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An effective potential for heavy quark antiquark bound system

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Abstract. A heavy quark antiquark potential is suggested connecting asymptotic freedom and quark confinement in a unified way. The $\alpha_s(q^2)$ calculated using Borel summation technique with three loop agrees with the two loop β-function up to $g^2/4\pi \simeq 1{\cdot}1$ but changes appreciably after $g^2/4\pi = 1{\cdot}5$. The potential so derived satisfactorily explains the $c\bar{c}$ and $b\bar{b}$ spectrum.

Keywords. Asymptotic freedom; confinement; Borel summation; Landau ghost; vacuum polarisation.

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1. Introduction

Over the past decade a successful description of ψ and γ families has been achieved and many predictions of charmonium as well as b-flavoured mesons have been confirmed experimentally. Theoretical efforts have been concentrated on the exploration of specific potential models as well as the application of rigorous methods derived from nonrelativistic quantum mechanics. At large and short distances however, a variety of asymptotic behaviours have been suggested all of which seem to be compatible with present experimental data. Theories based on strong and weak coupling expansions in quantum chromodynamics (QCD) tend to predict that the static $(q\bar{q})$ potential is coulombic at short distances. The potential obtained by a simple superposition of both asymptotic limits has been studied extensively by the Cornell group (Eichen *et al* 1978, 1980). There is as yet no clear understanding of the confinement mechanism in QCD. Recently numerical calculations in lattice gauge theory, especially by Creutz (Creutz 1980; Kogat *et al* 1979) have given a rather clear indication that there is perhaps a rapid crossover from the perturbative regime at short distance to the simple string-like situation at large distance.

A phenomenological potential which also connects the two regimes smoothly has been proposed by Richardson (1979). This potential provides an excellent description of both $c\bar{c}$ and $b\bar{b}$ spectra. However, such a potential does not indicate the typical behaviour of a crossover region found by lattice-gauge theory calculations. In this paper we have attempted to connect the two regimes. The potential so obtained has been used to study the spectrum of $c\bar{c}$ and $b\bar{b}$ systems.

2. The potential

Let us write the total potential in momentum space as

$$V(q) = V_g(q) + V_c(q), \tag{1}$$

where the QCD perturbative gauge potential $V_g(q)$ is dominant for values greater than a few $(\mathrm{GeV})^2$ say q_0^2 whereas the confining potential $V_c(q)$ dominates for $q^2 \ll q_0^2$.

For $q^2 \gg q_0^2$ the QCD result is well known i.e. the running coupling constant

$$\alpha_g(q^2) = \left[\frac{\beta_0}{4\pi}\log(q^2/\Lambda^2) + \eta f(q^2)\right]^{-1} \tag{2}$$

for first and second order perturbation expansion where

$$f(q^2) = \log\left[1 + \frac{\beta_0^2}{\beta_1}\log(q^2/\Lambda^2) + f(q^2)\right] \tag{3}$$

$$\beta_0 = 11 - \frac{2}{3}N_f$$

$$\eta = \frac{1}{4\pi\beta_0}, \quad \beta_1 = 102 - \frac{38}{3}N_f \quad \text{and} \quad N_f \tag{4}$$

being the number of flavours.

The determination of running coupling constant $\alpha_g(q^2)$ with first order (one gluon exchange) or second order terms is inadequate to explain the structure function data even in the region of 0·1 to 1 GeV^2. Theoretically there is the undesirable Landau ghost which prevents any sensible interpolation for $q^2 < \Lambda^2$. This has been shown in curve A and curve B of figures 1 and 2 plotted for two different Λ values. So a more convergent $\alpha_g(q^2)$ is needed to compare the theory with experiments. As a possible solution to the problem Sanda (1979) used the Borel-like summation technique proposed by Khuri (1979) in computing the second order β-function. Although calculation of $\alpha_g(q^2)$ by use of Borel summation with two loop β-function is free from Landau ghost and is consistent with experiment in both space-like and time-like regions, it cannot be trusted for $\alpha_g(q^2) \geq 1·5$. This is due to the fact that in the second order β-function b_4 is not known, and I_3 has a zero value at $g^2/4\pi = 1·1$ and remains negligible (compared to I_1 and I_2) for $g^2/4\pi \leq 1·5$. So for accurate determination of $\alpha_g(q^2)$ beyond $\alpha_g(q^2) \geq 1·5$ a higher order β-function is necessary.

In a recent paper Tarasov and coworkers (Tarasov *et al* 1980) have calculated the β-function in the three-loop approximation. The β-function reads as

$$g\beta(g) = -\frac{\beta_0 g^4}{16\pi^2} - \frac{\beta_1 g^6}{(16\pi^2)^2} - \frac{\beta_2 g^8}{(16\pi^2)^3}, \tag{5}$$

when $$\beta_2 = \left(\frac{2857}{2} - \frac{5033}{18}N_f + \frac{325}{54}N_f^2\right). \tag{6}$$

Using this three loop β-function we calculate $\alpha_g(q^2)$ with the Borel summation technique. Making full use of known and assumed analytic properties of the Borel

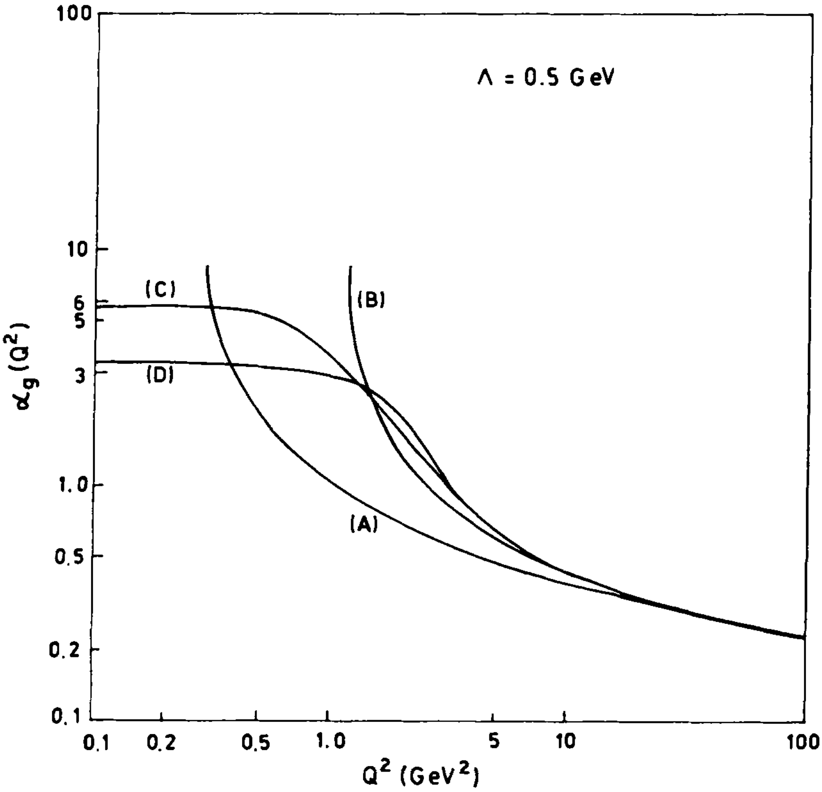

Figure 1. The running coupling constants based on one leading order term (curve A) and two leading order term (curve B) in the perturbation expansion of β-function. Curve C is obtained from Borel summation technique with two loop β-function. Curve D is with three loop β-function by Borel summation technique. All four curves are normalised so that α_g ($100\,\text{GeV}^2$) = 0·233 corresponding to $\Lambda = 0{\cdot}5$ GeV for curve A.

function $B(z)$ we define

$$g\beta(g) = \sum_{n=2}^{\infty} \frac{b_n}{(n-1)!} I_{n-1} \tag{7}$$

where

$$I_n = \int_0^{\infty} \text{Rel}\,(\omega^n) \exp(-z/g^2)\,\text{d}z, \tag{8}$$

with $\quad w = [a + i(z^2 - a^2)^{1/2}]/z$

and $\quad a = 32\pi^2/\beta_0$

Keeping the terms up to order g^8, I_1 becomes

$$I_1 = \frac{g^4}{2a} + \frac{3}{4a^3} g^8 \tag{9}$$

similarly

$$I_2 = \frac{g^6}{2a^2}, \quad I_3 = \frac{3g^8}{4a^3}. \tag{10}$$

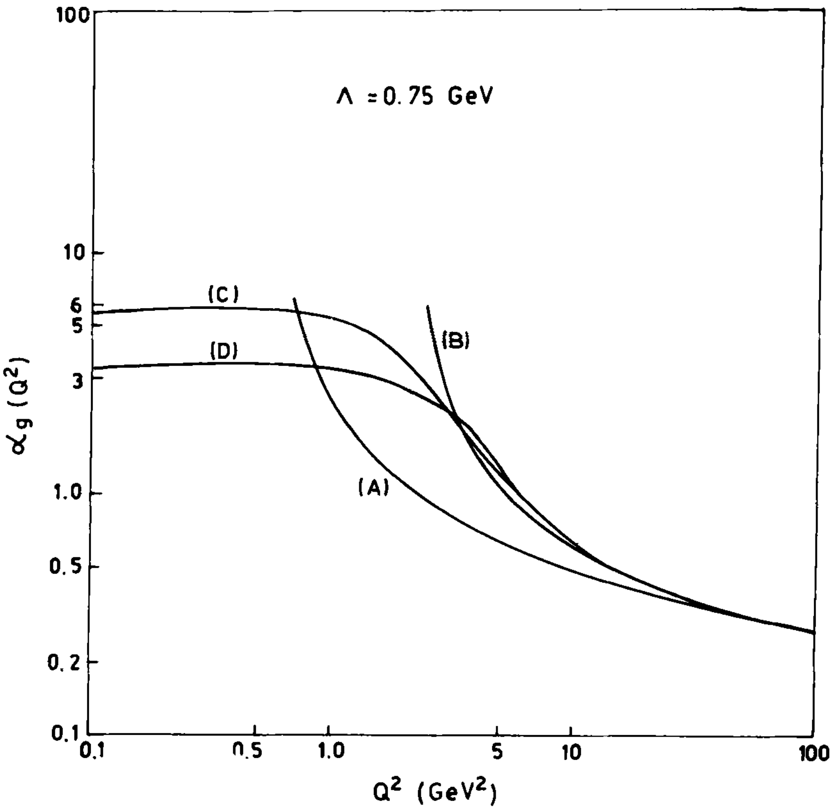

Figure 2. The running coupling constants based on one leading order term (curve A) and two leading order term (curve B) in the perturbation expansion of β-function. Curve C is obtained from Borel summation with two loop β-function curve D is with three loop β-function by Borel summation technique. All four curves are normalized so that α_g ($100\,\mathrm{GeV}^2$) = 0·27 corresponding to $\Lambda = 0{\cdot}75\,\mathrm{GeV}$.

With these values of I_1, I_2 and I_3, $g\beta(g)$ can be written as

$$g\beta(g) = \frac{b_2 g^4}{2a} + \frac{b_3 g^6}{4a^2} + \left(3b_2 + \frac{b_4}{2}\right)\frac{g^8}{4a^3} \tag{11}$$

comparing (5) with (11) b_2, b_3, b_4 are determined as follows:

$$b_2 = -4; \quad b_3 = -32a/\pi^2; \quad b_4 = \left(24 - \frac{15{\cdot}8971a}{\pi^2}\right). \tag{12}$$

Knowing the values of b_2, b_3, b_4, $g\beta(g)$ can be written as

$$g\beta(g) = b_2\left[\int_0^a \left(a - (a^2 - t^2 g^4)^{1/2}\right)\frac{\mathrm{d}t\, e^{-t}}{t} + a\int_a^\infty \frac{e^{-t}}{t}\mathrm{d}t\right]$$

$$+\frac{b_3}{2g^2}\left[\int_0^a \left(a - (a^2 - t^2 g^4)^{1/2}\right)^2 \frac{e^{-t}}{t^2}\mathrm{d}t + \int_a^\infty \frac{(2a^2 - g^4 t^2)e^{-t}\,\mathrm{d}t}{t^2}\right]$$

$$+\frac{b_4}{6g^4}\left[\int_0^a \left(a-(a^2-t^2g^4)^{1/2}\right)^3 \frac{e^{-t}\,dt}{t^3}+\int_a^\infty \frac{(4a^3-3ag^4t^2)e^{-t}\,dt}{t^3}\right] \tag{13}$$

Once $g\beta(g)$ is known, we calculate the value of $\alpha_g(q^2)$ for low q^2 region by using the expression

$$\int_{\bar{g}_0^2}^{\bar{g}^2} \frac{d\bar{g}^2}{g\beta(g)} = \log(q^2/q_0^2). \tag{14}$$

The value of $\alpha_g(q^2)$ comes up to a constant value 3·3358 instead of 5·778 (Deo and Barik 1983) as with second order β-function at low value of q^2 with $\Lambda = 0{\cdot}398$ GeV. The result obtained also agrees with second order calculations when $\alpha_g(q^2) \leqslant 1{\cdot}1$ (figures 1 and 2). It should also be noted that $\alpha_g(q^2)$ no longer diverges like (2), also the undesirable Landau ghost is absent in this formalism.

Then we follow the same procedure as has been done in our earlier paper (Deo and Barik 1983) to find the quark antiquark potential in configuration space to be of the same form but with a different $\alpha(q^2)$. This can be written as

$$V(r) = Kr - \frac{8}{3\pi}\int_0^\infty \left(\alpha(q^2) - \frac{3}{2}\frac{K}{q^2}\right)\frac{\sin qr}{qr}\,dq \tag{15}$$

with $\quad K = 20\pi M_q^2 \alpha_g(0)/BN_f$.

The graph of $\alpha(q^2)$ versus q^2 is shown in figure 3, which is compared with Richardson's $\alpha(q^2)$. The graph of $\alpha(q^2)$ versus q^2 shows a plateau connecting confinement to asymptotic region whereas Richardson's curve (I) joins smoothly between the two regions. This is the main important and interesting feature of our derived potential. This is unlike Richardson's potential which is purely a phenomenological potential. The potential so derived has only one adjustable parameter B. One notes that B can be self-consistently generated by taking a prescribed mass scale for the low q^2 region. But for simplification we have taken $K = 0{\cdot}14745$ (Richardson's value) and hence find the coupling strength

$$B = \alpha_c(m_q^2) = \left(\frac{30\pi\alpha_g(0)}{N_f}\right)^{1/2}(M_q/m_q). \tag{16}$$

The graph $V(r)$ versus r is plotted with different Λ value which is shown in figure 4.

3. Comparison with experiments

For the phenomenological study of the quark-antiquark bound systems in a flavour-independent non-relativistic potential model like equation (15) we have solved numerically the radial Schrödinger equation ($\hbar = c = 1$).

$$\frac{d^2R_l(r)}{dr^2} + \frac{2}{r}\frac{dR_l(r)}{dr} + 2\mu\left[E - V_{q_1\bar{q}_2}(r) - \frac{l(l+1)}{2r^2}\right]R_l(r) = 0. \tag{17}$$

The mass spectrum of heavy quark and antiquark bound state has the form

$$M_n(Q\bar{Q}) = 2m_Q + E_n(m_Q, V), \tag{18}$$

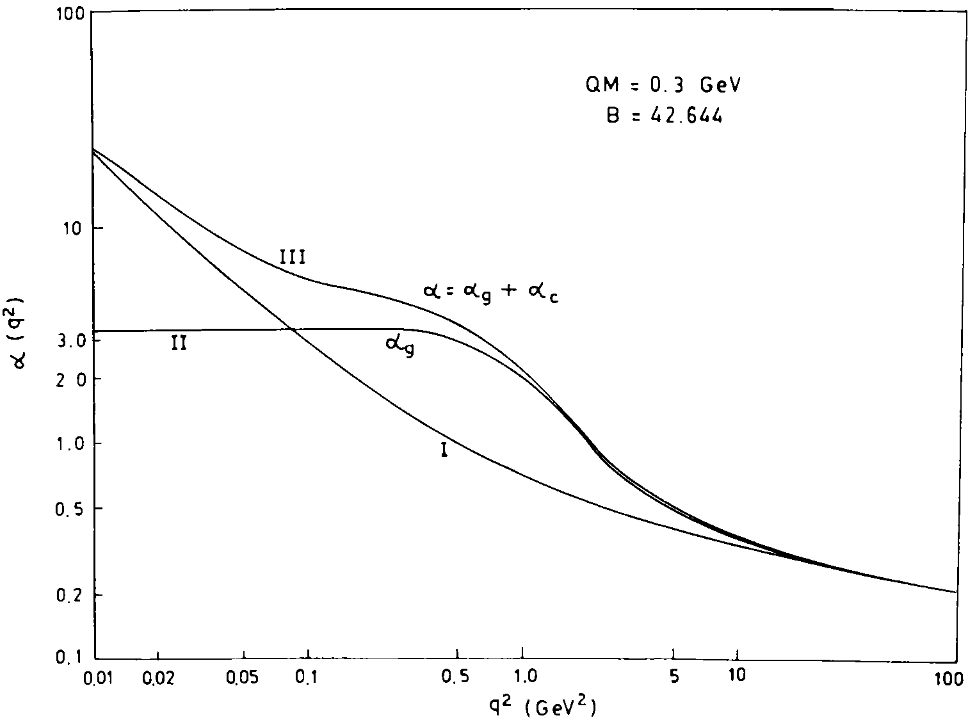

Figure 3. I: Plot of $\alpha_s(q^2)$ vs q^2 (Richardson's curve). II: Plot of $\alpha_g(q^2)$ vs q^2 (from Sanda-type analysis) with $M_q = 0{\cdot}3$ GeV and $\Lambda = 0{\cdot}398$ GeV. III: Plot of $\alpha(Q^2) = \alpha_g + \alpha_c$ vs q^2 with $\alpha = 0{\cdot}2165$ at $q^2 = 100\,\mathrm{GeV}^2$.

m_Q = quark mass, $E_n(m_Q, V)$ is the energy eigenvalue of the non-relativistic Schrödinger equation (17) with flavour-independent quark antiquark potential (15).

In our numerical calculation we fix the light quark mass $M_q = 0{\cdot}3$ and N_f = number of quark flavours = 3. Taking $K = 0{\cdot}14745$ we find the coupling strength $B = \alpha_c(m_q^2) = 42{\cdot}34$ with

$$\mathscr{L}t_{q^2 \to 0}\ \alpha_g(q^2) = 3{\cdot}3120.$$

Here we take the value of $\Lambda = 0{\cdot}16$ GeV (Barnett and Schlatter 1982). With these parameters we then fit the c-quark mass M_c and b-quark mass m_b to obtain the masses of $M(\psi) = 3{\cdot}097$ GeV and $M(\gamma) = 9{\cdot}46$ GeV. Then with the same parameters the location of some other states of $c\bar{c}$ and $b\bar{b}$ spectra is computed. The results are presented in tables 1 and 2 which are compared with experimental values.

(i) The quark masses obtained in the process of the fit are as follows

$$m_c = 1{\cdot}015\ \mathrm{GeV},$$

$$m_b = 4{\cdot}441\ \mathrm{GeV}.$$

(ii) We obtain the mass difference $\Delta M_\psi = (M'_\psi - M_\psi)$ and $\Delta M_r = (M'_r - M_r)$ for the $1S - 2S$ states. These can be compared with experimental values (Augustin *et al* 1974;

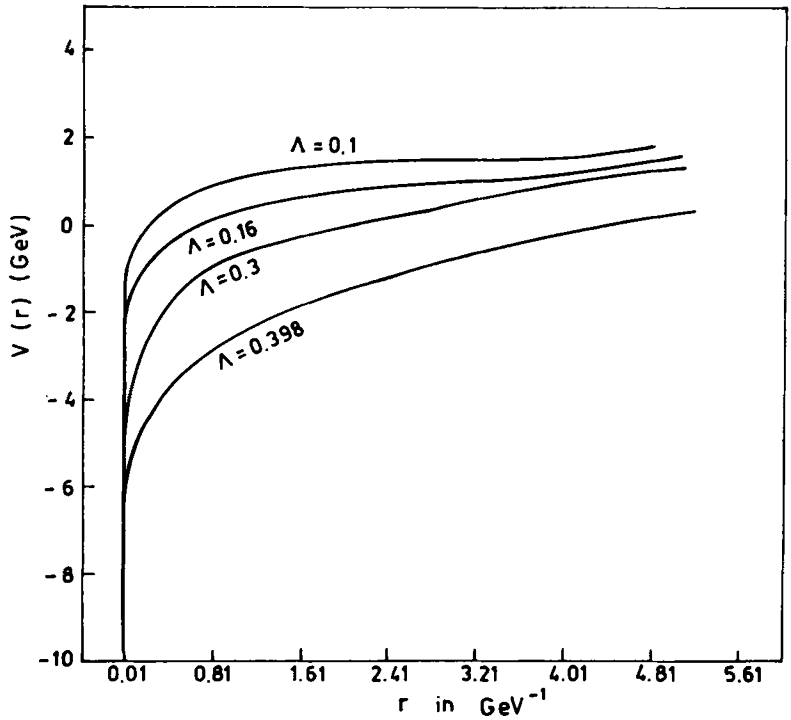

Figure 4. A graph of the potential $V(r)$ versus r with different Λ values.

Table 1. Mass (GeV) of the low lying S-states and D-states along with the mass of some P-states of $c\bar{c}$ family.

nl	Present model	Richardson's value	Experiment
1S	3·097	3·095	3·097 ± 0·001
2S	3·71	3·684	3·686 ± 0·003
3S	4·01	4·096	4·03 ± 0·01
4S	4·31	4·4498	—
5S	4·60	—	4·417 ± 0·01
1P	3·54	3·514	3·521 ± 0·003
2P	3·89	3·950	—
3P	4·20	4·308	—
1D	3·82	3·799	3·772 ± 0·006
2D	4·08	4·172	4·159 ± 0·02

Andrews *et al* 1980)

$$(\Delta M_{\psi})_{\text{cal}} = 0{\cdot}613 \text{ GeV} \qquad (\Delta M_{\psi})_{\text{exp}} = 0{\cdot}589 \text{ GeV},$$

$$(\Delta M_{r})_{\text{cal}} = 0{\cdot}52 \text{ GeV} \qquad (\Delta M_{r})_{\text{exp}} = 0{\cdot}534 \text{ GeV}.$$

Finally we extend our computation to obtain an estimate of some energy levels which

Table 2. Mass (GeV) of the low lying S-states and D-states along with the mass of some P-states of $b\bar{b}$ family.

nl	Present model	Richardson's value	Experiment
1S	9·46	9·452	$9{\cdot}46 \pm 0{\cdot}01$
2S	9·98	10·007	$9{\cdot}9944 \pm 0{\cdot}0004$
3S	10·35	10·338	$10{\cdot}35 \pm 0{\cdot}0001$
4S	10·55	10·598	$10{\cdot}546 \pm 0{\cdot}002$
5S	10·70	—	—
1P	9·84	9·888	—
2P	10·23	10·241	—
1D	10·09	10·137	—
2D	10·42	10·421	—

Table 3. Estimate of the masses (in GeV) of S-states along with the mass of some D-states and P-states of the hypothetical $t\bar{t}$ family with different t-quark masses.

nl	$m_t = 10$ GeV	$m_t = 14$ GeV	$m_t = 18$ GeV
1S	20·31	28·17	36·06
2S	20·80	28·69	36·62
3S	21·12	29·00	36·92
4S	21·39	29·25	37·12
5S	21·58	29·43	37·33
1P	20·70	28·60	36·52
2P	21·03	28·91	36·82
3P	21·29	29·14	37·04
1D	20·93	28·82	36·75
2D	21·20	29·08	36·99

are yet to be observed for heavier quarkonium like, $t\bar{t}$ family with different quark masses. These are presented in table 3.

4. Conclusion

In this work we have been successful in finding a quark-antiquark potential which connects asymptotic freedom on the one hand and confinement on the other in a unified way by a formal method of field theory unlike Richardson potential. The $\alpha_g(q^2)$ calculated by Borel summation technique using three loop β-function agrees with the two loop β-function upto a value of $g^2/4\pi \leqslant 1{\cdot}1$ but changes value appreciably after $g^2/4\pi \geqslant 1{\cdot}5$. The graph of $\alpha(q^2)$ versus q^2 shows a plateau connecting confinement to asymptotic region whereas the Richardson's curve joins smoothly between the two regions. This is the important interesting feature of our derived potential. The potential has some theoretical background unlike Richardson's potential which is purely a phenomenological one. Although there is a plateau between the two regions, it explains the spectra of $c\bar{c}$ and $b\bar{b}$ satisfactorily.

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