Pramāṇa – J. Phys., Vol. 25, No. 4, October 1985, pp. 369–375. © Printed in India.

Entropic formulation of uncertainty relations*

M D SRINIVAS
Department of Theoretical Physics, University of Madras, Guindy Campus, Madras 600 025, India

Abstract. We review the recent investigations on the improved formulation of uncertainty relations which employ the information-theoretic entropy rather than variance as a measure of uncertainty. We show that this formulation also brings out clearly the relation between the overall uncertainty and the quantum mechanical interference due to measurements.

Keywords. Uncertainty; variance; information-theoretic entropy; quantum mechanical interference.

1. The problem

The uncertainty relations as normally understood are supposed to provide an estimate of the minimum uncertainty expected in the outcome of a measurement of an observable B, given the uncertainty in the outcome of a measurement of another observable A. In the conventional Heisenberg formulation of the uncertainty relation (Heisenberg 1927; Robertson 1929) the variance of an observable is taken as a measure of uncertainty. As is well known we have

$$(\Delta^{\psi}A)^2(\Delta^{\psi}B)^2 \geqslant \tfrac{1}{4}|\langle\psi|AB-BA|\psi\rangle|^2, \tag{1}$$

where

$$(\Delta^{\psi}A)^2 = \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2. \tag{2}$$

When A, B are canonically conjugate to each other, then the right side of (1) will become independent of the state $|\psi\rangle$. Otherwise the right side of (1) is in general state-dependent. This is a serious inadequacy of the Heisenberg relation (1) and cannot be easily overcome by, say, taking the infinum of the right side over all states $|\psi\rangle$. The resulting relation,

$$(\Delta^{\psi}A)^2(\Delta^{\psi}B)^2 \geqslant \operatorname*{Inf}_{|\psi\rangle}\left\{\frac{1}{4}\left|\langle\psi|AB-BA|\psi\rangle\right|^2\right\} \tag{3}$$

would become trivial (the right side reduces to zero) even if one of the observables A, B has just one discrete eigenvalue, or equivalently a normalisable eigenvector.

The above remarks will become clear if we consider for instance the uncertainty relation for the components of angular momentum. From relation (1) we get

$$(\Delta^{\psi}J_x)^2(\Delta^{\psi}J_z) \geqslant \frac{\hbar^2}{4}\left|\langle\psi|J_y|\psi\rangle\right|^2. \tag{4}$$

* Lecture delivered at the International Symposium on Theoretical Physics, Bangalore, November 1984.

Now given the uncertainty in J_x we cannot say anything significant about the minimum possible uncertainty in J_z as the infinum of the right side in (4) is always zero. In fact as is well known, the right side of (4) vanishes for some states $|\psi\rangle$ even when the left side is non-zero.

Thus except for the case of canonically conjugate observables, the Heisenberg form of the uncertainty relation seems to be totally ineffective in providing an estimate of the minimum possible uncertainty in one observable given the uncertainty in the other.

2. Entropic formulation of uncertainty relations: Observables with purely discrete spectra

Recently Deutsch (1983) and Partovi (1983) have shown that for the case of observables with purely discrete spectra, the above inadequacy of the Heisenberg relation can be overcome by employing the information-theoretic entropy as a measure of uncertainty. Let A be an observable with a purely discrete spectrum with the spectral resolution

$$A = \sum_i a_i P^A(a_i), \tag{5}$$

where $\{a_i\}$ are the eigenvalues and $\{P^A(a_i)\}$ the associated eigen-projectors. Then the probability distribution of A in the density operator state ρ is given by

$$\Pr{}_A^\rho(a_i) = \mathrm{Tr}[\rho P^A(a_i)]. \tag{6}$$

Now an appropriate measure of the 'spread' of this probability distribution is the information-theoretic entropy given by (see for instance Khinchin 1957; Lindblad 1973; Ingarden 1976a, b; Wehrl 1978; Grabowski 1978a, b)

$$S^\rho(A) = -\sum_i \Pr{}_A^\rho(a_i) \log \Pr{}_A^\rho(a_i), \tag{7}$$

where it is assumed that $0 \log 0 = 0$. The above entropy $S^\rho(A)$ is a measure of the uncertainty of A in state ρ and should be clearly distinguished from the thermodynamic entropy (Von Neumann 1955)

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho). \tag{8}$$

$S(\rho)$ is a characteristic of the state ρ alone and is in fact a measure of the extent to which ρ is 'mixed' or 'chaotic' (Wehrl 1974).

To state the Deutsch-Partovi entropic formulation of uncertainty relation, let us consider two observables A, B with purely discrete spectra and the following spectral resolutions

$$A = \sum_i a_i P^A(a_i), \tag{8a}$$

$$B = \sum_j b_j P^B(b_j). \tag{8b}$$

Then it was shown by Partovi (1983) that

$$S^{\rho}(A)+S^{\rho}(B)\geqslant 2\log\frac{2}{\sup_{i,j}\|P^{A}(a_i)+P^{B}(b_j)\|}, \tag{9}$$

where $\|\,\|$ denotes the operator norm. Clearly the right side of relation (9) is independent of the state ρ and vanishes essentially only when both A and B have a common eigenvector—i.e., when $S^{\rho}(A)$ and $S^{\rho}(B)$ can both be made arbitrarily small by a suitable choice of the state ρ. Thus (9) gives an absolute lower bound on the uncertainty $S^{\rho}(B)$ given the uncertainty $S^{\rho}(A)$, whatever be the state ρ of the system.

Further when the spectra of both A, B are totally nondegenerate, then (9) reduces to the following more revealing form, originally derived by Deutsch (1983)

$$S^{\rho}(A)+S^{\rho}(B)\geqslant 2\log\frac{2}{1+\sup_{i,j}|\langle a_i|b_j\rangle|}. \tag{10}$$

where $\{|a_i\rangle\}$ are the normalised eigenvectors of A and $\{|b_j\rangle\}$ the normalised eigenvectors of B.

3. Entropic formulation of uncertainty relations: Canonically conjugate observables

It is indeed curious to note that an entropic formulation of the position-momentum uncertainty relation has been available in the literature for quite some time. Such a relation was conjectured more than two decades ago by Everett (1973) and independently by Hirschmann (1957) and was proved almost a decade back by Bialynicki-Birula and Mycielski (1975) and independently by Beckner (1975). If $\psi(x)$ is the wave function of a particle in one dimension and $\tilde{\psi}(k)$ is its Fourier transform given by

$$\tilde{\psi}(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-ikx)\,\psi(x)\,dx. \tag{11}$$

then the relation conjectured by Everett and Hirschmann is normally written in the form

$$-\int_{-\infty}^{\infty}|\psi(x)|^2\log|\psi(x)|^2\,dx-\int_{-\infty}^{\infty}|\tilde{\psi}(k)|^2\log|\tilde{\psi}(k)|^2\,dk\geqslant 1+\log\pi. \tag{12}$$

Here the two terms on the left side of (12) can be taken to be the entropies or the uncertainties in position and wave number respectively, but for the fact that they are dimensionally ill-defined. This is in fact a general problem which seems to arise with all the well-known information-theoretic definitions of the entropy of a continuous probability distribution. For instance if $p(x)$ is the probability density associated with a continuous random quantity x which has physical dimension D, then the entropy $S(x)$ as is usually defined (McElice 1977)

$$S(x)=-\int p(x)\log p(x)\,dx \tag{13}$$

has inadmissible physical dimension $\log 1/D$. This can be circumvented by defining the so-called exponential entropy $\varepsilon(x)$ given by (Padma 1984; Srinivas 1985)

$$\varepsilon(x) = \exp\{-\int p(x)\log p(x)\,\mathrm{d}x\}, \tag{14}$$

which is a monotonic function of $S(x)$, but also has physically admissible dimension D.

In terms of the exponential entropy, the Everett-Hirschman relation can be written in the form

$$\varepsilon^{\psi}(Q)\varepsilon^{\psi}(K) \geqslant \pi e. \tag{15}$$

Since the exponential entropy of momentum $\varepsilon^{\psi}(P)$ can be easily seen to be $\hbar\varepsilon^{\psi}(K)$, we get the (exponential) entropic form of the position-momentum uncertainty relation

$$\varepsilon^{\psi}(Q)\varepsilon^{\psi}(P) \geqslant \pi\hbar e. \tag{16}$$

It should firstly be noted that the above relation (16) is valid for any pair of canonically conjugate observables. What is of further interest is the fact, first demonstrated by Everett himself, that the entropic form (16) of the position-momentum uncertainty relation is stronger than the conventional Heisenberg relation

$$(\Delta^{\psi}Q)(\Delta^{\psi}P) \geqslant \tfrac{1}{2}\hbar. \tag{17}$$

This fact that (17) is indeed a consequence of (16) follows from the following inequality (Everett 1973; Bialynicki-Birula and Mycielski 1975) which holds between the variance and the exponential entropy for any observable A which has a purely continuous spectrum ranging over the whole real line

$$(\Delta^{\psi}A) \geqslant \frac{1}{(2\pi e)^{1/2}}\varepsilon^{\psi}(A). \tag{18}$$

Finally we may also note that the entropic formulation of the uncertainty relation for canonically conjugate observables can be extended to give a meaningful angle-angular momentum uncertainty relation (Bialynicki-Birula and Mycielski 1975). Further, phenomena like the spreading of a wave packet, etc can be much better discussed in terms of the entropic formulation (Grabowski 1981) than in terms of variances.

4. Uncertainty relations for successive measurements

We saw in the previous two sections that an entropic formulation of the uncertainty relation is available for both purely discrete and canonically conjugate observables and that in both cases the entropic uncertainty relation is much superior to the conventional Heisenberg relation. We shall now show that the entropic formulation can be extended so as to provide an elegant and satisfactory uncertainty relation for the case of successive measurements also.

The important feature that is common to both the relations (1) and (9) is that the quantities $(\Delta^{\rho}A)$, $(\Delta^{\rho}B)$ or $S^{\rho}(A)$, $S^{\rho}(B)$ refer to the uncertainties in the outcomes of A-measurement and B-measurement which are performed on identical but distinct or different ensembles of systems. If we want to give some content to the conventional wisdom associated with the uncertainty relations (that they indicate how there is an interference due to one measurement on the outcomes of another) then we have to obtain a new set of uncertainty relations which refer to a different experimental situation—where an ensemble of systems prepared in the density operator state ρ is first subjected to an A-measurement and the same ensemble is later subjected to a B-measurement. Such an 'uncertainty relation for successive measurements' was derived some time ago (Gnanapragasam and Srinivas 1979) employing, as usual, variance as a measure of uncertainty. Recently an entropic formulation of the 'uncertainty relation for successive measurements' has been obtained (Srinivas 1985) and this has considerably clarified some of the conventional wisdom associated with the uncertainty relations as being an expression of the interference between quantum mechanical measurements.

Consider two observables A, B as before with a totally discrete spectrum and spectral resolution as given by (8a) and (8b). Now if an ensemble of systems prepared in density operator state ρ is first subjected to an A-measurement and later to a B-measurement, then the joint probability that the outcome a_i is obtained in the first measurement and b_j is obtained in the second measurement, is given by (Wigner 1963, Srinivas 1975).

$$\mathrm{Pr}^{\rho}_{A,B}(a_i, b_j) = \mathrm{Tr}(P^B(b_j)P^A(a_i)\rho P^A(a_i)P^B(b_j)) \tag{19}$$

where we have assumed that we are working in the Heisenberg picture of time evolution. If we now employ the joint probability (19) for evaluating the variances $(\Delta^{\rho}A)^2_{A,B}$ and $(\Delta^{\rho}B)^2_{A,B}$ of the A-measurement and B-measurement respectively (in a situation when an ensemble of systems in state ρ is subjected to the sequence of measurements A, B) then we obtain the uncertainty relation (Gnanapragasam and Srinivas 1979)

$$(\Delta^{\rho}A)^2_{A,B}(\Delta^{\rho}B)^2_{A,B} \geqslant |\mathrm{Tr}(\rho A\varepsilon^A(B)) - \mathrm{Tr}(\rho A)\,\mathrm{Tr}(\rho\varepsilon^A(B))|^2, \tag{20}$$

where

$$\varepsilon^A(B) = \sum_i P^A(a_i)BP^A(a_i). \tag{21}$$

The uncertainty relation for successive measurements (20), being formulated in terms of variances, suffers from the same inadequacies as the Heisenberg relation (1). These disappear entirely once we choose the information-theoretic entropy as a measure of uncertainty. The natural definition of the entropics $S^{\rho}_{A,B}(A)$ and $S^{\rho}_{A,B}(B)$ will have to be in terms of the joint probability (19) as per the relations

$$S^{\rho}_{A,B}(A) = -\sum_i\left[\left\{\sum_j \mathrm{Pr}^{\rho}_{A,B}(a_i, b_j)\right\}\log\left\{\sum_j \mathrm{Pr}^{\rho}_{A,B}(a_i, b_j)\right\}\right], \tag{22a}$$

$$S^{\rho}_{A,B}(B) = -\sum_j\left[\left\{\sum_i \mathrm{Pr}^{\rho}_{A,B}(a_i, b_j)\right\}\log\left\{\sum_i \mathrm{Pr}^{\rho}_{A,B}(a_i, b_j)\right\}\right]. \tag{22b}$$

It can then be shown that we have the following entropic form of uncertainty relation for successive measurements (Srinivas 1985)

$$S^{\rho}_{A,B}(A) + S^{\rho}_{A,B}(B) \geqslant \log\frac{1}{\sup_{i,j}\|P^A(a_i)P^B(b_j)P^A(a_i)\|}. \tag{23}$$

It is easy to see that the right side of (23) (which, unlike in (20), is independent of state ρ) is non-negative and vanishes essentially only when A, B have a common eigenvector. Further, since it can be shown that

$$\frac{1}{\sup_{i,j} \| P^A(a_i) P^B(b_j) P^A(a_i) \|} \geqslant \left\{ \frac{2}{\sup_{i,j} \| P^A(a_i) + P^B(b_j) \|} \right\}^2 \tag{24}$$

the lower bound in (23) is indeed larger than the lower bound in the Deutsch-Partovi relation (9). In other words what we have shown is that, the lower bound on the 'overall uncertainty' is larger for the case of successive measurements (A-measurement followed by B-measurement) than for the case of distinct measurements (A-measurement and B-measurement performed on identical but distinct ensembles). This confirms the conventional wisdom that the interference of one measurement on another should contribute to a larger 'overall uncertainty' in the case of successive measurements.

5. An example

We can now come back to the example of the uncertainty relation between components of angular momentum. The Heisenberg relation as we wrote down earlier has the form

$$(\Delta^{\psi} J_x)^2 (\Delta^{\psi} J_z)^2 \geqslant \frac{\hbar^2}{4} |\langle \psi | J_y | \psi \rangle|^2. \tag{25}$$

For a spin $-\frac{1}{2}$ system it is easy to see that the right side vanishes for the state $|\psi\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ while the left side does not. On the other hand, we shall now show that the entropic form of the uncertainty relations really enables us to evaluate an absolute lower bound on the uncertainty in say J_z given the uncertainty in J_x.

For measurements of J_x and J_z performed on identical but distinct ensembles, we have to use the Deutsch-Partovi relation in the form (10), viz.

$$S^{\rho}(J_x) + S^{\rho}(J_y) \geqslant 2 \log \frac{2}{1 + \Delta_j}, \tag{26}$$

where

$$\Delta_j = \sup_{m_x, m_z} |\langle j m_x | j m_z \rangle|, \tag{27}$$

where $|jm_x\rangle$ are the eigenvectors of J_x and $|jm_z\rangle$ are the eigenvectors of J_z corresponding to angular momentum j. Δ_j can be found from the standard theory of rotation matrices (Brink and Satchler 1968) and can be shown to be given by

$$\Delta_j = \begin{cases} \left\{ \dfrac{(2j)!}{(j!)^2} \left(\dfrac{1}{2} \right)^{2j} \right\}^{1/2} & \text{for } j = 0, 1, 2, \ldots \\[2ex] \left\{ \dfrac{(2j+1)!}{((j+\frac{1}{2})!)^2} \left(\dfrac{1}{2} \right)^{2j+1} \right\}^{1/2} & \text{for } j = \frac{1}{2}, \frac{3}{2}, \ldots \end{cases}$$

For a spin $-1/2$ system, we get

$$S^{\rho}(J_x) + S^{\rho}(J_z) \geqslant 2 \log \frac{2}{1 + \frac{1}{\sqrt{2}}} \simeq \log 1{\cdot}37. \tag{29}$$

When we consider a situation where the same ensemble of systems is subjected to a sequence of measurements, first of J_x and then of J_z, then we should employ the uncertainty relation (23), which reduces in this case to

$$S^{\rho}_{J_x, J_z}(J_x) + S^{\rho}_{J_x, J_z}(J_z) \geqslant \log \frac{1}{\Delta_j^2}. \tag{30}$$

As we saw in relation (24), the lower bound $\log(1/\Delta_j^2)$ of (30) is in general larger than the lower bound $2\log[2/(1+\Delta_j)]$ of (26). For the case of a spin-1/2 system we get

$$S^{\rho}_{J_x, J_z}(J_x) + S^{\rho}_{J_x, J_z}(J_z) \geqslant \log 2 \tag{31}$$

which, when compared with the inequality (29), clearly shows what the quantum mechanical interference due to measurement is all about.

References

Beckner W 1975 *Proc. Natl. Acad. Sci. USA* **72** 638
Bialynicki-Birula I and Mycielski J 1975 *Commun. Math. Phys.* **44** 129
Brink D M and Satchler G R 1968 *Angular momentum* (Oxford: University Press)
Deutsch D 1983 *Phys. Rev. Lett.* **50** 631
Everett H 1973 in *The many-world interpretation of quantum mechanics* (eds) B S De Witt and N Graham (Princeton: University Press)
Gnanapragasam B and Srinivas M D 1979 *Pramana* **12** 699
Grabowski M 1978a *Rep. Math. Phys.* **14** 377
Grabowski M 1978b *Int. J. Theor. Phys.* **17** 635
Grabowski M 1981 *J. Math. Phys.* **22** 303
Heisenberg W 1927 *Z. Phys.* **43** 173
Hirschman I I 1957 *Am. J. Math.* **79** 152
Ingarden R S 1976a *Rep. Math. Phys.* **10** 43
Ingarden R S 1976b *Rep. Math. Phys.* **10** 131
Khinchin A I 1957 *Mathematical foundations of information theory* (New York: Dover)
Lindblad G 1973 *Commun. Math. Phys.* **33** 305
McElice R J 1977 *The theory of information and coding* (Boston: Addison Wesley)
Padma R 1984 *Entropic formulation of uncertainty relations* MSc Thesis (unpublished) Madras University
Partovi M H 1983 *Phys. Rev. Lett.* **50** 1883
Robertson H P 1929 *Phys. Rev.* **34** 163
Srinivas M D 1975 *J. Math. Phys* **16** 1676
Srinivas M D 1985 *Pramana* **24** 673
Von Neumann J 1955 *Mathematical foundations of quantum mechanics* (Princeton: University Press)
Wehrl A 1974 *Rep. Math. Phys.* **6** 15
Wehrl A 1978 *Rev. Mod. Phys.* **50** 221
Wigner E P 1963 *Am. J. Phys.* **31** 6