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A virial approach to soliton-like solutions of coupled non-linear differential equations including the 't-Hooft-Polyakov monopole equations

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Abstract. A virial theorem for solitons derived by Friedberg, Lee and Sirlin is used to reduce a system of second order equations to an equivalent first order set. It is shown that this theorem, when used in conjunction with our earlier observation that soliton-like solutions lie on the separatrix, helps in obtaining soliton-like solutions of theories involving coupled fields. The method is applied to a system of equations studied extensively by Rajaraman. The 't-Hooft-Polyakov monopole equations are then studied and we obtain the well-known monopole solutions in the Prasad-Sommerfeld limit ($\lambda = 0$); for the case $\lambda \neq 0$, we succeed in obtaining a non-trivial algebraic constraint between the fields of the theory.

Keywords. Nonlinear differential equation; virial theorem; solitons; monopole.

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1. Introduction

The study of non-linear differential equations (NLDES) has led to a phenomenal increase in their applications to problems of physical interest (Eilenberger 1981; Rajaraman 1982; Bullough and Caudrey 1982). This interest stems partly from the fact that some NLDEs have solutions which are identifiable as solitons. In an earlier paper (Malik *et al* 1983), we have shown how, in the case of equations involving only one field in 1 + 1 dimensions, one may obtain the soliton-like solutions in a systematic manner. Essentially, we used the observation that the soliton-like solutions of a given second-order NLDE define the separatrix of the equivalent autonomous system to reduce the original equation to a set of first-order equations. The present paper is a logical extension of our earlier work: we address ourselves to the problem of obtaining soliton-like solutions for coupled NLDEs. A celebrated example of the range of applications of such equations is the monopole solution. In the study of continuous chaos, we have another such example. In the latter case, one is interested in finding regions in parameter space where chaotic trajectories exist. In some cases (Neimark and Sil'nikov 1965; Sil'nikov 1965, 1969, 1970), these regions are known to start in the neighbourhood of homoclinic orbits—orbits which are precisely the ones on which the soliton-like solutions lie.

Our approach to the problem, again, is to reduce the given second-order coupled NLDEs of motion to first-order coupled equations of motion in a systematic manner. To do this we use the virial theorem for solitons as proved by Friedberg *et al* (1976) (FLS). Clearly, this method is applicable to systems which possess a Lagrangian. The virial theorem for solitons is shown to be identical to the equation for the separatrix for systems involving only one field in 1 + 1 dimensions. In higher dimensions, where it is not at all easy to write down the equation for the separatrix, the virial gives a constraint. This constraint, used along with the original equations of motion, then leads to the reduced equations of motion. We illustrate the procedure by obtaining all the solutions given by Rajaraman (1979) for the following set of coupled NLDEs:

$$\begin{aligned} \sigma'' &= -\sigma + \sigma^3 + d\rho^2\sigma \\ \rho'' &= f\rho + \lambda\rho^3 + d\rho(\sigma^2 - 1) \end{aligned} \tag{1}$$

Next, we deal with the equations describing the 't-Hooft (1974) and Polyakov (1974) theory of magnetic monopoles:

$$\begin{aligned} \xi^2 K''(\xi) &= KH^2 + K(K^2 - 1) \\ \xi^2 H''(\xi) &= 2K^2 H + \frac{\lambda}{e^2} H(H^2 - \xi^2) \end{aligned} \tag{2}$$

Specifically, we show how these equations may be solved in the Prasad-Sommerfeld (1975) limit without the use of the Bianchi identities and the Bogomolny bound. We also examine these equations in the limit $\lambda \neq 0$. In this case, we are able to obtain an algebraic relationship between the fields in the theory—a relationship which reduces to an identity in the Prasad-Sommerfeld limit. As before, this constraint, used along with the original equations of motion, leads to a set of first-order coupled NLDEs. While we have not been able to solve the reduced equations in the $\lambda \neq 0$ case, the fact that we succeed in obtaining a non-trivial algebraic relationship between the fields—hitherto unknown, does seem to signify a positive gain.

The paper is organized as follows. In §2, we rederive a virial theorem for solitons, originally derived by FLS. This is shown to be identical to the equation for the separatrix for 1 + 1 dimensional systems involving only one field. It is also observed that the equation for the separatrix for the case considered follows also from the requirement $H_F = 0$, where H_F is a suitably defined Hamiltonian for the motion of a fictitious particle. Section 3 is devoted to the set of equations (1): the virial theorem is used to obtain the various solutions that were earlier found by Rajaraman (1979) through his method of trial orbits. This section is closed with a brief comparison between Rajaraman's method and the procedure followed by us for solving simple coupled systems. In §4 we take up the monopole equations, given above as (2). The chief difference between this set and the set of equations (1) is that the former is a non-autonomous set. Using the virial approach, we show how this set may be solved in the Prasad-Sommerfeld limit, *i.e.*, when $\lambda = 0$. For this purpose, we do not use the Bianchi identities and the Bogomolny bound. The set of equations (2) is also examined for $\lambda \neq 0$. In this case, we succeed in reducing it to an equivalent set of first-order coupled NLDEs, equations (48), subject to an algebraic constraint between the fields, equations (49). Both, in §§3 and 4, in a heuristic spirit, we also use the condition $H_F = 0$, mentioned above. Section 5 is devoted to a summing up of our findings.

2. Equivalence between a virial theorem for solitons and the equation of separatrix for 1 + 1 dimensional systems

For the sake of completeness, we rederive below a virial theorem for solitons, originally derived by FLS. We then establish its equivalence with the equation for the separatrix for a class of 1 + 1 dimensional systems.

Consider a system the dynamics of which is described by the Lagrangian density

$$\mathscr{L} = \tfrac{1}{2}(\partial\mu\phi)^2 - V(\phi). \tag{3}$$

For the static case, we have

$$L_s = \int d^3\mathbf{r}\,\mathscr{L}_s(\mathbf{r}) = \int d^3\mathbf{r}\,[-\tfrac{1}{2}(\nabla_r\phi)^2 - V(\phi)], \tag{4}$$

and the equation of motion is

$$\nabla_r^2\phi = \frac{dV}{d\phi}. \tag{5}$$

Following FLS, let

$$\phi(\mathbf{r}) \to \phi(\mathbf{r}') \qquad \text{where} \qquad \mathbf{r}' = \lambda\mathbf{r}, \tag{6}$$

then

$$L_s \to \bar{L}_s = \int d^3\mathbf{r}'\,\bar{L}_s(\mathbf{r}')$$

$$= \int d^3\mathbf{r}'\left[-\frac{1}{2\lambda}(\nabla_{r'}\phi)^2 - \frac{1}{\lambda^3}V(\phi)\right]. \tag{7}$$

We now demand that

$$\left.\frac{\delta\bar{L}_s}{\delta\lambda}\right|_{\lambda=1} = 0. \tag{8}$$

A solution of (8) is

$$\tfrac{1}{2}\nabla^2\phi = -3V(\phi). \tag{9}$$

In the subsequent sections, the constraint following from the use of the virial theorem is similarly derived.

In one space dimension, the analogue of (9) would be

$$\tfrac{1}{2}(d\phi/dx) = V(\phi). \tag{10}$$

We note that for soliton-like solutions as $x \to \pm\infty$, $\phi(x) \to$ minima of $V(\phi)$. We can, without loss of generality, assume the minimum value of $V(\phi)$ to be 0.

Let us now reconsider (5) in one space dimension:

$$\frac{d^2\phi}{dx^2} = \frac{dV}{d\phi}, \tag{11}$$

which is equivalent to the following autonomous system

$$\begin{aligned}\phi_x &= \eta\\ \eta_x &= \frac{dV}{d\phi}\end{aligned} \tag{12}$$

If the critical points of the system (12) are saddle points, the separatrix of the system is easily seen to be

$$\tfrac{1}{2}(\mathrm{d}\phi/\mathrm{d}x)^2 = V(\phi) \tag{10'}$$

Thus, the equivalence between the virial theorem for solitons and the equation of separatrix for a class of 1 + 1 dimensional system is established.

To summarize, given a set of second-order non-linear differential equations, we may use the virial theorem to obtain an equivalent first-order set, as a first step towards obtaining solition-like solutions. For autonomous systems, the constraint obtained through the application of the virial theorem is seen to be equivalent to the vanishing of a certain fictitious Hamiltonian:

$$H_F = T_F + V = 0, \tag{13}$$

where

$$T_F = -\tfrac{1}{2}(\partial\phi/\partial x)^2 \tag{14}$$

and H_F is the Hamiltonian describing the motion of a fictitious unit-mass particle along the ϕ-axis, with x playing the role of "time". Further, for autonomous systems involving only one field, the constraint is, in fact, the equation of the separatrix.

For non-autonomous systems, one finds that the constraint arising from the use of virial theorem is different from that obtained by demanding that the fictitious Hamiltonian vanishes. Consequently, we use only the constraint from the virial theorem in order to effect simplification in the given set of second-order differential equations. In a heuristic spirit, however, one may use (13) in conjunction with the virial theorem. We show in later sections that this usage is not inconsistent with the virial theorem.

3. Soliton-like solutions of model field theory involving two scalar fields in 1 + 1 dimensions

We shall now apply the procedure outlined in § 2 to the set of equations given in (1). A restricted class of solutions for this set has been obtained earlier by Montonen (1976). This system has also been studied by Rajaraman (1979), who, through his method of trial orbits, succeeded in considerably enlarging the class of solutions given by Montonen. In what follows, we indicate how some additional solutions, not listed by Rajaraman, may be found.

As noted in Rajaraman, the set of equations (1) may be obtained from the Lagrangian

$$\begin{aligned} L = \int \mathrm{d}x\,[&\tfrac{1}{2}(\partial_\mu\sigma)^2 + \tfrac{1}{2}(\partial_\mu\rho)^2 - \{\tfrac{1}{4}(\sigma^2-1)^2 + \tfrac{1}{2}f\rho^2 \\ &+ \tfrac{1}{4}\lambda\rho^4 + \tfrac{1}{2}\mathrm{d}\rho^2(\sigma^2-1)\}] \end{aligned} \tag{15}$$

in the static limit.

Following FLS, let

$$\sigma(x) \to \sigma(x');\ \rho(x) \to \rho(x') \text{ where } x' = \mu x$$

then, for the static case

$$L_s \to \bar{L}_s = \int \mathrm{d}x' \mathscr{L}_s(x')$$

$$= \int \mathrm{d}x' [-\tfrac{1}{2}\mu(\partial\sigma/\partial x')^2 - \tfrac{1}{2}\mu(\partial\rho/\partial x')^2 - \{\tfrac{1}{4}(\sigma^2-1)^2 + + \tfrac{1}{2}f\rho^2 + \tfrac{1}{4}\lambda\rho^4 + \tfrac{1}{2}\mathrm{d}\rho^2(\sigma^2-1)\}] \tag{16}$$

If we now demand that

$$\left.\frac{\delta \bar{L}_s}{\delta\mu}\right|_{\mu=1} = 0. \tag{17}$$

we obtain

$$\tfrac{1}{2}(\partial\sigma/\partial x)^2 + \tfrac{1}{2}(\partial\rho/\partial x)^2 = \tfrac{1}{4}(\sigma^2-1)^2 + \tfrac{1}{2}f\rho^2 + \tfrac{1}{4}\lambda\rho^4 + \tfrac{1}{2}\mathrm{d}\rho^2(\sigma^2-1) \equiv g(\rho,\sigma). \tag{18}$$

which is the constraint following from the virial theorem. We note that the constraint following from the procedure in (13) is identical with the above constraint.

Differentiating the constraint in (18), we get.

$$\sigma'\sigma'' + \rho'\rho'' = \sigma'(\sigma^3 - \sigma + \mathrm{d}\rho^2\sigma) + \rho'[\lambda\rho^3 + f\rho + \mathrm{d}\rho(\sigma^2-1)], \tag{19}$$

which is trivially seen to be consistent with the original equations of motion. In general, this need not be so, as we shall see in the case of the equations describing the 't Hooft-Polyakov monopole.

The fact that (18) is consistent with the equations of motion, but is a first-order equation simplifies the task of solving the original equations. In order to obtain the solutions explicitly, we now need to split (18), so that we have separate first-order equations for σ and ρ. We then impose the requirement that these equations be consistent with the original second-order equations. This requirement, although trivial for the sum $\frac{1}{2}(\sigma')^2 + \frac{1}{2}(\rho')^2$, leads to a non-trivial relationship between σ and ρ, an orbit in Rajaraman's approach. It ought to be mentioned here that we have no guidelines for breaking up the constraint equation into two equations. This arbitrariness is possibly related to the fact that the equations under consideration allow for several soliton-like solutions to exist.

Consider now the following break up of the constraint equation (18):

$$(\rho')^2 = f\rho^2\sigma^2 \tag{20}$$

$$(\sigma')^2 = -f\rho^2\sigma^2 + \tfrac{1}{2}(\sigma^2-1)^2 + f\rho^2 + \tfrac{1}{2}\lambda\rho^4 + \mathrm{d}\rho^2(\sigma^2-1). \tag{21}$$

Differentiating (20) with respect to x, using (1) for ρ'' and squaring, we obtain

$$f^2\rho^2\sigma^2(\sigma')^2 = [f + \lambda\rho^2 + \mathrm{d}(\sigma^2-1) - f\sigma^2]^2(\rho')^2 \tag{22}$$

Substituting (20) and (21) into (22), we have

$$\frac{\lambda}{2}(f-2\lambda)\rho^4 - (1-\sigma^2)[-f^2 + fd + 2\lambda(f-d)]\rho^2 + (1-\sigma^2)^2[\tfrac{1}{2}f - (f-d)^2] = 0. \tag{23}$$

Equation (23) may be solved for ρ^2 in terms of σ^2, to yield

$$\rho^2 = \alpha(1-\sigma^2) \tag{24}$$

where

$$\alpha = \frac{1}{\lambda}\left[(d-f) \pm \left\{\frac{f^2(f-2d)+f(d^2-\lambda)}{(f-2\lambda)}\right\}^{1/2}\right]$$

Equation (24) corresponds to type-B orbit in Rajaraman's (1979) approach. Explicit solutions for σ and ρ may now be easily obtained: substituting (24) into (21), we get

$$(\sigma')^2 = \beta(1-\sigma^2)^2, \tag{25}$$

where

$$\beta = \tfrac{1}{2} - \alpha d + \alpha f + \tfrac{1}{2}\lambda\alpha^2.$$

Thus

$$\sigma(x) = \tanh(\sqrt{\beta}\cdot x). \tag{26}$$

Substituting (26) and (24) into the original second-order equations, we find that

$$\left.\begin{aligned} &\beta = f \\ \text{and}\quad & \\ &\lambda = \frac{d(d-2f)}{(1-2f)} \end{aligned}\right\}; \tag{27}$$

these restrictions on the parameters are identical to those obtained by Rajaraman (1979).

Consider another break up of (18):

$$\frac{1}{2}(\sigma')^2 = \frac{g(\rho,\sigma)}{(A+B\rho)} \tag{28}$$

$$\frac{1}{2}(\rho')^2 = \frac{(A+B\rho-1)g(\rho,\sigma)}{(A+B\rho)} \tag{29}$$

From the last two equations, we obtain

$$\frac{d\sigma}{d\rho} = \frac{1}{(A+B\rho-1)^{1/2}}, \tag{30}$$

or

$$\rho = \frac{1}{4}B\sigma^2 - \frac{1}{B}(A-1). \tag{31}$$

Consistency of the above break up with the original equations now leads to

$$\lambda = \frac{8d}{3};\ A-1 = \frac{2(d-3)}{d};\ f=2;$$

$$B^2 = \frac{8(d-3)}{d}. \tag{32}$$

These lead to the same orbit and solutions as given by Rajaraman.

The type-A orbit of Rajaraman (1979) may be obtained through a break up of (18) in which

$$(\sigma')^2 = \frac{A(\sigma+\delta)(\alpha - r\sigma)g(\rho,\sigma)}{A(\sigma+\delta)(\alpha - r\sigma) + B(L+M\sigma)^2}; \tag{33}$$

this leads to

$$\rho^2 = \frac{4B}{A}(\sigma+\delta)(\alpha - r\sigma), \text{ etc.} \tag{34}$$

An example of a solution not listed in Rajaraman is obtained through the break up

$$(\sigma')^2 = \alpha\rho^2\sigma^2; \tag{35}$$

we now have

$$f = 2, \qquad \lambda = (d-2)^2,$$

$$\sigma = \frac{2a_1}{(b_1^2 - 4a_1c_1)^{1/2}\cosh(\sqrt{a_1}\cdot x) + b_1} - a, \tag{36}$$

where

$$a_1 = (4a^2 - 2),\ b_1 = \tfrac{8}{3}a,\ c_1 = \tfrac{4}{3};$$

and a satisfies

$$\tfrac{4}{3}a^3 - 2a \pm \tfrac{2}{3} = 0,$$

where the $\pm$ signs correspond to two different homoclinic orbits, approaching, asymptotically. $\rho = 0$, $\sigma = \pm 1$

ρ is now obtained from the orbit

$$\sigma\left(1 - \frac{\sigma^2}{3} + \rho^2\right) = \pm\frac{2}{3}. \tag{37}$$

We note that the above solutions are valid for a different range of parameter values than that given by Rajaraman.

It is thus seen that when one compares the virial approach to the approach of Rajaraman, an element of arbitrariness is common to both approaches. In the latter, one has to choose suitable trial orbits, whereas in the former, one has to choose a suitable break up of the constraint following from the virial. For this set of equations the recipe, $H_F = 0$, carries the same information as the virial constraint, and hence cannot be exploited to reduce the element of arbitrariness in the break up. However, as we shall see in the next section, when studying non-autonomous systems the virial approach can be used quite profitably whereas the method of choosing trial orbits appears to be difficult.

4. The 't Hooft-Polyakov monopole equations

The search for soliton-like solutions in dimensions higher than 1 + 1 takes us beyond theories containing only scalar fields. A simple example of such a theory is provided by the following Lagrangian, which describes the interaction between an SO(3) gauge field and an isovector Higgs field $\boldsymbol{\phi}$ in $(3+1)$ dimensions

$$\mathscr{L} = -\tfrac{1}{4}\mathbf{G}_{\mu\nu}\cdot\mathbf{G}^{\mu\nu} + \tfrac{1}{2}\mathscr{D}_\mu\boldsymbol{\phi}\cdot D_\mu\boldsymbol{\phi} - \frac{\lambda}{4}(\phi^2 - a^2)^2 \tag{38}$$

where

$$\mathbf{G}^{\mu\nu} = \partial^\mu \mathbf{W}^\nu - \partial^\nu \mathbf{W}^\mu - e\mathbf{W}^\mu \times \mathbf{W}^\nu,$$

$$\mathscr{D}_\mu\boldsymbol{\phi} = \partial_\mu\boldsymbol{\phi} - e\mathbf{W}^\mu \times \boldsymbol{\phi}$$

and $\mathbf{W}^\mu$ is the gauge potential.

An ansatz which leads to the 't-Hooft-Polyakov monopole solution is

$$\phi_a = \frac{r_a}{er^2} H(aer)$$

$$W_a^i = -\frac{\varepsilon_{aij}}{er^2}[1 - K(aer)] \tag{39}$$

and

$$W_a^a = 0,$$

with the following boundary conditions on H and K (for details, see the lucid review by Goddard and Olive 1978):

$$\begin{array}{lll} (K-1) \leqslant 0(\xi) & H \leqslant 0(\xi) & \text{as } \xi \to 0 \\ K \to 0 & H \sim \xi & \text{sufficiently fast} \\ (\xi \equiv aer) & & \text{as } \xi \to \infty. \end{array} \tag{40}$$

The Lagrangian now becomes

$$L = \int d^3r\mathscr{L}$$

$$= \frac{4\pi a}{e}\int_0^\infty \frac{d\xi}{\xi^2}[\xi^2 K_\xi^2 + \tfrac{1}{2}(\xi H_\xi - H)^2 + \tfrac{1}{2}(K^2-1)^2 + \frac{\lambda}{4e^2}(H^2-\xi^2)^2 + K^2H^2], \tag{41}$$

and the equations of motion are

$$\xi^2 K_{\xi\xi} = KH^2 + K(K^2-1)$$

$$\xi^2 H_{\xi\xi} = 2HK^2 + \frac{\lambda}{e^2}H(H^2-\xi^2)^2 \tag{42}$$

It is seen from the boundary conditions in (40) that the field H does not behave as a soliton. Consequently, we cannot apply the FLS procedure directly to the Lagrangian in (41) to obtain the virial constraint. We introduce

$$J = \frac{H}{\xi}, \tag{43}$$

so that

$$L = \frac{4\pi a}{e} \int_0^\infty \frac{d\xi}{\xi^2} \left[\xi^2 K_\xi^2 + \tfrac{1}{2}\xi^4 J_\xi^2 + \tfrac{1}{2}(K^2-1)^2 + \right.$$

$$\left. + \xi^2 K^2 J^2 + \frac{\lambda}{4e^2} \xi^4 (J^2-1)^2 \right], \tag{44}$$

the FLS procedure now yields the following constraint

$$\xi^2 K_\xi^2 - \tfrac{1}{2}\xi^4 J_\xi^2 + \tfrac{1}{2}(K^2-1)^2 - K^2 J^2 \xi^2 - 3\Lambda \xi^4 (J^2-1)^2 = 0, \tag{45}$$

i.e.,

$$\xi^2 K_\xi^2 - \tfrac{1}{2}(\xi H_\xi - H)^2 - K^2 H^2 + \tfrac{1}{2}(K^2-1)^2 - 3\Lambda (H^2 - \xi^2) = 0 \tag{45a}$$

where

$$\Lambda = \frac{\lambda}{e^2}.$$

Let us now consider the Prasad-Sommerfeld limit, *i.e.*, $\lambda = 0$. A possible break up of (45a) is

$$\xi^2 K_\xi^2 = K^2 H^2$$

$$(\xi H_\xi - H)^2 = (K^2-1)^2 \tag{46}$$

These equations are consistent with the original second-order equations without any further constraint. We note that (46) have been obtained earlier by using the Bianchi identities and the Bogomolny bound, and have been solved analytically.

The question now naturally arises whether (46) could have been obtained in a unique manner. It is interesting to point out that (13), when applied to the Lagrangian in (44), leads to

$$\xi^2 K_\xi^2 + \tfrac{1}{2}(\xi H_\xi - H)^2 = \tfrac{1}{2}(K^2-1)^2 + K^2 H^2 \tag{47}$$

where we have used (43). If we now combine the last equation with the virial constraint, *i.e.*, (45a), we obtain (46) in a unique manner.

For the case when $\lambda \neq 0$, analytic solutions for the set of equations (42) have not yet been found, nor have these equations been reduced to an equivalent set of first-order equations. In our approach, if we are to be guided by the virial theorem alone, it is clear that any break up of (45a) will pose the problem of non-uniqueness in the equivalent set of first-order equations that it will lead to. Nevertheless, the following break up (45a) seems reasonable:

$$\begin{aligned} \xi^2 K_\xi^2 &= K^2 H^2 + \alpha\Lambda (H^2-\xi^2)^2 \\ \tfrac{1}{2}(\xi H_\xi - H)^2 &= \tfrac{1}{2}[(K^2-1)^2 + 2\beta\Lambda(H^2-\xi^2)^2] \end{aligned} \tag{48}$$

Consistency of the above break up with the original second-order equations now leads to

$$\alpha = 2; \quad \beta = -1$$

and

$$\begin{aligned} 2\Lambda(H^2-\xi^2) + [(K^2-1)^2 - 2\Lambda(H^2-\xi^2)^2]^{1/2}[HK^2 + 4\Lambda H(H^2-\xi^2)] \\ = K(K^2-1)[H^2K^2 + 2\Lambda(H^2-\xi^2)^2]^{1/2} \end{aligned} \tag{49}$$

Equation (49) is an algebraic constraint among H, K and ξ. When $\lambda = 0$, it reduces to an identity, as it should. Thus, the given set of equations (42) have been reduced to an equivalent set of first-order equations, subject to the above algebraic constraint. This, we feel, is a positive gain, even though we have not been able to solve the first-order equations that we obtained.

We turn now to the question of uniqueness of the break up of (45a) into (48). It is now natural to invoke the recipe of (13); we obtain, from (44) and (43)

$$\xi^2 K_\xi^2 + \tfrac{1}{2}(\xi H_\xi - H)^2 = K^2 H^2 + \tfrac{1}{2}(K^2 - 1)^2 + \Lambda (H^2 - \xi^2)^2 \tag{50}$$

If we now combine the last equation with the virial constraint, *i.e.*, (45a), we are lead to (48) in a unique manner (with $\alpha = 2$ and $\beta = -1$).

Thus, we see that using the virial leads to a simplification in the equations of motion of even non-autonomous systems. Further, it is interesting to note that, in the present case, the recipe $H_F = 0$ carries information which is different from that embodied in the constraint following from the application of the variational principle for solitons. When both these equations are used, the break up is achieved in a unique manner. Whether or not $H_F = 0$, in general, may also be a recipe for obtaining the equation for the separatrix of more complicated systems in higher dimensions is an open question.

5. Discussion

We had observed earlier that the soliton-like solutions of a given second-order NLDE involving only one field in $1+1$ dimensions may be obtained systematically from the equation for the separatrix of an equivalent set of first-order equations (Malik *et al* 1983). As a logical next step, we have tried in this paper to tackle the problem of finding soliton-like solutions of coupled second-order NLDEs in higher dimensions.

Our approach to the problem has been to try to reduce the original second-order equations to an equivalent set of first-order equations. The sets of coupled NLDEs we have dealt with are given in (1) and (2). In the separatrix-approach, one might try to cast each of these sets into an equivalent set of four first-order NLDEs. One would then immediately see how difficult it is to calculate the equation for the separatrix of such a system. This is a general feature for coupled systems in higher than $1+1$ dimensions. For this reason, we have relied on a variational principle for solitons given by FLS to bring about the desired reduction. The necessary virial theorem has been rederived for completeness of presentation. For systems involving only one field in $1+1$ dimensions, we find that it has the same content as the equation for the separatrix of an equivalent system. It is also observed that, formally, the equation for the separatrix of such a system may be obtained from the requirement that the Hamiltonian of an equivalent fictitious system should vanish ($H_F = 0$). The recipe $H_F = 0$ is applied, in a heuristic spirit, to complex systems involving more than one field in $1+1$ dimensions, even though the equivalence of the equation following from the use of this recipe and the equation for the separatrix has been established only for a system in $1+1$ dimension involving only one field.

For each of the sets of equations (1) and (2), the application of the variational principle for solitons yields a constraint involving only first-order derivatives of the fields. This virial constraint is split into two first-order equations, subject to consistency with the original equations. The last requirements leads to a non-trivial algebraic relationship between the fields in each case. For the set of equations (1), we then reobtain the various solutions that were earlier found by Rajaraman and indicate how some additional solutions may be found. It is pointed out that the chief limitation of our approach is the arbitrariness in splitting the virial-constraint, which is, indeed, akin to the limitation of Rajaraman's approach where there is an arbitrariness in choosing the trial orbits.

For the set of equations (2), which are non-autonomous, it is extremely difficult to guess a trial orbit. However, the virial-constraint can be calculated easily and, as mentioned above, we can obtain a set of first-order equations subject to an algebraic constraint between the fields. The case $\lambda = 0$ of these equations has been discussed earlier in the literature, where use is made of the Bianchi identities and the Bogomolny bound to reduce them to first-order equations; for the case $\lambda \neq 0$, such a reduction has not been achieved. Thus, (48) and (49) seem to be new results of this paper, which might be used to gain further insight into the monopole solutions when $\lambda \neq 0$.

We conclude by reemphasizing that eqs (48) and (49) can be obtained without the use of the recipe, $H_F = 0$, but with a certain amount of arbitrariness.

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