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Third and fourth order invariants for one-dimensional time-dependent classical systems

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Abstract. The construction of invariants up to fourth order in velocities has been carried out for one-dimensional, time-dependent classical dynamical systems. While the exact results are recovered for the first and second order integrable systems, the results for the third and fourth order invariants are expressed in terms of nonlinear *potential* equations. Noticing the separability of the potential in space and time variables these nonlinear equations are reduced to a tractable form. A possible solution for the third order case suggests a new integrable system $V(q,t) \sim t^{-4/3}q^{1/2}$.

Keywords. Dynamical invariants; time-dependent systems; integrable systems.

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1. Introduction

In the recent past there has been considerable interest in the construction of exact invariants for time-dependent (TD) classical dynamical systems. While very few attempts (Katzin and Levine 1982, 1983; Kaushal *et al* 1984a; Mishra *et al* 1984) are being made for the study of two- or higher-dimensional systems, various methods have been proposed (Lewis 1968; Lutzky 1978, 1979; Korsch 1979; Kaushal and Korsch 1981; Ray and Reid 1979, 1982; Lewis and Leach 1982) and used for the construction of invariants for one-dimensional systems described by the Lagrangian

$$L = \tfrac{1}{2}\dot{q}^2 - V(q,t). \tag{1}$$

Again for the one-dimensional case, the most studied system is the TD harmonic oscillator for which the constructed invariant (*cf.* (16)) turns out to be of second order in velocities. No doubt there has been some emphasis also on the study of TD non-harmonic systems (Kaushal and Korsch 1981; Ray and Reid 1979), but somehow only a few of them are integrable in the sense of second order invariants. In fact, once the invariants of a system (or the second constants of motion for a time-independent system) are known then the solution of the equation of motion merely reduces to the problem of quadrature.

In spite of the fact that third or higher order invariants (other than the total energy)

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are now constructed (Holt 1982; Hall 1983; Kaushal *et al* 1985) for some time-independent systems and quite a few new integrable systems are found somehow such attempts have not been made so far (to the best of our knowledge) for TD systems. As a matter of fact the study of time-independent systems even in higher-dimensions is somewhat easier as compared to that of TD systems mainly because, in this case, the over-determined set of the partial differential equations, required to be solved during the process of construction of invariant, does not involve (Holt 1982; Hall 1983) the coupling between the coefficients of even and odd powers of velocities in the invariant. As a result the second constant of motion for the system contains only either even or odd powers of velocities. However, this is not the case with TD systems for which such a coupling occurs through the time-derivatives of these coefficients. This not only increases the number of partial differential equations but also complicates their solutions, particularly for higher-dimensional systems and with regard to higher order invariants. For one-dimensional systems, however, the problem is somewhat simplified.

In the present work, we attempt to construct such third and fourth order invariants for the TD systems described by (1). In § 2, after describing the method in general, we recover the familiar results for the first and second order cases, particularly for the TD harmonic oscillator. In §§ 3 and 4, we respectively discuss the construction of third and fourth order invariants. The results are expressed in the form of "potential" equations which turn out to be nonlinear in nature and whose solutions would directly provide the integrable systems. A possible solution of the "potential" equation, in § 3, yields a new integrable system $V(q, t) \sim t^{-4/3} q^{1/2}$, which admits a third order invariant. Finally, the results are discussed and summarized in § 5.

2. The method

We consider the system described by the Lagrangian (1) with the concomitant equation of motion,

$$\ddot{q} = -\partial V/\partial q. \tag{2}$$

Let us consider the constant of motion, I, of the form

$$I = b_0 + b_1 \dot{q} + \frac{1}{2!} b_2 \dot{q}^2 + \frac{1}{3!} b_3 \dot{q}^3 + \frac{1}{4!} b_4 \dot{q}^4, \tag{3}$$

where b_i's $(i = 0, 1, 2, 3, 4)$, are functions of q and t. The invariance of I implies $\mathrm{d}I/\mathrm{d}t = 0$ and using (3), this gives

$$\frac{\partial b_0}{\partial t} + b_1 \ddot{q} + \left(\frac{\partial b_0}{\partial q} + \frac{\partial b_1}{\partial t} + b_2 \ddot{q}\right)\dot{q} + \left(\frac{\partial b_1}{\partial q} + \frac{1}{2}\frac{\partial b_2}{\partial t} + \frac{1}{2} b_3 \ddot{q}\right)\dot{q}^2$$
$$+ \left(\frac{1}{2}\frac{\partial b_2}{\partial q} + \frac{1}{6}\frac{\partial b_3}{\partial t} + \frac{1}{6} b_4 \ddot{q}\right)\dot{q}^3 + \left(\frac{1}{6}\frac{\partial b_3}{\partial q} + \frac{1}{24}\frac{\partial b_4}{\partial t}\right)\dot{q}^4 + \frac{1}{24}\frac{\partial b_4}{\partial q}\dot{q}^5 = 0. \tag{4}$$

On substituting $\ddot{q}$ from (2), (4) yields an expression in t, $\dot{q}$ which must vanish identically. This implies that the coefficients in (4) must vanish separately. Thus

$$\partial b_4/\partial q = 0, \tag{5a}$$

$$4\frac{\partial b_3}{\partial q} + \frac{\partial b_4}{\partial t} = 0, \tag{5b}$$

$$3\frac{\partial b_2}{\partial q}+\frac{\partial b_3}{\partial t}-b_4\frac{\partial V}{\partial q}=0, \tag{5c}$$

$$2\frac{\partial b_1}{\partial q}+\frac{\partial b_2}{\partial t}-b_3\frac{\partial V}{\partial q}=0, \tag{5d}$$

$$\frac{\partial b_0}{\partial q}+\frac{\partial b_1}{\partial t}-b_2\frac{\partial V}{\partial q}=0, \tag{5e}$$

$$\frac{\partial b_0}{\partial t}-b_1\frac{\partial V}{\partial q}=0. \tag{5f}$$

Note that this is an overdetermined set of coupled partial differential equations and the coupling between the coefficients corresponding to even and odd powers of q in (3) arises due to the time-derivatives of b_i's. This is not, however, the case with time-independent systems (see Holt 1982; Hall 1983).

To obtain the systems which admit first order invariants (*i.e.* linear in velocities), we set $b_2 = b_3 = b_4 = 0$ in (5) and hence obtain the following set of equations:

$$\frac{\partial b_1}{\partial q}=0;\quad \frac{\partial b_0}{\partial q}=-\frac{\partial b_1}{\partial t};\quad \frac{\partial b_0}{\partial t}=b_1\frac{\partial V}{\partial q}.$$

While the solutions of the first two equations are $b_1 = \rho_1(t)$; $b_0 = -\dot{\rho}_1 q + \rho_2(t)$, the third yields the "potential" equation,

$$\frac{\partial V}{\partial q}+\frac{\ddot{\rho}_1}{\rho_1}q-\frac{\dot{\rho}_2}{\rho_1}=0, \tag{6}$$

whose only solution is

$$V(q,t)=-\frac{\ddot{\rho}_1}{2\rho_1}q^2+\frac{\dot{\rho}_2}{\rho_1}q+\rho_3(t).$$

Here ρ_i's are some arbitrary functions of time. If we set $\rho_3(t) = -\dot{\rho}_2^2/(2\rho_1\ddot{\rho}_1)$, then the integrable system is the *TD rotating harmonic oscillator*,

$$V(q,t)=\tfrac{1}{2}\omega^2(t)[q-\alpha(t)]^2, \tag{7}$$

with the invariant,

$$I=\rho_2+(\rho_1\dot{q}-\dot{\rho}_1 q), \tag{8}$$

where ρ_1 and ρ_2 are given by

$$\ddot{\rho}_1+\omega^2(t)\rho_1=0;\qquad \dot{\rho}_2+\omega^2(t)\alpha(t)\rho_1=0.$$

For the systems admitting second order invariants we set $b_3 = b_4 = 0$ in (5a)–(5f) and obtain,

$$\frac{\partial b_2}{\partial q}=0;\qquad 2\frac{\partial b_1}{\partial q}+\frac{\partial b_2}{\partial t}=0; \tag{9a,b}$$

along with (5e) and (5f). Equations (9a) and (9b) imply

$$\begin{aligned} b_2 &= \sigma_1(t),\\ b_1 &= -\tfrac{1}{2}\dot{\sigma}_1 q+\sigma_2(t), \end{aligned} \tag{10}$$

where σ_1 and σ_2 are some arbitrary functions of t. Using these results in (5e) and (5f) and noting that $(\partial^2 b_0/\partial q \cdot \partial t) = (\partial^2 b_0/\partial t \cdot \partial q)$, we obtain a general "potential" equation of the form,

$$\left(-\frac{1}{2}\dot{\sigma}_1 q + \sigma_2\right)\frac{\partial^2 V}{\partial q^2} - \sigma_1\left(\frac{\partial^2 V}{\partial t \cdot \partial q}\right) - \frac{3}{2}\dot{\sigma}_1\left(\frac{\partial V}{\partial q}\right) + \left(-\frac{1}{2}\dddot{\sigma}_1 q + \ddot{\sigma}_2\right) = 0. \tag{11}$$

This is a linear, second order partial differential equation whose solutions, in principle, would provide the integrable systems admitting second order invariants. We shall be obtaining similar 'potential' equations for third and fourth order invariants in the next two sections. However, they will be nonlinear in nature. As such the solution of these nonlinear equations will be a difficult task, therefore, with the help of some simple and known cases we demonstrate here a method of using these "potential" equations for constructing the invariants:

Case (*a*): TD harmonic oscillator:

$$V(q,t) = \Omega(t)q^2. \tag{12}$$

On substituting the derivatives of V in (11) we obtain an identity,

$$-2(\sigma_1\dot{\Omega} + 2\dot{\sigma}_1\Omega + \tfrac{1}{4}\dddot{\sigma})q + 2\sigma_2\Omega + \ddot{\sigma}_2 = 0, \tag{13}$$

which is satisfied if we choose

$$\tfrac{1}{4}\dddot{\sigma}_1 + 2\dot{\sigma}_1\Omega + \sigma_1\dot{\Omega} = 0, \tag{14a}$$

$$\ddot{\sigma}_2 + 2\sigma_2\Omega = 0. \tag{14b}$$

Now, we set

$$\sigma_1 = \rho^2, \qquad \sigma_2 = 0,$$

which reduces (14a) to

$$\ddot{\rho} + 2\Omega(t)\rho = k/\rho^3, \tag{15}$$

where k is an arbitrary constant of integration. The coefficients b_1 and b_2 in (3) now become

$$b_2 = \rho^2, \quad b_1 = -\rho\dot{\rho}q,$$

and the coefficient b_0 can be obtained from (5f) and (15), in the form

$$b_0 = \tfrac{1}{2}k(q/\rho)^2 + \tfrac{1}{2}q^2\dot{\rho}^2.$$

Finally, the invariant (3) turns out to be

$$I = \tfrac{1}{2}k(q/\rho)^2 + \tfrac{1}{2}(q\dot{\rho} - \dot{q}\rho)^2, \tag{16}$$

a well known result in literature (Lewis 1968; Korsch 1979; Ray and Reid 1979, 1982).

Case (*b*): Let us consider a general power form of V as

$$V(q,t) = \beta(t)q^m. \tag{17}$$

Using this form of V in (11) we obtain the identity

$$[-\tfrac{1}{2}\dot{\sigma}_1 m(m-1)\beta - \sigma_1 m\dot{\beta} - \tfrac{3}{2}\dot{\sigma}_1 m\beta]q^{m-1} + \sigma_2 m(m-1)\beta q^{m-2}$$
$$-\tfrac{1}{2}\dddot{\sigma}_1 q + \ddot{\sigma}_2 = 0,$$

which must be satisfied if we require that the coefficients of powers of q must vanish (since $m \neq 0$),

$$\dot{\sigma}_1 \beta\left(\frac{m}{2}+1\right)+\sigma_1 \dot{\beta}=0; \quad \sigma_2(m-1)\beta=0, \qquad \ddot{\sigma}_1=0; \qquad \ddot{\sigma}_2=0. \tag{18}$$

The solution to these equations for $(m \neq 1)$ are

$$\sigma_2=0, \quad \sigma_1=(a_1 t^2+a_2 t+a_3), \quad \beta(t)=k_1 \sigma_1^{-(m+2)/2}, \tag{19}$$

where a_1, a_2, a_3 and k_1 are some arbitrary constants of integration. The coefficient b_0 from (5f) turns out to be $b_0=\sigma_1 \beta q^m+k_2$, and the invariant (3), in this case, can be written as

$$I=\sigma_1 \beta q^m+\tfrac{1}{2}\dot{q}(\sigma_1 \dot{q}-\dot{\sigma}_1 q), \tag{20}$$

where the arbitrary constant of integration k_2 is taken as zero.

If $m=1$, then $\sigma_2 \neq 0$ and (18) implies that $\sigma_2=c_1 t+c_2$. Choosing the arbitrary constants of integration c_1 and c_2 as $c_1=2a_1$, and $c_2=a_2$, the invariant for the linear TD potential, $V(q,t)=\beta(t)q$, can be obtained as

$$I=\sigma_1(q-2)\beta+\tfrac{1}{2}\dot{q}[\sigma_1 \dot{q}-\dot{\sigma}_1(q-2)]. \tag{21}$$

In the above cases we have considered the potential $V(q,t)$ to be separable in q, t. Alternatively, using (10) in (5e) we notice that

$$\sigma_1(t)=(\partial^3 b_0/\partial q^3)/(\partial^3 V/\partial q^3),$$

which suggests that both b_0 and V are still separable in q and t to the extent that the ratio $(\partial^3 b_0/\partial q^3)/(\partial^3 V/\partial q^3)$ is just a function of t. Therefore, we choose

$$b_0(q,t)=h(t)u(q), \qquad V(q,t)=\frac{h(t)}{\sigma_1}u(q). \tag{22}$$

Using these forms in (5f) and integrating the resultant expression we find the unknown function $u(q)$ in (22) as

$$u(q)=k_3(\sigma_2-\tfrac{1}{2}\dot{\sigma}_1 q)^{-(2\sigma_1 \dot{h}/\dot{\sigma}_1 h)}, \tag{23}$$

with the corresponding invariant

$$I=hu(q)+\sigma_2 \dot{q}+\tfrac{1}{2}\dot{q}(\sigma_1 \dot{q}-\dot{\sigma}_1 q), \tag{24}$$

where σ_1 and σ_2 must satisfy $\ddot{\sigma}_1=\dot{\sigma}_2=0$.

3. Third order invariants

For this case we set $b_4=0$ in (5a)–(5f). The set of equations to be solved are

$$\frac{\partial b_3}{\partial q}=0 \quad \text{and} \quad \frac{\partial b_2}{\partial q}=-\frac{1}{3}\frac{\partial b_3}{\partial t}, \tag{25a,b}$$

along with (5d)–(5f). Equations (25a) and (25b), respectively, provide

$$b_3=\psi_1(t) \quad \text{and} \quad b_2=-\tfrac{1}{3}\dot{\psi}_1 q+\psi_2(t), \tag{26}$$

where ψ_1 and ψ_2 are some arbitrary functions of time. Substituting these expressions for b_3 and b_2 in (5d) and integrating we get

$$b_1 = \tfrac{1}{12}\ddot{\psi}_1 q^2 - \tfrac{1}{2}\dot{\psi}_2 q + \tfrac{1}{2}\psi_1 V + \psi_3(t), \tag{27}$$

where ψ_3 is another arbitrary function of t. Now, with these results for b_1 and b_2, (5e) and (5f) can be used to eliminate b_0 in favour of V by noting the fact that $(\partial^2 b_0/\partial q \cdot \partial t) = (\partial^2 b_0/\partial t \cdot \partial q)$. This will lead to a general "potential" equation of the type

$$\begin{aligned}\frac{1}{2}\psi_1 V\frac{\partial^2 V}{\partial q^2} + \left(\frac{1}{12}\ddot{\psi}_1 q^2 - \frac{1}{2}\dot{\psi}_2 q + \psi_3\right)\frac{\partial^2 V}{\partial q^2} + \frac{1}{2}\psi_1(\partial V/\partial q)^2 \\ + \frac{1}{2}(\ddot{\psi}_1 q - 3\dot{\psi}_2)\frac{\partial V}{\partial q} + \left(\frac{1}{3}\dot{\psi}_1 q - \psi_2\right)\frac{\partial^2 V}{\partial q \cdot \partial t} + \frac{1}{2}\psi_1\frac{\partial^2 V}{\partial t^2} \\ + \dot{\psi}_1\frac{\partial V}{\partial t} + \frac{1}{2}\ddot{\psi}_1 V + \left(\frac{1}{12}\ddddot{\psi}_1 q^2 - \frac{1}{2}\dddot{\psi}_2 q + \ddot{\psi}_3\right) = 0,\end{aligned} \tag{28}$$

whose solution would provide an integrable TD system admitting the third order invariants.

As such the solution of the nonlinear equation (28) is difficult. However, as for the second order case, we also notice here from (27) that $\psi_1(t) = 2(\partial^3 b_1/\partial q^3)/(\partial^3 V/\partial q^3)$, which implies the separability of b_1 and V in q and t variables to the extent that the ratio $(\partial^1 b_1/\partial q^3)/(\partial^3 V/\partial q^3)$ is only a function of t. For this reason we choose

$$b_1(q,t) = f(t)v(q), \quad V(q,t) = \frac{2f(t)}{\psi_1}v(q). \tag{29}$$

Substituting these forms of b_1 and V in (27) we obtain the identity

$$\tfrac{1}{12}\ddot{\psi}_1 q^2 - \tfrac{1}{2}\dot{\psi}_2 q + \psi_3 = 0. \tag{30}$$

In order that this identity be satisfied we must have

$$\ddot{\psi}_1 = 0, \qquad \dot{\psi}_2 = 0, \qquad \psi_3 = 0, \tag{31a}$$

which yield

$$\psi_1 = c_3 t + d_1, \qquad \psi_2 = c_4. \tag{31b}$$

Here c_3, c_4, d_1 are the arbitrary constants of integration and now-onward we choose $d_1 = 0$.

For the choices (29) and (31), the "potential" equation (28) reduces to a somewhat simpler form, *viz*,

$$\frac{1}{v}\frac{\mathrm{d}}{\mathrm{d}q}\left(v\frac{\mathrm{d}v}{\mathrm{d}q}\right) + \frac{(\dot{f}\psi_1 - c_3 f)}{\psi_1 f^2}\cdot\left(\frac{1}{3}c_3 q - c_4\right)\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}q} + \frac{\psi_1\ddot{f}}{2f^2} = 0. \tag{32}$$

Now we look for possible solutions of this nonlinear equation.

(i) Let

$$\frac{1}{v}\frac{\mathrm{d}}{\mathrm{d}q}\left(v\frac{\mathrm{d}v}{\mathrm{d}q}\right) = 0, \tag{33a}$$

Equation (33a) implies that

$$v(q) = (k_4 q + k_5)^{1/2}, \tag{33b}$$

where k_4 and k_5 are some arbitrary constants. If we set $k_4 = \frac{1}{3}c_3$ and $k_5 = -c_4$, then the remaining q-dependent factors in (32) cancel out and we are left with an ordinary, linear differential equation of second order in f, namely

$$3\ddot{f}+t^{-1}\dot{f}-t^{-2}f=0, \tag{34}$$

whose solutions are

$$f(t)=at^{-1/3}, \quad f(t)=at. \tag{35}$$

For the reasons given below in (ii) we accept here the first solution in (35) and obtain the eliminated constant b_0 as

$$b_0 = -\tfrac{1}{2}a^2t^{-2/3},$$

and finally, the invariant (3) as

$$I = -\frac{1}{2}\left[f-\left(\frac{1}{3}c_3q-c_4\right)^{1/2}\dot{q}\right]^2+\frac{1}{6}\psi_1\dot{q}^3. \tag{36}$$

(ii) We can also choose

$$f\psi_1 - c_3 f = 0,$$

in (32) which implies $f(t) \sim t$. This solution, though consistent with the second solution of (34) and satisfies (32) along with (33a) (since $\ddot{f}\psi_1/2f^2$ turns out to be zero), however, is not acceptable because the potential $V(q, t)$, in this case, becomes time-independent. Thus, we find that the possibility (ii) does not give anything new except that it supplements the possibility (i).

4. Fourth order invariants

To find the integrable systems admitting fourth order invariants one has to solve the whole set of equations (5). As before the solutions of (5a) and (5b), lead to

$$b_4 = s_1(t) \quad \text{and} \quad b_3 = -\tfrac{1}{4}\dot{s}_1q+s_2(t), \tag{37}$$

which in turn provide the solution of (5c) as

$$b_2 = \frac{1}{24}\ddot{s}_1q^2-\frac{1}{3}\dot{s}_2q+\frac{1}{3}s_1V+s_3(t), \tag{38}$$

where s_i's are some arbitrary functions of t. Further using these expressions for b_2, b_3 and b_4 the integration of (5d) yields

$$b_1 = -\frac{1}{144}\dddot{s}_1q^3+\frac{1}{12}\ddot{s}_2q^2-\frac{1}{24}\dot{s}_1\gamma$$
$$-\frac{1}{6}s_1\frac{\partial\gamma}{\partial t}-\frac{1}{2}\dot{s}_3q-\frac{1}{2}\left(\frac{1}{4}\dot{s}_1q-s_2\right)V+s_4(t), \tag{39}$$

where s_4 is another arbitrary function of time and $\gamma = \int V\mathrm{d}q$. Now, with these results for b_1 and b_2 (5e) and (5f) can be used to eliminate b_0 in favour of V again by noting the fact that $(\partial^2 b_0/\partial q\cdot\partial t) = (\partial^2 b_0/\partial t\cdot\partial q)$. This will, in fact, determine the "potential" equation of the type

$$\left[\frac{1}{144}s_1^{(3)}q^3 - \frac{1}{12}s_2^{(2)}q^2 + \frac{1}{2}\dot{s}_3 q - s_4 + \frac{1}{24}\dot{s}_1\gamma + \frac{1}{6}s_1\frac{\partial\gamma}{\partial t} + \frac{1}{2}\left(\frac{1}{4}\dot{s}_1 q - s_2\right)V\right]\cdot\frac{\partial^2 V}{\partial q^2} + \frac{1}{2}\left(\frac{1}{4}\dot{s}_1 q - s_2\right)(\partial V/\partial q)^2$$
$$+\frac{1}{2}\left[\frac{1}{8}s_1^{(3)}q^2 - s_2^{(2)}q + 3s_3 + \dot{s}_1 V + s_1\frac{\partial V}{\partial t}\right]\frac{\partial V}{\partial q}$$
$$+\left(\frac{1}{24}s_1^{(2)}q^2 - \frac{1}{3}\dot{s}_2 q + s_3 + \frac{1}{3}s_1 V\right)\frac{\partial^2 V}{\partial q\cdot\partial t}$$
$$+\left[\frac{1}{144}s_1^{(5)}q^3 - \frac{1}{12}s_2^{(4)}q^2 + \frac{1}{2}s_3^{(3)}q - s_4^{(2)} + \frac{1}{24}s_1^{(3)}\gamma + \frac{1}{4}s_1^{(2)}\frac{\partial\gamma}{\partial t} + \frac{3}{8}\dot{s}_1\frac{\partial^2\gamma}{\partial t^2} + \frac{1}{6}s_1\frac{\partial^3\gamma}{\partial t^3} + \frac{1}{2}\left(\frac{1}{4}s_1^{(3)}q - s_2^{(2)}\right)V\right.$$
$$\left.+\left(\frac{1}{4}s_1^{(2)}q - \dot{s}_2\right)\frac{\partial V}{\partial t} + \frac{1}{2}\left(\frac{1}{4}\dot{s}_1 q - s_2\right)\frac{\partial^2 V}{\partial t^2}\right] = 0, \tag{40}$$

where the numbers in the paranthesis on superscripts of s_i's represent the order of time-derivatives of s_i's. Note that (40) is a nonlinear, integro-partial differential equation whose solutions, in principle, would directly provide the integrable systems admitting fourth order invariants.

As before we give here the reduced form of the "potential" (40) by noting the fact from (38) that the ratio $3(\partial^3 b_2/\partial q^3)/(\partial^3 V/\partial q^3)$, is a function of time only. This implies the separability of b_2 and V in q and t variables and hence we write

$$b_2(q,t) = g(t)w(q), \quad V(q,t) = \frac{3g(t)}{s_1}w(q). \tag{41}$$

Substituting these forms of b_2 and V in (38) we again obtain an identity

$$\frac{1}{24}\ddot{s}_1 q^2 - \frac{1}{3}\dot{s}_2 q + s_3 = 0,$$

which would imply

$$\ddot{s}_1 = 0 \qquad \dot{s}_2 = 0 \qquad s_3 = 0$$

and hence $s_1 = c_5 t + d_2$, $s_2 = c_6$. Here c_5, c_6, d_2 are the arbitrary constants of integration and also we have set $d_2 = 0$. The choice (41) reduces the potential equation (40) to a somewhat simpler form,

$$\left(\dot{g}s_1 - \frac{3}{4}g\dot{s}_1\right)Ww'' + 3g\left(\frac{1}{4}\dot{s}_1 q - s_2\right)(ww')' + (5\dot{g}s_1 - 2g\dot{s}_1)ww'$$
$$+\frac{1}{g}\left(\frac{1}{3}\dddot{g}\, s_1^2 - \frac{1}{4}\ddot{g}\dot{s}_1 s_1 + \frac{1}{2}\dot{g}\dot{s}_1^2 - \frac{1}{2s_1}\dot{s}_1^3 g\right)W$$
$$+\frac{1}{g}\left(\frac{1}{4}\dot{s}_1 q - s_2\right)\left(\ddot{g}s_1 - 2\dot{s}_1\dot{g} + \frac{2g}{s_1}\dot{s}_1^2\right)w = 0, \tag{42}$$

where $W = \int w\,dq$, and $s_4(t)$ is taken to be zero. The primes on w represent the derivatives of w w.r.t. q. Equation (42) appears difficult to handle even for a trivial case like the one discussed in § 3. However, the use of rationalization method (see §5) can provide integrable TD systems in this case.

5. Discussion and summary

The method described in § 2 is subsequently used in reproducing the results for some known cases. This not only provides a check on the method but also outlines a procedure for the use of nonlinear "potential" equations obtained in §§ 3 and 4, corresponding to the third and fourth order invariants. The only system admitting first order invariants is a TD rotating harmonic oscillator (*cf.* (7)). In addition to the familiar case of TD harmonic oscillator the second order invariants are constructed (*cf.* § 2) for (i) a TD linear potential, $V(q,t) = \beta(t)q$ (*cf.* (21)), and (ii) a potential of the type $(c_2 - c_1 q)^{-F(t)}$, in which the time-dependence occurs through the exponent.

To obtain a general solution of the potential equation (28) or (40) is a difficult task. However, for a given TD system its integrability can be checked by using what we call the method of rationalization. This method, outlined in § 2 and detailed elsewhere (Katzin and Levine 1982, 1983; Kaushal *et al* 1984a), suggests a way out for determining unknown arbitrary functions of time which, normally, are found to satisfy some tractable, ordinary differential equations. A possible solution of the nonlinear equation (32) provides a new integrable system, $V(q,t) \sim t^{-4/3} q^{1/2}$, which admits a third order invariant (*cf.* (36)).

Finally, for the purposes of constructing further higher order invariants we obtain here a recursion relation involving the coefficients $b_n(q,t)$, where $b_n(q,t)$ occur in a general form of the invariant I, namely

$$I = \sum_{n=0}^{\infty} b_n(q,t) \frac{p^n}{n!}. \tag{43}$$

The Hamiltonian corresponding to the Lagrangian (1) is now written as

$$H = \frac{1}{2} p^2 + V(q,t), \tag{44}$$

and the invariance of I implies

$$\frac{dI}{dt} \equiv \frac{\partial I}{\partial t} + [I, H]_{\text{P.B.}} = 0. \tag{45}$$

Using (43) and (44) in (45) it is now easy to obtain a recursion relation of the type

$$\dot{b}_n + n \frac{\partial b_{n-1}}{\partial q} - b_{n+1} \frac{\partial V}{\partial q} = 0, \tag{46}$$

which conforms the set of equations (5) for fourth order invariants.

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