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Fractionally charged non-leaking dyons and fermions in a bag

S M ROY and VIRENDRA SINGH
Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

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Abstract. We consider a fermion of charge e confined to a spherical bag with a Dirac monopole of strength g at its centre. We find that the boundary conditions making the lowest angular momentum hamiltonian self-adjoint are characterized by a unitary matrix U, and the corresponding vacuum charge has a fractional part $2|eg|\alpha/\pi$ where $\det U = -\exp(2i\alpha)$. Boundary conditions for conservation of helicity, CP, CT and PT are displayed. We demonstrate the possibility of a fractionally charged dyon whose interaction with a fermion conserves helicity. We also show that *the simultaneous validity of helicity, CP, CT and PT requires integer vacuum charge.*

Keywords. Non-leaking dyons; fermions; spherical bag; unitary matrix.

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The Jackiw-Rebbi (1976) discovery of half-integral fermion number of fermion-monopole systems is further dramatised by Witten's (1979) result that in the presence of a CP-violating angle θ_0 the monopole acquires a charge $-e\theta_0/(2\pi)$ where e is the fermion charge. In an apparently different line of research Kazama *et al* (1977), Callias (1977) and Goldhaber (1977) discovered that the fermion-Dirac monopole Hamiltonian in the lowest angular momentum state is self-adjoint only when a boundary condition at the origin (monopole position) involving the CP-violating parameter θ_0 is imposed. Two major consequences are the "inevitable failure of helicity conservation" (Goldhaber 1977) (intimately related to the Rubakov-Callan effect in the non-abelian case (Rubakov 1981, 1982; Callan 1982a, b, 1983)), and the confirmation by Grossman and Yamagishi (1983) of the Witten effect with a precise connection to the $r = 0$ boundary condition. The monopole becomes a helicity leaking dyon of fractional charge, the fraction being irrational in general but a half-integer or an integer when CP is conserved.

The present work demonstrates that these conclusions get radically altered when the fermion monopole system is enclosed in a spherical bag of finite radius R. In particular it is possible to have a helicity conserving dyon of fractional charge. CP violation forces fractional charge but does not force helicity violation. Further, *the simultaneous conservation of helicity, CP, CT and PT forces the monopole charge to be integral.* For the lowest angular momentum hamiltonian we find a simple formula relating the vacuum charge to the boundary conditions. The charge eigenvalues are independent of R but depend non-trivially on boundary conditions at $r = R$ (as well as $r = 0$) and hence have non-unique $R \to \infty$ limit. A similar boundary condition dependence of the vacuum charge was obtained recently (Roy and Singh 1984a, b) for the 1 + 1 dimensional Jackiw-Rebbi and Goldstone-Wilczek (1981) hamiltonians. We generalize here the

Witten-Grossman-Yamagishi results connecting fractional charge to violation of discrete symmetries.

We use the Wu-Yang (1975) vector potentials $\mathbf{A} = \mathbf{A}_a(\mathbf{r})$ for $\mathbf{r} \in R_a$ $(\theta \neq \pi)$, and $\mathbf{A}_b(\mathbf{r})$ for $\mathbf{r} \in R_b$ $(\theta \neq 0)$ where

$$\mathbf{A}_a(\mathbf{r}) = \mathbf{A}_b(-\mathbf{r}) = g\frac{\hat{z}\times\hat{r}}{r(1+\hat{r}\cdot\hat{z})}, \tag{1}$$

with g = monopole strength. The Dirac wave section ψ $(= \psi_a$ in R_a, ψ_b in $R_b)$ obeys,

$$H\psi(\mathbf{r}, t) = i\frac{\partial\psi(\mathbf{r}, t)}{\partial t},\ H = \boldsymbol{\alpha}\cdot\boldsymbol{\pi} + \beta m, \tag{2}$$

where

$$\boldsymbol{\pi} = -i\nabla - e\mathbf{A},\ \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix},\ \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3}$$

$\boldsymbol{\sigma}$ are Pauli matrices and e and m denote charge and mass of the fermion, confined to the bag $|\mathbf{r}| \leqslant R$. Before defining the boundary conditions which make H self-adjoint, we note, following Goldhaber (1977) the following "formal" symmetry properties

$$\boldsymbol{\Sigma}\cdot\boldsymbol{\pi} H \boldsymbol{\Sigma}\cdot\boldsymbol{\pi} = H,\ (CP)H(CP)^{-1} = -H,$$
$$(CT)H(CT)^{-1} = -H,\ (PT)H(PT)^{-1} = H. \tag{4}$$

Here $\boldsymbol{\Sigma}\cdot\boldsymbol{\pi}$ is related to the helicity, with $\boldsymbol{\Sigma} = \mathrm{diag}(\boldsymbol{\sigma}, \boldsymbol{\sigma})$. The discrete transformations C, P, T are defined by

$$\psi_{a,b}(\mathbf{x}, t) \xrightarrow{P} \psi^P_{a,b}(\mathbf{x}, t) = \eta_P \beta \psi_{b,a}(-\mathbf{x}, t), \tag{5}$$

$$\psi(\mathbf{x}, t) \xrightarrow{C} \psi^C(\mathbf{x}, t) = \eta_C \beta \alpha_2 \psi^*(\mathbf{x}, t), \tag{6}$$

$$\psi(\mathbf{x}, t) \xrightarrow{T} \psi^T(\mathbf{x}, t) = \eta_T \Sigma_2 \psi^*(\mathbf{x}, -t), \tag{7}$$

$$P\mathbf{A}(\mathbf{x})P^{-1} = C\mathbf{A}(\mathbf{x})C^{-1} = T\mathbf{A}(\mathbf{x})T^{-1} = -\mathbf{A}(\mathbf{x}), \tag{8}$$

where $|\eta_P| = |\eta_C| = |\eta_T| = 1$. The subscripts a, b in (5) refer to regions R_a, R_b. The corresponding subscripts in (6)–(8) are omitted, since the same subscript occurs throughout each equation. It follows that the wave sections,

$$\psi^{PT}_{a,b}(\mathbf{x}, t) = \eta_P \eta_T \beta \Sigma_2 \psi^*_{b,a}(-\mathbf{x}, -t), \tag{9}$$

$$\psi^{CT}(\mathbf{x}, t) = -\eta_C \eta^*_T \beta \alpha_2 \Sigma_2 \psi(\mathbf{x}, -t), \tag{10}$$

and

$$\psi^{CP}_{a,b}(\mathbf{x}, t) = -\eta_C \eta^*_P \alpha_2 \psi^*_{b,a}(-\mathbf{x}, t) \tag{11}$$

obey the same Dirac equation as $\psi(\mathbf{x}, t)$. In the lowest angular momentum state $j = |q| - \frac{1}{2}$, $q \equiv eg$,

$$\psi(\mathbf{x}, t) = \frac{1}{r}\begin{pmatrix} F(r) & \eta_{j,m}(\theta, \phi) \\ G(r) & \eta_{j,m}(\theta, \phi) \end{pmatrix} \exp(-iEt)$$
$$\equiv \frac{1}{r}\chi(r) \otimes \eta_{jm}(\theta, \phi) \exp(-iEt), \tag{12}$$

where

$$H_0\chi(r) = E\chi(r), \; H_0 \equiv -i\,sgn(q)\sigma_1\frac{\mathrm{d}}{\mathrm{d}r} + M\sigma_3. \tag{13}$$

Using the expressions (Kazama *et al* 1977) for the 2 component angular sections η_{jm} in terms of the monopole harmonics Y_{qlm}, and the formulae

$$(Y^*_{qlm}(-\hat{r}))_b = (-1)^{l+m}[Y_{q,l,-m}(\hat{r})]_a, \tag{14}$$

and

$$\sigma_2(\eta^*_{j,m}(-\hat{r}))_b = i(-1)^{j+m+1}(\eta_{j,-m}(\hat{r}))_a, \tag{15}$$

we have the PT, CT and CP transforms of (12)

$$\psi^{PT}(\mathbf{x}, t) = i(-1)^{j+m+1}\eta_T\eta_P\frac{1}{r}\chi^{PT}(r)\otimes\eta_{j,-m}(\hat{r})\exp(-iEt), \tag{16}$$

$$\psi^{CT}(\mathbf{x}, t) = -i\eta_C\eta^*_T\frac{1}{r}\chi^{CT}(r)\otimes\eta_{j,m}(\hat{r})\exp(+iEt), \tag{17}$$

$$\psi^{CP}(\mathbf{x}, t) = i(-1)^{j+m}\eta_C\eta^*_P\frac{1}{r}\chi^{CP}(r)\otimes\eta_{j,-m}(\hat{r})\exp(+iEt), \tag{18}$$

where

$$(\chi^{PT}(r), \chi^{CT}(r), \chi^{CP}(r)) \equiv [\sigma_3\chi^*(r), \sigma_2\chi(r), \sigma_1\chi^*(r)]. \tag{19}$$

A consequence of $\psi^{PT,CT,CP}(\mathbf{x}, t)$ obeying the Dirac equation is

$$H_0(\chi^{PT}, \chi^{CT}, \chi^{CP}) = E(\chi^{PT}, -\chi^{CT}, -\chi^{CP}). \tag{20}$$

Similarly from the formal commutation of $\mathbf{\Sigma}\cdot\boldsymbol{\pi}$ with H, and the relation in the lowest j state

$$\mathbf{\Sigma}\cdot\boldsymbol{\pi}\psi(\mathbf{x}, t) = -\frac{i}{r}sgn(q)\frac{\mathrm{d}\chi(r)}{\mathrm{d}r}\otimes\eta_{jm}(\theta, \phi), \tag{21}$$

we have the formal helicity conservation,

$$[h_0, H_0] = 0, \; h_0 \equiv -i\frac{\mathrm{d}}{\mathrm{d}r}sgn(q). \tag{22}$$

To go from formal symmetries to actual symmetries we discuss now the boundary conditions. For infinite space, Callias (1977) and Goldhaber (1977) deduced that whereas h_0 cannot be made self-adjoint with any boundary condition, self-adjointness of H_0 requires the boundary condition

$$F(0) + G(0) = (F(0) - G(0))\exp\left[i\left(\theta_0 - \frac{\pi}{2}\right)\right]. \tag{23}$$

This does not allow even hermiticity (symmetry) of h_0. The bag boundary conditions are essentially different.

Self-adjointness of the Hamiltonian

From the equation

$$(\chi_2, H_0\chi_1) - (H_0\chi_2, \chi_1) = 0, \tag{24}$$

the domain of self-adjointness $D(H_0)$ is calculated by requiring that if $\chi_1 \in D(H_0)$ then

(24) must imply that $\chi_2 \in D(H_0)$. We thus find that $D(H_0)$ consists of χ which apart from obeying,

$$\int_0^R dr\chi^\dagger(r)\chi(r) < \infty, \int_0^R dr(H_0\chi)^\dagger(H_0\chi) < \infty, \tag{25}$$

also obey the boundary conditions

$$\begin{pmatrix} F(R)+G(R)sgnq \\ F(0)-G(0)sgnq \end{pmatrix} = U\begin{pmatrix} F(R)-G(R)sgnq \\ F(0)+G(0)sgnq \end{pmatrix} \tag{26}$$

where U is the 2×2 unitary matrix

$$U \equiv \exp(i\alpha)\begin{pmatrix} \cos\lambda \exp[i(\beta+\pi/2)] & \sin\lambda \exp(i\gamma) \\ \sin\lambda \exp(-i\gamma) & -\cos\lambda \exp[-i(\beta+\pi/2)] \end{pmatrix}, \tag{27}$$

with α, β, γ, λ being arbitrary real parameters. These boundary conditions define a four parameter family of self-adjoint hamiltonians similar to the Jackiw-Rebbi and Goldstone-Wilczek one space dimension cases considered previously (Roy and Singh 1984a, b).

Self-adjointness of helicity

Similarly, the boundary conditions which make h_0 self-adjoint are

$$\begin{pmatrix} F(R)+G(R)sgnq \\ F(R)-G(R)sgnq \end{pmatrix} = V\begin{pmatrix} F(0)+G(0)sgnq \\ F(0)-G(0)sgnq \end{pmatrix}, \tag{28}$$

where V is an arbitrary 2×2 unitary matrix.

Simultoneous self-adjointness of hamiltonian and helicity

For this purpose the parameters α, β, γ, λ must be such that (26) and (28) are equivalent (*i.e.*, they imply each other). This happens if and only if $\sin\lambda = \pm 1$. Hence the common domain of self-adjointness of the hamiltonian and helicity is specified by the simple boundary condition

$$\begin{pmatrix} F(R) \\ G(R)sgnq \end{pmatrix} = \exp(i\gamma_1 + i\alpha\sigma_1)\begin{pmatrix} F(0) \\ G(0)sgnq \end{pmatrix}, \tag{29}$$

where

$$\exp(i\gamma_1) \equiv \exp(i\gamma)\sin\lambda,\ \sin\lambda = \pm 1. \tag{30}$$

Both (26) and (28) are equivalent to (29) with,

$$U = \exp(i\alpha)\begin{pmatrix} 0 & \exp(i\gamma_1) \\ \exp(-i\gamma_1) & 0 \end{pmatrix}, \quad V = \exp(i\gamma_1)\begin{pmatrix} \exp(i\alpha) & 0 \\ 0 & \exp(-i\alpha) \end{pmatrix}. \tag{31}$$

The departure of the boundary condition (29) from the quasi-periodic ones mentioned by Grossman (1983) corresponding to $\alpha = n\pi$ (n = integer) will be crucial for the existence of fractional Witten charge.

Discrete symmetries

For CP invariance (4) to hold if $\chi \in D(H_0)$, then $\chi^{CP} \in D(H_0)$, and vice versa. Similar conditions hold for CT and PT invariance. Thus conditions for the discrete symmetries to hold are given by

$$PT\colon U = U^T;\quad CT\colon U = -\sigma_3 U^\dagger \sigma_3;\quad CP\colon U = -\sigma_3 U^* \sigma_3. \tag{32}$$

The infinite space results follow from (32) by setting $|U_{22}| = 1$, and other $U_{ij} = 0$, showing that PT invariance allows arbitrary θ_0 in (23) whereas CT and CP invariance give identical conditions $\theta_0 = n\pi$. For finite R, (32) yields

$$\begin{aligned} &PT\colon \sin\lambda \sin\gamma = 0 \\ &CT\colon \text{either } \sin\alpha = \cos\lambda \sin\beta = 0 \text{ or } \cos\alpha = \cos\beta = \sin\lambda = 0 \\ &CP\colon \text{either } \sin\alpha = \cos\lambda \sin\beta = \sin\lambda \sin\gamma = 0, \\ &\quad \text{or } \cos\alpha = \cos\lambda\cos\beta = \sin\lambda\cos\gamma = 0. \end{aligned} \tag{33}$$

For CT or CP invariance we need $\sin\alpha\cos\alpha = 0$. If $\sin\alpha = 0$, CP invariance implies CT invariance. If $\cos\alpha = 0$, CT invariance implies CP invariance. The most symmetric case, with helicity self-adjoint and conserved, and all the discrete symmetries PT, CT, CP valid corresponds to:

$$h_0,\ PT,\ CT,\ CP\colon \sin\alpha = \cos\lambda = \sin\gamma = 0, \text{ i.e. } U = \pm\sigma_1, \tag{34}$$

i.e., periodic or antiperiodic boundary conditions.

Vacuum charge eigenvalues

The Dirac field operator $\Psi(\mathbf{x}, t)$ in the Heisenberg representation obeys the same differential equation as the c-no. wave function $\Psi(\mathbf{x}, t)$ and has the charge conjugation property

$$\Psi_\alpha(\mathbf{x}, t) \xrightarrow{C} \Psi^C_\alpha(\mathbf{x}, t) = \eta_C(\beta\alpha_2)_{\alpha\beta}\Psi^\dagger_\beta(\mathbf{x}, t). \tag{34}$$

The total charge operator N (odd under C) and its eigenvalue N_0 in a vacuum state (with a convenient regularization (Goldstone and Jaffe 1983; Paranjape and Semenoff 1983; Niemi and Semenoff 1983)) are given by

$$N = \frac{1}{2}\int d^3x [\Psi^\dagger_\alpha(\mathbf{x}, t), \Psi_\alpha(\mathbf{x}, t)], \tag{35}$$

$$N_0 = \lim_{\varepsilon\to 0+}\left[-\frac{1}{2}\sum_{\substack{n\\ E_n\neq 0}} (sgn E_n)\exp(-\varepsilon|E_n|)\right] + N_{0,0}, \tag{36}$$

where $N_{0,0}$ is the contribution of zero energy levels to vacuum charge. When there are ν zero energy c-no. wave functions the vacuum state is 2^ν fold degenerate and the corresponding $N_{0,0}$ vary from $-\nu/2$ to $\nu/2$ in steps of unity. In such a case we shall calculate the N_0 corresponding to the choice of the vacuum with $N_{0,0} = -\nu/2$; the other vacua will have charges differing from this by integers.

We shall only calculate the contribution to vacuum charge N_0 of the lowest angular momentum levels. The results will however give the full vacuum charge whenever the boundary conditions for the higher j levels respect CP or CT leading to $E \to -E$ symmetry of these levels and hence their zero contribution to N_0.

For the most general boundary condition given by (26) and (27), we find that the energy levels are given by

$$\cos(kR)\cos\alpha + \frac{\sin(kR)}{k}[E\sin\alpha + M\cos\lambda\cos\beta] = \cos\zeta, \tag{37}$$

where

$$k^2 \equiv E^2 - M^2, \cos\zeta \equiv \sin\lambda \cos\gamma. \tag{38}$$

Vacuum charge for $M = 0$

In this case,

$$E(M=0) = \frac{\alpha+\zeta+2n\pi}{L} \text{ and } \frac{\alpha-\zeta+2n\pi}{L}, n = 0, \pm 1, \pm 2, \ldots. \tag{39}$$

Each of these levels is $2j+1 = 2|eg| = 2|q|$ fold degenerate. A simple calculation yields

$$N_0(M=0) = 2|q|\left(\frac{\alpha}{\pi} - 1 - \mathrm{Int}\left(\frac{\alpha+\zeta}{2\pi}\right) - \mathrm{Int}\left(\frac{\alpha-\zeta}{2\pi}\right)\right), \tag{40}$$

with $\mathrm{Int}\, x \equiv$ largest integer $\leqslant x$.

Vacuum charge for $M > 0$

In this case, for a given U, $E = E(M=0) + 0(1/n)$, for $n \to \infty$. We then show that $\mathrm{d}N_0(M)/\mathrm{d}M = 0$ except at those values $M = M_i$ where one or more eigenvalues E pass through zero leading to an integer jump in $N_0(M) \equiv N_0$ (Roy and Singh 1984a, b). We find,

$$N_0(M) = N_0(M=0) + I, \tag{41}$$

where I is an integer given by

$$I = -2|q| \sum_{i=1}^{2} sgn\left(\frac{\mathrm{d}E}{\mathrm{d}M}\right)_{M=M_i} \theta(M-M_i)\theta(x_i)\theta(-y_i), \tag{42}$$

with $\theta(x)$ being the step function, and

$$x_i \equiv \frac{1}{D}(\cos\lambda\cos\beta\cos\zeta + (-1)^i S\cos\alpha), \tag{43}$$

$$y_i \equiv \frac{1}{D}(\cos\alpha\cos\zeta + (-1)^i S\cos\lambda\cos\beta), \tag{44}$$

$$S \equiv +(\cos^2\zeta + \cos^2\lambda\cos^2\beta - \cos^2\alpha)^{1/2}, \tag{45}$$

$$D \equiv \cos^2\lambda\cos^2\beta - \cos^2\alpha, \tag{46}$$

$$M_i \equiv \frac{1}{R}\sin h^{-1} x_i, \tag{47}$$

and

$$\left(\frac{\mathrm{d}E}{\mathrm{d}M}\right)_{M=M_i} \equiv -\frac{M_i R}{\sin\alpha}(\cos\alpha + \cos\lambda\cos\beta \coth(M_i R)). \tag{48}$$

Equation (41) gives the exact formula for the vacuum charge $N_0(M)$. Its fractional part is given by the simple formula

$$\text{Fractional part of } (N_0(M) - 2|eg|\alpha/\pi) = 0. \tag{49}$$

Helicity conservation allows arbitrary α and hence arbitrary fractional value for vacuum (monopole) charge. So does PT conservation. For either CP or CT to be conserved $\sin\alpha\cos\alpha = 0$, and hence the vacuum charge is $\frac{1}{2} \times$ integer, where the integer

Table 1. Dependence of fractional part of vacuum charge (equation (49)) on symmetries of the lowest angular momentum fermion-monopole bag hamiltonian assuming CP (or CT) invariance of the higher angular momentum hamiltonians.

Conserved quantities	Helicity	CP	CT	PT	Helicity, CP, CT, PT
Fractional part of vacuum charge	arbitrary	$0, \frac{1}{2}$	$0, \frac{1}{2}$	arbitrary	0

can be even or odd. This is due to the fact that the non-zero energy levels then occur in pairs due to $E \to -E$ symmetry and do not contribute to the vacuum charge. If helicity, CP, CT and PT are conserved then $\sin\alpha = 0$, and the vacuum charge must be integral (table 1).

Our extension of the Callias-Goldhaber and Witten-Grossman-Yamagishi results demonstrate that the change from infinite to finite space volume has interesting consequences for the vacuum charge eigenvalues and symmetry properties *e.g.*, if we consider $U = \text{diag} \exp[i\eta(\theta_0' - \pi/2)], \exp[-i\eta(\theta_0 - \pi/2)]$ which Yamagishi used in discussing the $R \to \infty$ limit, we find that N_0 has a fractional part $2q(\theta_0' - \theta_0 + \pi)/(2\pi)$, with a non-trivial dependence on the boundary condition parameter θ_0' at $r = R$, no matter how large R may be. This fact escaped notice previously because N_0 was not calculated for finite R. Further, more symmetry properties are possible to satisfy for finite R than for $R = \infty$, each symmetry restricting boundary conditions and vacuum charge eigenvalues.

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