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An isotropic cosmological model in general scalar tensor theory

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Abstract. An isotropic homogeneous cosmological model with Robertson-Walker line element is studied in general scalar tensor theory where the parameter ω is a function of the scalar field. The model consists of perfect fluid with the equation of state $p = \varepsilon\rho$. Exact solutions are obtained in Dicke's conformally transformed units for $\varepsilon = 1$ and $\varepsilon = 1/3$ assuming a functional relationship between ω and the scalar field ϕ. The properties are compared with vacuum models in this theory.

Keywords. Cosmology; isotropic model; general scalar tensor theory.

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1. Introduction

In the general scalar tensor theory proposed by Nordtvedt (1970), the parameter ω is taken to be a function of the scalar field ϕ instead of being a constant. In this variable ω formalism, many a cosmological solution is already available. Isotropic (Bishop 1976; Barker 1978; Van den Bergh 1983) and anisotropic (Banerjee and Santos 1981a, b) models have been studied.

In the present paper, we investigate a homogeneous and isotropic cosmological model in the framework of Nordtvedt. Special attention is paid to Schwinger's (1970) scalar-tensor theory which was later shown to be a special case of Nordtvedt's theory with $(2\omega + 3) = 1/(\alpha\phi)$ where α is a constant (Van den Bergh 1982). In this theory, vacuum isotropic models have been discussed by Schwinger (1970) and Kimball and Yee (1974). We study this model in the presence of matter in the form of a perfect fluid with an equation of state connecting pressure p and density ρ. Two cases are considered, one with an equation of state $p = \rho$, representing a stiff fluid, and the other with $p = (1/3)\rho$, representing a radiation universe. The choice of non-zero p and ρ seems to be a better description of the state than the *a priori* assumption that the early universe is scalar-dominated.

We use Dicke's conformally transformed units (Dicke 1962), where G remains fixed and only the particle masses vary. Though less appealing compared to the atomic units where masses remain fixed, it makes the calculations much simpler. In this framework, standard Einstein equations are satisfied with the scalar field playing the role of an additional material source. It is found that at very early stages of the evolution, the energy density due to the scalar field dominates over the matter density for both $p = \rho$ and $p = (1/3)\rho$. Moreover, it is observed that the solution for the metric in the stiff fluid case is similar to the case of the vacuum model. Very recently, the radiation universe has been studied by Van den Bergh (1983). In this paper, Van den Bergh's results are confirmed in Dicke's revised units.

In § 2, we present the field equations in the revised units. In § 3 we integrate them in the case of a stiff fluid and study the nature of the singularity at the initial epoch in this theory. In § 4, we briefly indicate the behaviour of the model if the universe is filled with radiation.

2. Field equations

Nordtvedt field equations in Dicke's conformally transformed units are given by

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \tfrac{1}{2} R g_{\alpha\beta}$$
$$= -T_{\alpha\beta} - \frac{(2\omega+3)}{2}\frac{1}{\phi^2}(\phi_{,\alpha}\phi_{,\beta} - \tfrac{1}{2}\phi^{,\mu}\phi_{,\mu}g_{\alpha\beta}), \tag{1}$$

and the wave equation for the scalar field is

$$\Box(\ln\phi) \equiv (\ln\phi)^{;\mu}_{;\mu} = \frac{1}{(2\omega+3)}\left[T - \frac{1}{\phi}\phi^{,\mu}\phi_{,\mu}\frac{d\omega}{d\phi}\right]. \tag{2}$$

Here $8\pi G_0$ and c are taken to be unity. For a perfect fluid, the energy momentum tensor $T_{\alpha\beta}$ is given by

$$T_{\alpha\beta} = (\rho + p)v_\alpha v_\beta - p g_{\alpha\beta} \tag{3}$$

where ρ and p are matter density and pressure respectively and v_α is the component of the four-velocity.

The Robertson-Walker line element for an isotropic homogeneous cosmological model is

$$dS^2 = dt^2 - \frac{R^2(t)}{(1+kr^2/4)}(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\Phi^2), \tag{4}$$

where the parameter k can assume the values 0 or ± 1 and R is a function of time t alone. By choosing co-moving coordinates and suitably normalizing v^μ, one can have $v^\mu v_\mu = 1$ and $v^\mu = \delta^\mu_0$.

Consider only the case when $k = 0$. With this, the non-trivial field equations are

$$\frac{3\dot{R}^2}{R^2} = \rho + \rho_\phi, \tag{5}$$

$$\frac{\dot{R}^2}{R^2} + \frac{2\ddot{R}}{R} = -(p + p_\phi), \tag{6}$$

where $\rho_\phi = p_\phi = \dfrac{(2\omega+3)}{4}(\dot{\phi}/\phi)^2$ represent the energy density and pressure due to the scalar field (Raychaudhuri 1979). A dot represents differentiation with respect to t. The wave equation (2) becomes

$$(\ln\phi)^{\cdot\cdot} + \frac{3\dot{R}}{R}(\ln\phi)^{\cdot} = \frac{1}{(2\omega+3)}\left[(\rho - 3p) - \frac{1}{\phi}\dot{\phi}^2\frac{d\omega}{d\phi}\right]. \tag{7}$$

Though not independent, a very useful relation is the non-trivial divergence equation

$$(\rho R^3) + 3pR^2\dot{R} + \frac{1}{2}\left(\frac{\dot{\phi}}{\phi}\right)R^3(\rho - 3p) = 0. \tag{8}$$

Equation (8) is identical with the corresponding relation in Brans-Dicke theory where ω is a constant parameter (Brans and Dicke 1961).

3. Behaviour of the model with a stiff fluid

Assume that the universe is filled with a perfect fluid with equation of state $p = \rho$. Adding (5) and (6) one obtains

$$\frac{2\ddot{R}}{R} + \frac{4\dot{R}^2}{R^2} = 0, \tag{9}$$

which readily integrates to

$$R^3 = at + b, \tag{10}$$

where a and b are constants of integration. Suitable choice of time origin will lead to $b = 0$ so that

$$R^3 = at. \tag{11}$$

Subtracting (6) from (5) one obtains

$$2(\rho + \rho_\phi) = 2\left(\frac{\dot{R}}{R}\right)^2 - \frac{2\ddot{R}}{R}. \tag{12}$$

Writing ψ for $\ln \phi$, we get from (7)

$$\ddot{\psi} + \frac{3\dot{R}}{R}\dot{\psi} = \frac{1}{(2\omega + 3)} \cdot [-2\rho - \dot{\omega}\dot{\psi}]. \tag{13}$$

Replacing 2ρ from (12) in (13) and using

$$\rho_\phi = \frac{(2\omega + 3)}{4}\left(\frac{\dot{\phi}}{\phi}\right)^2,$$

one can write

$$(2\omega + 3)\left[\ddot{\psi} + \frac{3\dot{R}}{R}\dot{\psi} + \frac{(2\omega + 3)^{\cdot}}{2(2\omega + 3)}\dot{\psi}\right] = \frac{(2\omega + 3)}{2}\dot{\psi}^2 + 2\left(\frac{\dot{R}}{R}\right)^{\cdot} \tag{14}$$

This equation can be integrated if the exact functional dependence of ω on ϕ is known. We shall use Schwinger's form of ω, where

$$2\omega + 3 = \frac{1}{\alpha\phi} = \frac{1}{\alpha}e^{-\psi}, \quad \alpha \text{ being a constant.}$$

With this choice and using (11), one can integrate (14) to obtain

$$\frac{1}{\phi} = -\frac{\alpha a}{3}(\ln t)^2 + p_1 \ln t + p_2, \tag{15}$$

where p_1 and p_2 are constants of integration. Therefore, from (11) and (15) one obtains the complete solution for the model with a stiff fluid.

Equation (15) shows that as $t \to 0$, $1/\phi$ explodes to infinity as $(\ln t)^2$ and hence ϕ goes to zero. Again, for $p = \rho$ equation (8) integrates to yield

$$\rho = A\phi/R^6, \tag{16}$$

where A is a constant of integration. Now R^6 goes to zero as t^2, *i.e.*, at a faster rate than $(\ln t)^2$, which indicates that the matter density becomes very large at the initial epoch. The energy density corresponding to the scalar field can be obtained from (5) and (11). One finds that

$$\rho_\phi = 3\left(\frac{\dot{R}}{R}\right)^2 - \rho = \frac{a^2}{3R^6} - \rho. \tag{17}$$

Near the singularity, ρ_ϕ goes to infinity as R^{-6}, *i.e.* as t^{-2}, which indicates that the energy density due to the scalar field dominates over the matter density at the initial epoch $t \to 0$. For substantially large R, however, the matter density ρ becomes more prominent and the scalar field becomes insignificant. Also, from (11) one finds that $\ddot{R}/R < 0$ and $\dot{R}/R \neq 0$ for any finite magnitude of t, *i.e.* there is no possibility for a bounce in this model. Equation (11) also enables one to find the Hübble constant (H) and the deceleration parameter (q),

$$H = \dot{R}/R = \frac{1}{3t}, \tag{18}$$

$$q = -\frac{\ddot{R}}{RH^2} = 2. \tag{19}$$

For empty space, *i.e.*, with $p = \rho = 0$, these expressions for the cosmological parameters H and q had been obtained earlier in Schwinger's scalar tensor theory (Kimball and Yee 1974). Here we obtain the same results with an equation of state $p = \rho$ without the *a priori* assumption that the early universe is scalar-dominated. Moreover, that ρ_ϕ dominates over ρ as $t \to 0$ comes as a consequence of the field equations.

4. Behaviour of the model with radiation

With $p = \frac{1}{3}\rho$, the wave equation (7) becomes independent of ρ and p whereas (8) becomes independent of ϕ. In other words, the divergence relations for the scalar field and matter become independent of each other (Raychaudhuri 1979). Van den Bergh (1983) used this property to obtain a solution for radiation universe in this model in atomic units where particle masses remain fixed.

Using $p = \frac{1}{3}\rho$, (7) and (8) can be integrated to yield

$$(2\omega + 3)^{1/2}\frac{\dot{\phi}}{\phi} = B/R^3, \tag{20}$$

and
$$\rho = C/R^4, \tag{21}$$

respectively, where B and C are constants of integration. From (20) one obtains

$$\rho_\phi = \frac{(2\omega+3)}{4}\left(\frac{\dot{\phi}}{\phi}\right)^2 = \frac{B^2}{4R^6}. \tag{22}$$

So we see that $\rho \propto R^{-4}$ and $\rho_\phi \propto R^{-6}$ and hence, when $R \to 0$, ρ_ϕ will dominate over ρ. But when R becomes large, the scalar field will become insignificant and matter density dominates over the former. These features, however, are already obtained in Van den Bergh's work.

We now proceed to find the exact solutions for this model in the revised units. Multiplying (6) by 3 and adding the result to (5) one obtains

$$\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{B^2}{12R^6} = 0. \tag{23}$$

Defining $u = \dot{R}$ and replacing the time derivatives by derivatives with respect to R, (23) yields

$$\frac{\mathrm{d}}{\mathrm{d}R}(u^2) + \frac{2}{R}(u^2) + \frac{B^2}{6}\cdot\frac{1}{R^5} = 0,$$

which has the solution

$$u^2 = \dot{R}^2 = \frac{D}{R^2} + \frac{B^2}{12R^4}, \tag{24}$$

where D is a constant of integration. From (24) one can obtain the solution for the metric,

$$t + E = \begin{cases} \frac{1}{2D}\{R(DR^2 + B^2/12)^{1/2}\} - \frac{B^2}{24D}\ln[RD^{1/2} + (R^2D + B^2/12)^{1/2}], & \text{when } D > 0; \\ \frac{1}{2D}\{R(DR^2 + B^2/12)^{1/2}\} - \frac{B^2}{24D}\left(-\frac{1}{D}\right)^{1/2}\sin^{-1}\left[R\left(-\frac{12D}{B^2}\right)^{1/2}\right] & \text{when } D < 0; \end{cases} \tag{25}$$

when $D = 0$, the behaviour is similar to the case of a stiff fluid,

$$R^3 = \frac{\sqrt{3}}{2}Bt + F. \tag{26}$$

Here E and F are constants of integration. With $(2\omega + 3) = 1/(\alpha\phi)$ (20) yields

$$-2/\sqrt{\alpha\phi} = \int \frac{B\,\mathrm{d}t}{R^3}. \tag{27}$$

Equations (25) to (27) provide the complete set of solutions for the radiation universe in Dicke's revised units. Equation (24) brings out an interesting feature of this model. When $R = R_0$ such that

$$D/R_0^2 = -B^2/12R_0^4, \tag{28}$$

one obtains $\dot{R} = 0$, *i.e.*, there is a bounce. This possibility is absent if one considers vacuum or a stiff fluid in this model. From (28), it is clear that the possibility of bounce is obtained only if $D < 0$. If $D \geqslant 0$, then $\dot{R} \neq 0$ and there is no maximum at any stage of the evolution with $R \to 0$ as the point of singularity.

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