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Exponentiation problem in the construction of an effective low-momentum Hamiltonian for bosons

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Abstract. The problem of exponentiation of connected-graph contributions C, when one carries out only a partial trace of the density matrix of an assembly of bosons in order to construct an effective, low-momentum Hamiltonian, is examined. It is found that besides accounting for the exponentiation of connected graphs, disconnected graphs contribute certain terms D to connected-graph contributions. The D-terms diminish as the number of iterations increases in the Singh's renormalization-group theory for the present system. Therefore, these terms play no role in determining critical behaviour of the system.

Keywords. Exponentiation; low-momentum Hamiltonian; bosons; connected graph; disconnected graph; partial trace; renormalization group; time-ordered product; double-commutator.

PACS No. 05·30

1. Introduction

A renormalization-group (RG) theoretic study of critical and multicritical behaviour of a system described by a suitable Hamiltonian H_0 requires constructing an effective Hamiltonian H_e involving small-momentum fields eliminating those corresponding to large momenta in H_0 (see e.g. Wilson 1971; Riedel and Wegner 1972; Wilson and Kogut 1974). Whether the field is classical or quantum mechanical, to obtain H_e by using diagrammatic perturbation theory one has to perform a partial trace on the density matrix to integrate out large momentum modes and show that contributions of connected graphs get exponentiated by disconnected graph contributions. While for the classical case Wilson and Kogut (1974) have stated that it is straightforward, though cumbersome, to prove the exponentiation of connected graphs, proving the same for a quantum system can be difficult on account of the fact that contributions of connected and disconnected graphs contain time-ordered products (T-products) of uncontracted small-momentum operators and a T-product, in general, cannot be factorized into two or more than two T-products. The aim of the present work is to examine the problem of exponentiation for a quantum system, viz an assembly of weakly interacting bosons, by carrying out only a partial trace of the density matrix.

Several years ago, Singh (1975, 1976, 1978) had shown how critical behaviour in a pure Bose system may be studied by performing RG-transformations directly on a quantum-mechanical Hamiltonian. Following Singh's approach Olinto (1985) has recently discussed crossover in interacting systems. For a dilute, weakly interacting Bose-gas at low-temperature, the crossover from critical behaviour to ideal gas behaviour had been examined earlier by Rasolt *et al* (1984). In the works of Singh and Olinto, while the calculations were made to second order in interactions, the problem, whether higher-order disconnected graphs would lead to exponentiation of contributions of second-order connected graphs, was not examined. As the validity of the RG approach presented in these studies rests upon proving the exponentiation, the importance of the present problem is obvious.

Although, as already stated, a T-product cannot be written as a product of two or more T-products, the former differs from the latter by a certain number of "remainder terms" involving commutators. Contributions of disconnected graphs up to the fifth-order have been examined in this paper using this fact. It is found that, besides giving exponentiation of connected graphs, disconnected graphs give contributions D which must be added to the contributions C of the connected graphs. The contributions D arise from the "remainder terms" and, to the extent the present author has been able to verify, they get exponentiated in higher-order graphs. These contributions are irrelevant in the context of Singh's RG approach. The reasons, as will be explained in § 5, are (i) with respect to the RG transformations the parameter s in Singh's theory is irrelevant and (ii) a remainder term in D arising from a given disconnected graph has at least two operators less than the graph. It will also become clear in § 5 that the irrelevance of D-terms refers to ideal Bose gas fixed point as well as one in the case of an interacting assembly.

A brief outline of the paper is as follows: In § 2, the problem is formulated. In § 3 contributions of first-order graphs and second-order connected graphs are written down. Exponentiation of the former up to second-order is also shown. The discussion concerning exponentiation of contributions of connected graphs and contributions D arising from disconnected graphs forms the content of § 4. The paper ends with some concluding remarks in § 5.

2. Formulation of the problem

The system under consideration is an assembly of spinless, interacting bosons of mass (m/2) contained in a box volume V. In units such that $\hbar = 1$, the Hamiltonian of the system can be written in the form (cf. equation (9) in Singh 1975)

$$\beta H = H_0 = H_F(q) + H_F(p) + h_I, \tag{1}$$

$$H_F(q) = \beta \sum_q \left(\frac{q^2}{m} - \mu\right) a_q^\dagger a_q, \tag{2}$$

$$H_F(p) = \beta \sum_p \left(\frac{p^2}{m} - \mu\right) a_p^\dagger a_p, \tag{3}$$

$$h_I = \frac{\beta u_0}{4V} \sum_{k_1 \ldots k_4} a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} \delta_{k_1+k_2, k_3+k_4}. \tag{4}$$

Here, u_0 denotes the interaction constant, and μ the chemical potential of boson. β denotes the inverse of the product of the Boltzman constant k_B and the absolute temperature T. p's denote momenta larger than $p_c\zeta^{-1}$ and q's momenta less than $p_c\zeta^{-1}$; p_c is a momentum cut-off large in comparison with boson thermal momentum

$$\lambda^{-1}(T) = \left(\frac{m}{4\pi\beta}\right)^{1/2} \tag{5}$$

and $\zeta \gg 1$. The four k's in any term of h_I can either be all p's, or all q's, or a combination of p's and q's.

The parameters (s, r, v), defined by

$$s = \beta p_c^2/m \quad r = -m\mu p_c^{-2}, \quad v = s^{-2}\beta u_0 p_c^d, \tag{6}$$

can be introduced, as in Singh's work, to write

$$H_F(q) = \sum_{|q|<p_c\zeta^{-1}} s(q^2 p_c^{-2} + r) a_q^\dagger a_q, \tag{7}$$

$$H_F(p) = \sum_{|p|>p_c\zeta^{-1}} s(p^2 p_c^{-2} + r) a_p^\dagger a_p, \tag{8}$$

$$h_I = \frac{s^2 v p_c^{-d}}{4V} \sum_{k_1..k_4} a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} \delta_{k_1+k_2,k_3+k_4}. \tag{9}$$

Here d denotes the dimensionality of the system. As will become clear later (see § 5), the parameter s plays an important role in the present problem.

The quantity of interest is the grand partition function Z of the system, defined by

$$Z = \mathrm{Tr}\exp(-H_0). \tag{10}$$

Upon treating $(H_F(q) + H_F(p))$ as the unperturbed Hamiltonian, and h_I as a perturbation, the partition function can be expanded in powers of h_I in the usual manner (Abrikosov *et al* 1963). The Hilbert space of the system can be written as the direct product

$$h_0 \otimes h_1, \tag{11}$$

where h_0 denotes the subspace on which a_q's act and h_1 the subspace on which a_p's act. The partition function consequently can be written in the form

$$Z = Z_0 \underset{h_0}{\mathrm{Tr}} \exp[-H_F(q)] \Bigg[1 + \sum_{n=1}^{\infty} (-1)^n (n!)^{-1} \times \int_0^1 \mathrm{d}\tau_1 \ldots \int_0^1 \mathrm{d}\tau_n \langle T\{h_I(\tau_1) \ldots h_I(\tau_n)\} \rangle \Bigg], \tag{12}$$

$$Z_0 = \underset{h_1}{\mathrm{Tr}} \exp(-H_F(p)), \tag{13}$$

$$h_I(\tau) = \exp\{\tau[H_F(q) + H_F(p)]\} h_I \exp\{-\tau[H_F(q) + H_F(p)]\}. \tag{14}$$

In (12), T denotes the imaginary time ordering operator, and $\langle \ldots\ldots \rangle$ denotes thermodynamic average calculated with the density matrix $\exp(-H_F(p))$.

The present paper attempts to examine whether contributions of connected graphs arising from the expansion in (12) get exponentiated by second- and higher-order disconnected graph contributions. In other words, the aim is to determine whether it is possible to write

$$Z = Z_0 \underset{h_0}{\mathrm{Tr}} \exp(-H_F(q) + h(q) + C(q))], \tag{15}$$

where $h(q)$ denotes the part of h_I corresponding to all the four k's being less than $p_c\zeta^{-1}$ and $C(q)$ denotes contributions of connected graphs.

3. First- and second-order graphs

The graphs corresponding to the first-order term in (12) are depicted in figure 1. The external lines represent low-momentum ($|q| < p_c\zeta^{-1}$) boson operators and the internal lines high-momentum ($|p| > p_c\zeta^{-1}$) boson operators. The contribution F_1 of these graphs is easily written down. One gets

$$F_1 = -\int_0^1 d\tau[h(q, \tau) + t_1(q, \tau) + C_1], \tag{16}$$

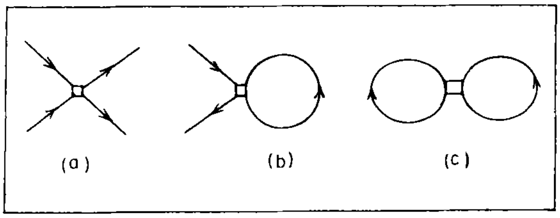

Figure 1. Graphs corresponding to the first-order term in the expansion in (12).

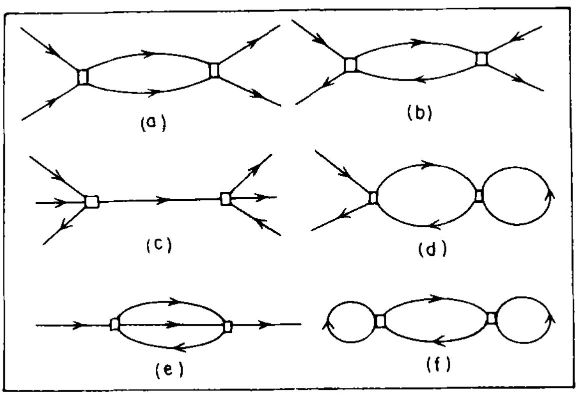

Figure 2. Connected graphs corresponding to the second-order term in the expansion in (12).

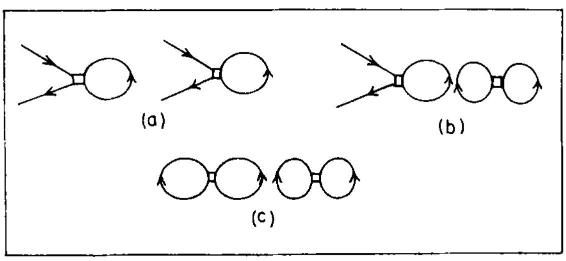

Figure 3. Disconnected graphs corresponding to the second-order term in the expansion in (12).

where

$$t_1(q,\tau)=\sum_q K_1(s,r)\tilde{a}_q(\tau)a_q(\tau), \tag{17}$$

$$K_1(s,r)=s^2vf_1(s,r), \tag{18}$$

$$f_1(s,r)=\int\limits_{\zeta^{-1}\leqslant|q|\leqslant 1}\frac{d^dq}{(2\pi)^d}(\exp s(q^2+r)-1)^{-1}, \tag{19}$$

$$C_1=\tfrac{1}{2}Vvs^2p_c^df_1^2(s,r). \tag{20}$$

The first, second and third terms in the integrand in (16) correspond to the graphs 1(*a*), 1(*b*) and 1(*c*), respectively. Graph 1(*b*) has been associated with a numerical factor 4 corresponding to the number of ways in which the external lines can be chosen and the graph 1(*c*) with a numerical factor 2 corresponding to two possible *p*-pairings.

The second-order term in (12) gives connected as well as disconnected graphs. The connected graphs are shown in figure 2, while the disconnected ones are shown in figure 3. Graphs 2(*a*) and 2(*b*) give contributions similar to $h(q)$, whereas 2(*d*) and 2(*e*) give contributions similar to $H_F(q)$. The contribution of graph 2(*c*) contains a product of six operators.

The contribution $F(2a)$ of graph 2(*a*) can be written in the form

$$F(2a)=-\int_0^1 d\tau g_2(q,\tau), \tag{21}$$

$$g_2(q,\tau)=-4\left(\frac{s^2vp_c^{-d}}{4V}\right)^2\frac{1}{2!}\int_0^1 d\tau_1\int_0^1 d\tau_2$$

$$\times\sum_p\sum_{q_1..q_4}\mathscr{G}(p,\tau_1-\tau_2)\mathscr{G}(-p+q_1+q_2,\tau_1-\tau_2)$$

$$f_a(q_1\dots q_4;\tau_1,\tau_2;\tau), \tag{22}$$

$$f_a=\exp\{(\varepsilon_{q_1}+\varepsilon_{q_2})\tau_1-(\varepsilon_{q_3}+\varepsilon_{q_4})\tau_2\}\delta_{q_1+q_2,q_3+q_4}$$

$$\chi^{-1}(q_1..q_4)\{\tilde{a}_{q_1}(\tau)\tilde{a}_{q_2}(\tau)a_{q_3}(\tau)a_{q_4}(\tau)+[a_{q_3}(\tau)a_{q_4}(\tau),\tilde{a}_{q_1}(\tau)\tilde{a}_{q_2}(\tau)]\,\theta(\tau_2-\tau_1)\}, \tag{23}$$

$$\mathscr{G}(p,\tau)=-\langle T\{a_p(\tau)\tilde{a}_p(0)\}\rangle, \tag{24}$$

$$\chi(q_1..q_4)=\int_0^1 d\tau\exp\{(\varepsilon_{q_1}+\varepsilon_{q_2}-\varepsilon_{q_3}-\varepsilon_{q_4})\tau\}, \tag{25}$$

$$\varepsilon_q=s(q^2p_c^{-2}+r). \tag{26}$$

The numerical factor 4 in the right side of (22) arises due to two possible *p* pairings and the fact that τ values assigned to two vertices in figure 2a can be interchanged. In writing (23), the *T*-product

$$T\{\tilde{a}_{q_1}(\tau_1)\tilde{a}_{q_2}(\tau_1)a_{q_3}(\tau_2)a_{q_4}(\tau_2)\},$$

has been broken into a simple product and a commutator term. The factor χ^{-1} arises in the process of writing all the operators at the same τ value in order that g_2 may appear like $h(q,\tau)$.

The contributions of other graphs in figure 2 can be similarly written down. The total contribution F_2 of the second-order connected graphs is found to be

$$F_2 = -\int_0^1 \mathrm{d}\tau[g_2(q,\tau)+I_2(q,\tau)+J_2(q,\tau)+t_2(q,\tau)+C_2], \tag{27}$$

$$-\int_0^1 \mathrm{d}\tau I_2(q,\tau) = 16\left(\frac{s^2 v p_c^{-d}}{4V}\right)^2 \frac{1}{2!}\int_0^1 \mathrm{d}\tau_1 \int_0^1 \mathrm{d}\tau_2 \sum_p \sum_{q_1\ldots q_4} \mathscr{G}(p,\tau_1-\tau_2)$$
$$\mathscr{G}(-q_1+q_3+p,\tau_2-\tau_1)T\{\tilde{a}_{q_1}(\tau_1)a_{q_3}(\tau_1)$$
$$\tilde{a}_{q_2}(\tau_2)a_{q_4}(\tau_2)\}\delta_{q_1+q_2,q_3+q_4}, \tag{28}$$

$$-\int_0^1 \mathrm{d}\tau J_2(q,\tau) = -8\left(\frac{s^2 v p_c^{-d}}{4V}\right)^2 \frac{1}{2!}\int_0^1 \mathrm{d}\tau_1 \int_0^1 \mathrm{d}\tau_2 \sum_p \sum_{q_1\ldots q'_3} \mathscr{G}(p,\tau_1-\tau_2)$$
$$T\{\tilde{a}_{q_1}(\tau_1)\tilde{a}_{q_2}(\tau_1)a_{q_3}(\tau_1)\tilde{a}_{q'_3}(\tau_2)a_{q'_1}(\tau_2)a_{q'_2}(\tau_2)\}\delta_{p,q_1+q_2-q_3}\delta_{p,q'_1+q'_2-q'_3}, \tag{29}$$

$$t_2 = \sum_q K(q)\tilde{a}_q(\tau)a_q(\tau), \tag{30}$$

$$K(q) = -s^4v^2 f_1(s,r)f_2(s,r) - K_2(q), \tag{31}$$

$$K_2(q) = -16\left(\frac{s^2 v p_c^{-d}}{4V}\right)^2 \frac{1}{2!}\int_0^1 \mathrm{d}\tau_1 \int_0^1 \mathrm{d}\tau_2 \sum_{p_1 p_2} \mathscr{G}(p_1,\tau_1-\tau_2)$$
$$\mathscr{G}(p_2,\tau_1-\tau_2)\mathscr{G}(p_1+p_2-q,\tau_2-\tau_1)\exp[\varepsilon_q(\tau_1-\tau_2)], \tag{32}$$

$$f_2(s,r) = \int_{\zeta^{-1}\leqslant|q|\leqslant 1} \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{\exp[s(q^2+r)]}{(\exp[s(q^2+r)]-1)^2}, \tag{33}$$

$$C_2 = -\frac{1}{2}s^4v^2 f_1^2(s,r)f_2(s,r)p_c^d V - \sum_q K'_2(q), \tag{34}$$

$$K'_2(q) = -16\left(\frac{s^2 v p_c^{-d}}{4V}\right)^2 \frac{1}{2!}\int_0^1 \mathrm{d}\tau_2 \int_0^{\tau_2} \mathrm{d}\tau_1 \sum_{p_1 p_2} \mathscr{G}(p_1,\tau_1-\tau_2)$$
$$\times\,\mathscr{G}(p_2,\tau_1-\tau_2)\mathscr{G}(p_1+p_2-q,\tau_2-\tau_1)\exp[\varepsilon_q(\tau_1-\tau_2)]. \tag{35}$$

While the second and third terms in the integrand in (27), respectively, correspond to graphs 2(*b*) and 2(*c*), the sum of the last two terms correspond to graphs 2(*d*), 2(*e*) and 2(*f*). The numerical factors in the contributions above correspond to the number of ways in which the external lines can be chosen, different possible p-pairings and the number of ways in which τ values assigned to vertices in a graph can be interchanged, e.g. the factor 16 in the right side of (28) corresponds to the number of ways in which the external lines can be chosen.

The contribution of the disconnected graphs of figure 3 is

$$\frac{1}{2!}\int_0^1 \mathrm{d}\tau_1 \int_0^1 \mathrm{d}\tau_2\, T(t_1(q,\tau_1)+C_1)\cdot(t_1(q,\tau_2)+C_1). \tag{36}$$

Upon collecting all the first- and second-order terms, the expression within the parantheses in (12) becomes

$$1-\int_0^1 \mathrm{d}\tau[h(q,\tau)+t_1(q,\tau)+C_1+g_2(q,\tau)+I_2(q,\tau)+J_2(q,\tau)+t_2(q,\tau)+C_2]$$

$$+\frac{1}{2!}\int_0^1 \mathrm{d}\tau_1\int_0^1 \mathrm{d}\tau_2\, T[(h(q,\tau_1)+t_1(q,\tau_1)+C_1)$$

$$\cdot(h(q,\tau_2)+t_1(q,\tau_2)+C_1)]. \tag{37}$$

This expression shows exponentiation of first-order graphs up to second order. Exponentiation of second-order contributions g_2, I_2, etc can be seen in third- and higher-order disconnected graphs only.

4. Third- and higher-order disconnected graphs

A few typical disconnected graphs of the third-order are shown in figure 4. The graph 4(a) will be considered for illustration. One finds

$$F(4a)=8\left(\frac{s^2vp_c^{-d}}{4V}\right)^3\int_0^1 \mathrm{d}\tau_1\int_0^1 \mathrm{d}\tau_2\int_0^1 \mathrm{d}\tau_3\sum_{pp_1}\sum_{q_1\cdots q_4}\mathscr{G}(p,\tau_1-\tau_2)$$

$$\mathscr{G}(-p+q_1+q_2,\tau_1-\tau_2)\mathscr{G}(p_1,-\delta)T\{A(\tau_1)B(\tau_2)C(\tau_3)\}$$

$$\times\,\delta q_1+q_2,q_3+q_4, \tag{38}$$

$$A=a^\dagger_{q_1}a^\dagger_{q_2}, \tag{39}$$

$$B=a_{q_3}a_{q_4}, \tag{40}$$

$$C=\sum_{q_5}a^\dagger_{q_5}a_{q_5}. \tag{41}$$

To bring (38) into a form which may look like the product of the contributions of the graphs 1(b) and (2a), one can use the identity

$$T(A_1B_2C_3)=T(A_1B_2)C_3+R_3(C_3,A_1B_2), \tag{42}$$

$$R_3(C_3,A_1B_2)=A_1[C_3,B_2](132)+[C_3,A_1B_2](312)$$
$$+B_2[C_3,A_1](231)+[C_3,B_2A_1](321), \tag{43}$$

where A_1 stands for $A(\tau_1)$, and (ijk) denotes $\theta(\tau_i-\tau_j)\theta(\tau_j-\tau_k)$. It can be noted that R_3 is, in general, a four-operator term similar to the interaction $h(q,\tau)$.

Upon using (42) in (38), one gets

$$F(4a)=\int_0^1 \mathrm{d}\tau_1\int_0^1 \mathrm{d}\tau_2 g_2(\tau_1)t_1(\tau_2)-\int_0^1 \mathrm{d}\tau\, R_3(\tau). \tag{44}$$

$$R_3(\tau)=-8\left(\frac{s^2vp_c^{-d}}{4V}\right)^3\int_0^1 \mathrm{d}\tau_1\int_0^1 \mathrm{d}\tau_2\int_0^1 \mathrm{d}\tau_3\sum_{pp_1}\sum_{q_1\cdots q_4}\mathscr{G}(p,\tau_1-\tau_2)$$
$$\times\,\mathscr{G}(-p+q_1+q_2,\tau_1-\tau_2)\mathscr{G}_1(p_1,-\delta)R_3(\tau_1,\tau_2,\tau_3;\tau)$$
$$\times\,\exp[(\varepsilon_{q_1}+\varepsilon_{q_2})\tau_1-(\varepsilon_{q_3}+\varepsilon_{q_4})\tau_2]\,\delta_{q_1+q_2,q_3+q_4+}, \tag{45}$$

where $R_3(\tau_1,\tau_2,\tau_3;\tau)$ is obtained from (43) by replacing A_1, B_2 and C_3 by $A(\tau)$, $B(\tau)$ and $C(\tau)$, respectively, and dividing the resulting expression by $\chi(q_1\ldots q_4)$ given by (25).

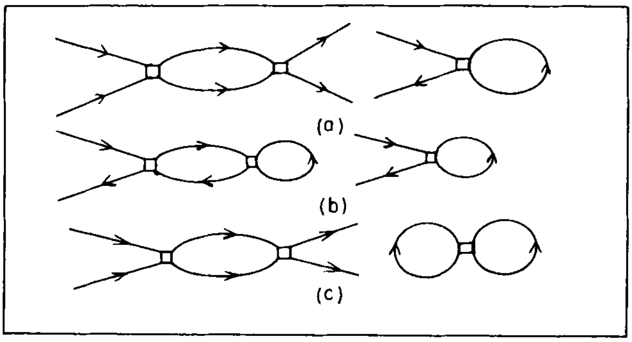

Figure 4. A few typical disconnected graphs corresponding to the third-order term in the expansion in (12).

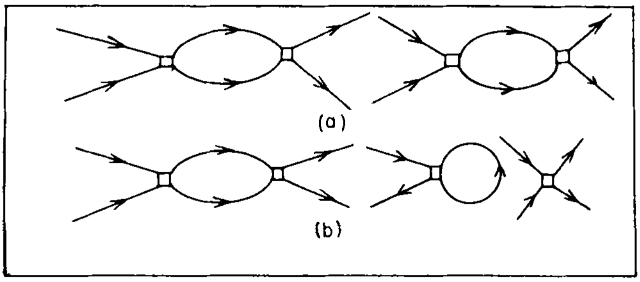

Figure 5. A few typical disconnected graphs corresponding to the fourth-order term in the expansion in (12).

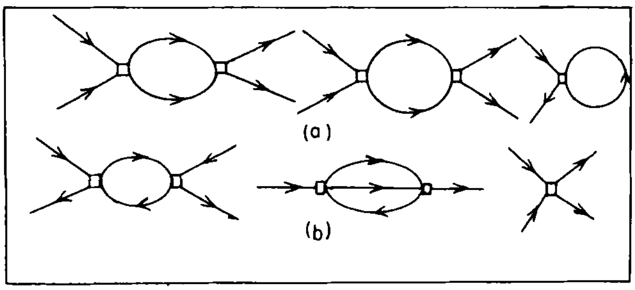

Figure 6. A few typical disconnected graphs corresponding to the fifth-order term in the expansion in (12).

The term containing $R_3(\tau)$ in (44) adds a contribution of order (v^3) to the low-momentum interaction $h(q, \tau)$. The point to note is that this contribution has arisen from a disconnected graph. As will be seen below, such disconnected graph contributions get exponentiated in higher orders much like the contributions of connected graphs.

One needs to write the first term in (44) as a T-product to display exponentiation of g_2 and t_1 in second-order. This is achieved by using the result

$$T(A_1 B_2) = A_1 B_2 + [B_2, A_1]\theta(\tau_2 - \tau_1). \tag{46}$$

One is then left with a remainder term

$$-\int_0^1 \mathrm{d}\tau_1 \int_0^1 \mathrm{d}\tau_2[t_1(\tau_2), g_2(\tau_1)]\theta(\tau_2-\tau_1) = -\int_0^1 R_3'(\tau)\mathrm{d}\tau. \tag{47}$$

Like $R_3(\tau)$, $R_3'(\tau)$ is a four-operator term which gets added to $h(q,\tau)$. It is to be examined whether R_3' is also exponentiated in higher orders.

A few typical fourth-order disconnected graphs are shown in figure 5. The contribution of graph 5(a) is

$$F(5a) = 2\left(\frac{s^2 v p_c^{-d}}{4V}\right)^4 \int_0^1 \mathrm{d}\tau_1 \ldots \int_0^1 \mathrm{d}\tau_4 \sum_{pp'} \sum_{q_1 \ldots q_4'} \mathscr{G}(p, \tau_1-\tau_2)\mathscr{G}(-p+q_1+q_2, \tau_1-\tau_2)$$
$$\times \mathscr{G}(p', \tau_3-\tau_4)\mathscr{G}(-p'+q_1'+q_2', \tau_3-\tau_4)$$
$$\times T(A_1 B_2 A_3' B_4')\delta_{q_1+q_2, q_3+q_4}\delta_{q_1'+q_2', q_3'+q_4'}, \tag{48}$$

where A' stands for $(a^\dagger_{q_1'} a_{q_2'})$ and B' for $(a_{q_3'} a_{q_4'})$. The contribution involves the product $T\{A_1 B_2 A_3' B_4'\}$. It is slightly tedious but straightforward to prove the following identity:

$$T(A_1 B_2 C_3 D_4) = T(A_1 B_2)T(C_3 D_4) + R_3(C_3, A_1 B_2)D_4$$
$$+ R_3(D_4, A_1 B_2)C_3 + R_4, \tag{49}$$

where R_3's are given by (43) and R_4 involves 16 double-commutator terms. One finds

$$\begin{aligned} R_4 = {} & A_1[D_4, [C_3, B_2]](1432) + B_2[D_4, [C_3, A_1]](2431) \\ & + A_1[C_3, [D_4, B_2]](1342) + B_2[C_3, [D_4, A_1]](2341) \\ & + [D_4, A_1[C_3, B_2]](4132) + [D_4, B_2[C_3, A_1]](4231) \\ & + [C_3, A_1[D_4, B_2]](3142) + [C_3, B_2[D_4, A_1]](3241) \\ & + A_1[D_4, [C_3, B_2]](4312) + B_2[D_4, [C_3, A_1]](4321) \\ & + A_1[C_3, [D_4, B_2]](3412) + B_2[C_3, [D_4, A_1]](3421) \\ & + [D_4, [C_3, A_1]B_2](4312) + [D_4, [C_3, B_2]A_1](4321) \\ & + [C_3, [D_4, A_1]B_2](3412) + [C_3, [D_4, B_2]A_1](3421). \end{aligned} \tag{50}$$

R_4 contains at least four operators $(a, a^\dagger)$ less than $(A_1 B_2 C_3 D_4)$. Upon substituting (49) in (48), one gets

$$F(5a) = \frac{1}{2!}\int_0^1 \mathrm{d}\tau_1 \int_0^1 \mathrm{d}\tau_2 g_2(\tau_1) g_2(\tau_2) - \int_0^1 \mathrm{d}\tau\, R_{4a}(\tau), \tag{51}$$

where

$$-\int_0^1 \mathrm{d}\tau\, R_{4a}(\tau) = 2\left(\frac{s^2 v p_c^{-d}}{4V}\right)^4 \int_0^1 \mathrm{d}\tau_1 \ldots \int_0^1 \mathrm{d}\tau_4$$
$$\sum_{pp'} \sum_{q_1 \ldots q_4'} \mathscr{G}(p, \tau_1-\tau_2)\mathscr{G}(-p+q_1+q_2, \tau_1-\tau_2)$$
$$\times \mathscr{G}(p', \tau_3-\tau_4)\mathscr{G}(-p'+q_1'+q_2', \tau_3-\tau_4)$$
$$\times \{R_3(A_3', A_1 B_2)B_4' + R_3(B_4', A_1 B_2)A_3' + R_4\}$$
$$\times \delta_{q_1+q_2, q_3+q_4}\delta_{q_1'+q_2', q_3'+q_4'}. \tag{52}$$

It is evident from (51) that the first term on the right side gives exponentiation of g_2 with itself in second order. The second term does not admit of any such interpretation. It is clear from (52) that the terms involving R_3's can be combined with the six-operator terms arising from the second-order graph. 2(*c*) and the third-order connected (or disconnected) graphs whereas the term involving R_4 gives a contribution of fourth order to $h(q, \tau)$. It can be noted that on writing the first term of (51) as a T-product of $g_2(\tau_1)$ and $g_2(\tau_2)$, one is left with an additional six-operator term $R'_{4a}(\tau)$ given by

$$-\int_0^1 d\tau\, R'_{4a}(\tau) = -\frac{1}{2!}\int_0^1 d\tau_1 \int_0^1 d\tau_2 [g_2(\tau_2), g_2(\tau_1)]\theta(\tau_2-\tau_1). \tag{53}$$

More interesting is the contribution of graph 5(*b*). One finds

$$\begin{aligned} F(5b) = &-8\left(\frac{s^2 v p_c^{-d}}{4V}\right)^4 \int_0^1 d\tau_1 \ldots \int_0^1 d\tau_4 \sum_{pp'} \sum_{q_1 \ldots q'_4} \mathscr{G}(p, \tau_1-\tau_2) \\ &\times \mathscr{G}(-p+q_1+q_2, \tau_1-\tau_2)\mathscr{G}(p', -\delta)T(A_1B_2C_3h_4) \\ &\delta_{q_1+q_2, q_3+q_4}\delta_{q'_1+q'_2, q'_3+q'_4}, \end{aligned} \tag{54}$$

where C is defined by (40) and h denotes the four-point vertex of figure 1a. Upon using (49) in (54), with the replacement of D_4 in the former by h_4, one can write

$$\begin{aligned} F(5b) = &-\int d\tau_2 d\tau_3 d\tau_4 g_2(\tau_2) T(t_1(\tau_3)h(\tau_4)) \\ &-8\left(\frac{s^2 v p_c^{-d}}{4V}\right)^4 \int_0^1 d\tau_1 \ldots \int_0^1 d\tau_4 \sum_{pp'} \sum_{q_1 \ldots q'_4} \mathscr{G}(p, \tau_1-\tau_2) \\ &\times \mathscr{G}(-p+q_1+q_2, \tau_1-\tau_2)\mathscr{G}(p', -\delta) \\ &\times \{R_3(C_3, A_1B_2)h_4 + R_3(h_4, A_1B_2)C_3 + R'_4\} \\ &\delta_{q_1+q_2, q_3+q_4}\delta_{q'_1+q'_2, q'_3+q'_4}, \end{aligned} \tag{55}$$

where R'_4 is obtained from R_4 given by (50) with the replacement of D_4 by h_4. It is not difficult to see that

$$\begin{aligned} T(A_1B_2C_3) = &A_1 T(B_2C_3) + [B_2, A_1]C_3\theta(\tau_2-\tau_1) \\ &+ [C_3, A_1]B_2\theta(\tau_3-\tau_1) \\ &+ [C_3, [B_2, A_1]]\,(321) + [B_2, [C_3, A_1]]\,(231). \end{aligned} \tag{56}$$

In view of (56), the first term in (55) can be written as follows:

$$\begin{aligned} &-\int d\tau_2 d\tau_3 d\tau_4 g_2(\tau_2) T(t_1(\tau_3)h(\tau_4)) \\ &= \int d\tau_2 d\tau_3 d\tau_4 \{-T(g_2(\tau_2)t_1(\tau_3)h(\tau_4)) + [t_1(\tau_3), g_2(\tau_2)]h(\tau_4)\theta(\tau_3-\tau_2) \\ &+ [h(\tau_4), g_2(\tau_2)]t_1(\tau_3)\theta(\tau_4-\tau_2) + R'_{4b}(\tau_2, \tau_3, \tau_4)\}, \end{aligned} \tag{57}$$

$$\begin{aligned} R'_{4b}(\tau_2, \tau_3, \tau_4) = &[h(\tau_4), [t_1(\tau_3), g_2(\tau_2)]]\,(432) \\ &+ [t_1(\tau_3), [h(\tau_4), g_2(\tau_2)]]\,(342). \end{aligned} \tag{58}$$

The first term in (57) represents exponentiation of (g_2, t_1, h) in third order. The second term corresponds to the exponentiation in second order of h and R'_3 (cf. (47)); a similar interpretation holds for the third term in (57). The double commutators generated by writing the second and third terms as T-products, together with R'_{4b}, constitute a six-operator term to be added to the six-operator terms mentioned above.

The second term of (55) corresponds to exponentiation of R_3 (cf. (45)) and h and a similar interpretation holds for the third term. One thus concludes that the remainder terms R_3 and R'_3 defined in connection with the third-order graph 4(a), get properly exponentiated in fourth-order graphs. The last term in (55), which involves R'_4, corresponds to a six-operator term to be added to the six-operator terms alluded to above.

To obtain further confirmation for the exponentiation of the remainder terms (R_3, R'_3), (R_{4a}, R'_{4a}) etc., the fifth-order disconnected graphs shown in figure 6 has been examined. The contribution of graph 6(a) is

$$\begin{aligned} F(6a) = 8\left(\frac{s^2 v p_c^{-d}}{4V}\right)^5 & \int_0^1 \mathrm{d}\tau_1 \ldots \int_0^1 \mathrm{d}\tau_5 \sum_{pp'p''} \sum_{q_1 \ldots q'_4} \mathscr{G}(p, \tau_1 - \tau_2) \\ & \times \mathscr{G}(-p + q_1 + q_2, \tau_1 - \tau_2) \mathscr{G}(p', \tau_3 - \tau_4) \\ & \times \mathscr{G}(-p' + q'_1 + q'_2, \tau_3 - \tau_4) \mathscr{G}(p'', -\delta) \\ & \times T(A_1 B_2 A'_3 B'_4 C_5) \delta_{q_1+q_2, q_3+q_4} \delta_{q'_1+q'_2, q'_3+q'_4}. \end{aligned} \tag{59}$$

The contribution involves the T-product $T(A_1 B_2 A'_3 B'_4 C_5)$. By a procedure similar to that leading to (49), one obtains the identity

$$\begin{aligned} T(A_1 B_2 C_3 D_4 E_5) = & \, T(A_1 B_2 C_3 D_4) E_5 + R_3(E_5, A_1 B_2) T(C_3 D_4) \\ & + R_3(E_5, C_3 D_4) T(A_1 B_2) + R_5, \end{aligned} \tag{60}$$

where R_5, like R_4 earlier, contains double commutators. Upon using (49), the first term in (60) in this case can be written as

$$\begin{aligned} T(A_1 B_2 A'_3 B'_4) C_5 = & \, T(A_1 B_2) T(A'_3 B'_4) C_5 \\ & + \{R_3(A'_3, A_1 B_2) B'_4 + R_3(B'_4, A_1 B_2) A'_3 + R_4\} C_5. \end{aligned} \tag{61}$$

The contribution of graph 6(a) corresponding to the first term in (60) accordingly becomes

$$\begin{aligned} & -\tfrac{1}{2} \int \mathrm{d}\tau_1 \, \mathrm{d}\tau_2 \mathrm{d}\tau_3 \, g_2(\tau_1) g_2(\tau_2) t_1(\tau_3) \\ & -2\left(\frac{s^2 v p_c^{-d}}{4V}\right)^4 \int_0^1 \mathrm{d}\tau_1 \ldots \int_0^1 \mathrm{d}\tau_5 \sum_{pp'} \sum_{q_1 \ldots q'_4} \\ & \times \mathscr{G}(p, \tau_1 - \tau_2) \mathscr{G}(-p + q_1 + q_2, \tau_1 - \tau_2) \\ & \times \mathscr{G}(p', \tau_3 - \tau_4) \mathscr{G}(-p' + q'_1 + q'_2, \tau_3 - \tau_4) \\ & \times \{R_3(A'_3, A_1 B_2) B'_4 + R_3(B'_4, A_1 B_2) A'_3 + R_4\} t_1(\tau_5) \\ & \times \delta_{q_1+q_2, q_3+q_4} \delta_{q'_1+q'_2, q'_3+q'_4}. \end{aligned} \tag{62}$$

The simple product $(g_2 g_2 t_1)$ in (62) can be converted into a T-product by using the identity

$$
\begin{aligned}
-g_2(\tau_1)g_2(\tau_2)t_1(\tau_3) = & -T(g_2(\tau_1)g_2(\tau_2)t_1(\tau_3)) \\
& + [g_2(\tau_2), g_2(\tau_1)]t_1(\tau_3)\theta(\tau_2-\tau_1) \\
& + g_2(\tau_1)[t_1(\tau_3), g_2(\tau_2)]\theta(\tau_3-\tau_2) \\
& + g_2(\tau_2)[t_1(\tau_3), g_2(\tau_1)]\theta(\tau_3-\tau_1) \\
& + R'_{6a}(\tau_1, \tau_2, \tau_3), \qquad (63)
\end{aligned}
$$

$$
\begin{aligned}
R'_{6a}(\tau_1, \tau_2, \tau_3) = & [[t_1(\tau_3), g_2(\tau_2)], g_2(\tau_1)]\,(321) \\
& + [[t_1(\tau_3), g_2(\tau_1)], g_2(\tau_2)]]\,(312). \qquad (64)
\end{aligned}
$$

Besides the exponentiation of (g_2, g_2, t_1) in third-order, the expression (62) thus represents exponentiation of the remainders R_{4a} and R'_{4a} of the graph 5(a) with t_1, and of R'_3 with g_2. The second and third terms together in (60) correspond to exponentiation of R_3 with g_2.

It is evidently not possible to extend calculations of the above type to disconnected graphs of arbitrary order. To establish an identity of the type (60) for six operators, for example, requires handling 720 permutations. The results obtained above, however, suggest that (12) is equivalent to

$$
Z = Z_0 \operatorname*{Tr}_{h_0} \exp[-(H_F(q)+h(q)+C(q)+D(q))], \qquad (65)
$$

where $C(q)$ denotes the contribution of all connected graphs and

$$
D(q) = (R_3(q)+R'_3(q)) + (R_4(q)+R'_4(q)) + \ldots . \qquad (66)
$$

denotes the contribution from the remainder terms of the disconnected graphs.

5. Concluding remarks

The analysis of the previous sections leads one to the fact that the present system can be described by the effective, low-momentum Hamiltonian $(H_F(q)+h(q)+C(q)+D(q))$. It may be noted that the contributions $D(q)$ eventually become unimportant when this Hamiltonian is used for RG theoretic discussion of the critical behaviour of the system. The reason is that with respect to the RG transformation the parameter s is irrelevant which enables one to replace Bose distribution factor $(\exp s(q^2+r)-1)^{-1}$, associated with an internal line, by $(s(q^2+r))^{-1}$ (Singh 1975, 1976, 1978). Remembering that each vertex carries a factor s^2, it is easy to check that a connected or disconnected graph with $2n$ external operators carries a factor s^n. Since from the preceding section it is clear that a remainder term in $D(q)$ arising from a given disconnected graph has at least two operators less than the graph, it follows that a $2(n-1)$ operator term in $D(q)$ is associated with a factor s^n (or a higher power of s) and hence can be ignored in comparison with the contribution of the connected graphs having $2(n-1)$ operators.

The irrelevance of $D(q)$, as already stated in § 1, refers to both ideal and interacting Bose gas fixed points. The reason is that the parameter s is not only irrelevant in the case of an interacting Bose gas (see Singh 1975, 1976, 1978), it is also irrelevant in the ideal Bose gas case (see Singh 1980). It was shown by Singh, in the references cited above, how scaling behaviour arose as a consequence of the irrelevant nature of s. One now finds that the same property helps to restore the connected-graph theorem (Abrikosov *et al*

1963), when one takes only a partial trace of the density matrix, after a large number of renormalizations.

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