

Exact energy eigenvalues by summation of the JWKB series

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Abstract. For the potential $V(x) = V_0 \tan^2 x$, the corrections to the lowest order JWKB (Bohr-Sommerfeld) energy quantization rule are non-zero. These higher order corrections are explicitly computed using the formalism of Dunham. The resulting JWKB series for the energy eigenvalues is summable, and yields the exact bound state spectrum.

Keywords. Higher order JWKB corrections; exact energies.

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1. Introduction

The Bohr-Sommerfeld quantization rule

$$\int_{x_1}^{x_2} dx [2m(E - V(x))]^{1/2} = (K + \frac{1}{2})\pi\hbar \quad (1)$$

provides a method of calculating approximately the energy eigenvalues of one-dimensional bound state problems. As shown in most texts on quantum mechanics (see, for instance, Landau and Lifshitz 1977), this formula results when the Schrödinger eigenvalue problem is solved in the lowest order of the JWKB approximation. It is an interesting fact that the above approximate formula yields the exact eigenvalue spectrum for the harmonic oscillator potential $V = V_0 x^2$ and the Morse potential $V = V_0 [\exp(-2ax) - 2\exp(-ax)]$. In these cases it turns out that all the higher order JWKB corrections vanish. There are certain other potentials for which the Schrödinger bound state problem can be solved exactly, but for which the exact energies are not reproduced by the lowest order quantization rule. It is then natural to investigate whether in such cases the JWKB analysis yields the exact spectrum when higher order corrections are included. To demonstrate that this is indeed true in the case of the exactly solvable potential $V(x) = V_0 \tan^2 x$ is the purpose of the present paper. For this potential we compute explicitly all the higher order corrections to (1), and show how the JWKB series can be summed to all orders to yield the exact energies.

For computing the higher order corrections, we find it most convenient to make use of the quantization formula of Dunham (1932) which is a generalization of (1) to all orders. The Dunham formula is particularly suited to this purpose, since it is formulated in terms of complex contour integration which makes all the higher order

JWKB integrals well-defined. For an illustration of this method, the work of Bender *et al* (1977) on the anharmonic oscillator potential $V = x^{2n}$ may be consulted.

2. Dunham's quantization formula

We briefly present the analysis of Dunham for completeness and to establish notation. Consider the Schrödinger eigenvalue problem

$$-\hbar^2 \psi'' + [V(x) - E]\psi = 0, \quad \psi(\pm \infty) = 0, \quad (2)$$

which corresponds to the bound motion of a particle of mass $m = 1/2$ in the potential $V(x)$. In the JWKB method of solving (2), one writes

$$\psi(x) = \exp \left[\frac{1}{\hbar} \int^x dx' \sum_{n=0}^{\infty} \hbar^n y_n(x') \right]. \quad (3)$$

Substituting into (2) and equating like powers of \hbar , one gets

$$y_0^2(x) = V(x) - E, \quad (4)$$

$$2y_0 y_n + \sum_{m=1}^{n-1} y_m y_{n-m} + dy_{n-1}/dx = 0, \quad n \geq 1. \quad (5)$$

The recurrence relation (5) yields all the y_n 's, once y_0 is determined from (4). For a potential with a single minimum (and hence with two classical turning points for physical bound states), Dunham showed that the bound state energies are determined by the quantization formula (with $\hbar = 1$)

$$\frac{1}{2i} \oint \sum_{n=0}^{\infty} y_n(z) dz = K\pi \quad (K = 0, 1, 2, \dots). \quad (6)$$

Here z is a complex variable with $\text{Re} z = x$, and the contour of integration encircles the two classical turning points, which are branch points of y_0 . A branch cut is introduced along the real axis between the turning points. All the y_n 's are then single-valued in the cut plane. For definiteness, the branch of y_0 that is negative imaginary on the upper lip of the cut is chosen. The contour must then be traversed in the positive sense.

For all odd $n > 1$, the y_n 's turn out to be total derivatives of single-valued functions, and hence their integrals over the closed contour vanish. For $n = 1$, one gets $y_1 = -y_0'/2y_0$, and its integral around the specified contour has the value $-\pi/2$. Therefore, all the y_n 's with odd n drop out of the sum in (6) which now becomes

$$\frac{1}{2i} \oint \sum_{n=0}^{\infty} y_{2n}(z) dz = (K + 1/2)\pi, \quad (7)$$

If the $n = 0$ term alone is retained in the above sum, one obtains the familiar lowest order quantization formula (1).

3. Application to the potential $V(x) = V_0 \tan^2 x$

We wish to determine the bound state energies of a particle acted on by the potential

$$V(x) = V_0 \tan^2 x, \quad V_0 > 0, \quad (8)$$

which confines the particle to move between $x = \pi/2$ and $x = -\pi/2$. Clearly, in the region $|x| < \pi/2$, this potential has a single minimum, and there are two classical turning points corresponding to any bound state. Further, the nature of the potential requires that the wave function ψ vanish at $x = \pm \pi/2$. The fact that in the present problem there is a classically allowed region flanked on both sides by forbidden regions enables the entire analysis of Dunham to be carried through, there being no change due to the imposition of boundary conditions at finite points, rather than at $\pm \infty$ as in the original derivation of (7). We may therefore apply the quantization formula (7) to compute the energy levels for the potential (8).

3.1 Expression for y_n

Using the recurrence relation (5), we first obtain an expression for the functions y_n which occur in (7). For this purpose it proves convenient to change variable to $t = \tan z$. Explicit evaluation gives

$$y_1 = y_0^{-2}(a_1 t^3 + a_2 t),$$

$$y_2 = y_0^{-5}(b_1 t^6 + b_2 t^4 + b_3 t^2 + b_4),$$

where the a 's and b 's are constants. These expressions suggest that we make the *ansatz*

$$y_n = y_0^i - 3n \sum_{s=0}^{[3n/2]} A_{n,s} t^{3n-2s}, \tag{9}$$

where $[3n/2] = 3n/2$ for even n , and $(3n-1)/2$ for odd n . It is not difficult to show by induction that this form for y_n solves the recurrence relation (5), provided that the constants $A_{n,s}$ satisfy the relation

$$2A_{n+1,s} = -V_0(1-2s)A_{n,s} - \sum_{m=1}^n \sum_{k=0}^s A_{n-m+1,s-k} A_{m,k}, \tag{10}$$

where $A_{0,0} = 1$. It turns out, as will be shown below, that in order to calculate the energies, we need to know only the coefficients $A_{2n,0}$. These are determined as follows: With $C_n = A_{n,0}$, we get from (10) the following recurrence relation:

$$2C_{n+1} = -V_0 C_n - \sum_{m=1}^n C_{n-m+1} C_m, \tag{11}$$

with $C_0 = 1$. It is easily checked that

$$C_1 = -V_0/2, \quad C_3 = C_5 = 0.$$

In fact, $C_{2n+3} = 0$ for all n , as can be verified by induction. Using this fact, (11) can be put in the form

$$\sum_{m=0}^k C_{2m} C_{2k-2m} = 0, \quad k \geq 2, \tag{12}$$

which involves only the even coefficients. The solution to (12) is

$$C_{2n} = a^n \binom{1/2}{n},$$

where a is an arbitrary constant. The value of a is determined so that the correct value of

$C_2 = V_0^2/8$ is reproduced. We get

$$a = (V_0/2)^2. \text{ Finally,}$$

$$C_{2n} = A_{2n,0} = (V_0/2)^{2n} \binom{1/2}{n}. \tag{13}$$

3.2 Evaluation of the integrals in (7)

In terms of the t variable, $y_0 = -(V_0 t^2 - E)^{1/2}$. With y_n given by (9), the quantization condition (7) takes the form

$$(K + 1/2)\pi = \sum_{n=0}^{\infty} \sum_{s=0}^{3n} A_{2n,s} I_{n,s}. \tag{14}$$

$$I_{n,s} = \frac{i}{2} \oint dt (1 + t^2)^{-1} t^{6n-2s} (V_0 t^2 - E)^{1/2-3n}. \tag{15}$$

Here the contour of integration encircles the branch cut along the real axis between $t = \pm (E/V_0)^{1/2}$, and no other singularities of the integrand. Because of the simple structure of the integrand, the integral is best evaluated by pushing the contour to infinity and applying the residue theorem. The result is

$$I_{n,s} = -\pi V_0^{1/2-3n} \delta_{s,0} + \pi (-1)^s (V_0 + E)^{1/2-3n}. \tag{16}$$

The first term in (16) is the contribution from the contour at infinity, and the second term is due to the two simple poles at $t = \pm i$. Substituting (16) in (14), we have

$$K + \frac{1}{2} = (V_0 + E)^{1/2} - \sqrt{V_0} \sum_{n=0}^{\infty} A_{2n,0} V_0^{-3n} + \sum_{n=1}^{\infty} T_{2n} (V_0 + E)^{1/2-3n}, \tag{17}$$

where we have defined

$$T_{2n} = \sum_{s=0}^{3n} (-)^s A_{2n,s}.$$

3.3 Energy eigenvalues

We now simplify (17) by showing that T_{2n} in the last term vanishes. We observe that all the y_n for $n > 0$ vanish at $t = i$. This is easily demonstrated by induction. The value of y_{2n} at $t = i$ is readily seen from (9) to be equal to T_{2n} , apart from a non-zero factor. Hence, $T_{2n} = 0$ for $n > 0$. Consequently, the quantization formula (17) becomes

$$K + \frac{1}{2} = (V_0 + E)^{1/2} - \sqrt{V_0} \sum_{n=0}^{\infty} (V_0/2)^{2n} \binom{1/2}{n} V_0^{-3n}. \tag{18}$$

It is interesting to note here that the energy E occurs only in the first term which is the contribution from the lowest order. Upon summing the series in (18), we get

$$K + \frac{1}{2} = (V_0 + E)^{1/2} - \sqrt{V_0} \left(1 + \frac{1}{4V_0}\right)^{1/2}.$$

Inverting this we obtain

$$E = -V_0 + [K + \frac{1}{2} + (V_0 + 1/4)^{1/2}]^2 \quad (K = 0, 1, 2, \dots). \tag{19}$$

The spectrum of energy levels given by (19) is identical to that obtained by solving the Schrödinger equation exactly (Gol'dman *et al* 1956).

4. Discussion

We have shown how the exact energy spectrum of the potential $V_0 \tan^2 x$ can be computed by summing the JWKB series. For this potential, the higher order corrections to (1) turn out to be non-zero. Other exactly solvable potentials for which the JWKB series is non-trivial but summable to yield the exact energies are $-V_0 (\cosh x)^{-2}$ (noted by Bender *et al* 1977) and $V_0 (x/a - a/x)^2$, $x > 0$. In all cases, the energy E occurs only in the lowest order term, the higher order corrections being independent of it. It is interesting to observe that for all known exactly solvable potentials, the higher order corrections form binomial series which can be formally summed.

The use of Dunham's method has enabled us to determine in an elegant way the higher order integrals in a closed form. This is of course due to the simple nature of $V(x)$: specifically, the quantity $V(x) - E$ has only two simple zeroes (both real) in the complex x plane. Consequently, one could push the Dunham contour to infinity without encountering any other cut. For potentials which give rise to additional zeros of $V - E$, such a deformation of the contour will not simplify the calculations.

We close our discussion with a few observations. That the JWKB series is summable to give the exact energies in the present problem should not be construed to mean that for a general $V(x)$ the accuracy of the energy values can be increased indefinitely by including more and more terms of the series. In general, the JWKB series is not summable; it is in fact an asymptotic series. The accuracy of the computed energy values increases with order at first, but eventually decreases. The asymptotic nature of the JWKB series can also be demonstrated in the present case, if we calculate the energy not from (19), but by truncating the series in (18) at some finite value of n . We then find that if $V_0 > 1/4$, the accuracy of E increases with n , but if $V_0 < 1/4$, it shows the behaviour described above. The binomial series in (18) converges only if $V_0 > 1/4$, but by analytic continuation the final result is true for all values of V_0 .

References

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Note added in proof: Yet another potential for which the JWKB series can be summed to yield the exact energy spectrum is the Pöschl-Teller potential

$$V(x) = a(a-1) \operatorname{cosec}^2 x + b(b-1) \operatorname{sec}^2 x, \quad a, b > 1$$