1\lambda+1; t) = -\frac{i\beta}{\hbar\nu}[1\lambda, \lambda q]\sin\nu t - \frac{\beta}{\hbar^2\nu^2}\{-\alpha[0\lambda, \lambda q] + \beta[1\lambda+1, \lambda q] + \gamma[1\lambda-1, \lambda q]\}(1-\cos\nu t) + [1\lambda+1, \lambda q], \tag{51}$$

$$f_q^\lambda(1\lambda-1; t) = -\frac{i\gamma}{\hbar\nu}[1\lambda, \lambda q]\sin\nu t - \frac{\gamma}{\hbar^2\nu^2}\{-\alpha[0\lambda, \lambda q] + \beta[1\lambda+1, \lambda q] + \gamma[1\lambda-1, \lambda q]\}(1-\cos\nu t) + [1\lambda-1, \lambda q] \tag{52}$$

where the angular frequency is given by

$$\nu = \left[\frac{(2j_2+1)}{2j_2(j_2+1)}\right]^{1/2}\frac{\varepsilon_0^0(11)}{\hbar} = -\left[\frac{3(2j_2+1)}{2j_2(j_2+1)}\right]^{1/2}\frac{a_1}{\hbar}. \tag{53}$$

The various time-averaged parameters for this case are given by

$$F_q^\lambda(1\lambda) = 0, \tag{54}$$

$$F_q^\lambda(0\lambda) = \left\{1 - \frac{\alpha^2}{\hbar^2\nu^2}\right\}[0\lambda, \lambda q] + \frac{\alpha\beta}{\hbar^2\nu^2}[1\lambda+1, \lambda q] + \frac{\alpha\gamma}{\hbar^2\nu^2}[1\lambda-1, \lambda q], \tag{55}$$

$$F_q^\lambda(1\lambda+1) = \frac{\alpha\beta}{\hbar^2\nu^2}[0\lambda, \lambda q] + \left\{1 - \frac{\beta^2}{\hbar^2\nu^2}\right\}[1\lambda+1, \lambda q] - \frac{\beta\gamma}{\hbar^2\nu^2}[1\lambda-1, \lambda q], \tag{56}$$

$$F_q^\lambda(1\lambda-1) = \frac{\alpha\gamma}{\hbar^2\nu^2}[0\lambda, \lambda q] - \frac{\beta\gamma}{\hbar^2\nu^2}[1\lambda+1, \lambda q] + \left\{1 - \frac{\gamma^2}{\hbar^2\nu^2}\right\}[1\lambda-1, \lambda q]. \tag{57}$$

The instantaneous and the time-averaged parameters of the first system alone are given respectively by (52) and (57) on substituting $\lambda = 1$ while the corresponding parameters of the second system are given by (50) and (55). All others represent the various correlations which exist between the two systems.

In the context of muon and nucleus hyperfine interactions Mukhopadhyay and coworkers (Mukhopadhyay and Hintermann 1979; Hambro and Mukhopadhyay 1975, 1977) have used cartesian operators to obtain the final muon polarization. We wish to mention here that our equation (57) with $\lambda = 1$ gives the vector polarization of the muon while (55) gives the various multipole-parameters of the nucleus corresponding to $\lambda = 1, \ldots, 2j_2$. In their method, Mukhopadhyay and Hintermann represent that part of the density matrix containing tensors of rank greater than 2 as $\rho'(t)$ and set $\rho'(t=0) = 0$. This ρ' does not contribute anything to the final vector polarization as indicated by them. However it has to be taken into consideration for evaluating the higher multipole parameters of the nucleus which is completely carried out here without making any assumptions.

We also give the expressions for the populations of the hyperfine states $|F_{\pm} m\rangle$ in the context of muon absorption which take the form

$$N_{\pm}(m) = \sum_{k_1 k_2 k} \begin{bmatrix} j_1 & j_2 & F_{\pm} \\ j_1 & j_2 & F_{\pm} \\ k_1 & k_2 & k \end{bmatrix} C(F_{\pm} k F_{\pm};\, m 0 m) \frac{[k]}{[F_{\pm}]} [k_1 k_2, k0]. \tag{58}$$

Expressions for $N_{\pm}(m)$ ignoring higher rank (rank greater than 1) initial parameters are given by Hambro and Mukhopadhyay (1975) which are readily seen to follow from the more general expression given above.

4.3 $j_1 = j_2 = 1$

In this case both dipole and quadrupole scalar interactions can occur and consequently α takes the values 1 and 2. We give below only the time-averaged parameters for this case. They are

$$F_q^1(10) = \tfrac{1}{2}\{[10, 1q] + [01, 1q]\} = F_q^1(01), \tag{59}$$

$$F_q^2(20) = \frac{1}{3\sqrt{3}}[11, 2q] + \frac{5}{18}\{[20, 2q] + [02, 2q]\} - \frac{\sqrt{7}}{9}[22, 2q]$$
$$= F_q^2(02). \tag{60}$$

It is interesting to note that the vector polarization parameters of the final systems do not depend on the initial second rank parameters of the individual systems even though such parameters can couple to the vector parameters to give again vector parameters i.e., to say that $F_q^1(10)$ and $F_q^1(01)$ do not depend on the parameters $[21, 1q]$ and $[12, 1q]$. This essentially means that the initial alignment of either system has nothing to do with the final vector polarizations of the two systems. On the other hand the final alignment parameters do depend on the initial vector parameters which is evident from (60). Further as is clear from (59) and (60) the corresponding final polarization parameters are equal.

We now consider few ways of obtaining systems with specific types of polarization which can be achieved by appropriately choosing the initial systems.

Suppose the system (2) is unpolarized while the system (1) is say purely vector-polarized initially. From (59) and (60) it is clear that after the interaction both the systems are purely vector-polarized. The system (1) gets depolarized by 50% while the system (2) gets polarized by the same amount. If on the other hand the system (1) is purely tensor-polarized and is interacting with an unpolarized system (2) then both will possess only tensor polarizations after interaction. When both systems are purely vector-polarized initially then they will exhibit both vector and tensor polarizations after the interaction.

The above discussion is not exhaustive but is meant to be illustrative of the general utility of the method outlined in this paper. We have also demonstrated that the method yields as special cases some of the well known results in the literature for which more generalised and explicit forms are exhibited here.

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