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Schwinger's action principle, supersymmetry and path integrals

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Abstract. We use Schwinger's action principle in quantum mechanics to obtain the quantisation from Lagrangian for the fermionic variables, as well as when it contains auxiliary coordinates. We illustrate this with a supersymmetric Lagrangian which naturally includes auxiliary variables. We further show that the action principle also leads to Feynman's path integral quantisation, which is aesthetically pleasing.

Keywords. Schwinger's action principle; path integrals; supersymmetry.

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1. Introduction

In path-integral (Feynman and Hibbs 1965) or functional integration methods (Rzewuski 1969) the Lagrangian of the dynamical system plays the essential role. Schwinger developed his action principle (Schwinger 1951, 1953) which was also naturally based on the Lagrangian. This however had ambiguities while dealing with fermions (Schwinger 1970). We have recently generalised (Misra and Pattnaik 1983) action principle to superspace which is useful in many contexts (Dimopoulos and Georgi 1981; Sakai and Yanagida 1982; Weinberg 1982; Salam and Strathdee 1978; Wess 1981) and had seen that by using the constraints from field equations, one can remove the above ambiguities for the fermions, as well as treat auxiliary variables. In the present paper we illustrate the same technique in ordinary space with a supersymmetric Lagrangian* and establish a close link between Schwinger's action principle and Feynman's path integrals.

2. Schwinger's action principle

For the sake of clarity, we shall very briefly note the basic features of Schwinger's action principle for quantum mechanics as well as introduce specific modifications. Here, the variation of the amplitudes $\langle q', t | q'_0, t_0 \rangle$ is given as (Schwinger 1951, 1953; Dirac 1935)

$$|q', t\rangle + \delta |q', t\rangle = \exp(iF)|q', t\rangle \tag{1}$$

where F is hermitian. This leads to

$$\delta\langle q'_2, t_2 | q'_1, t_1\rangle = -i\langle q'_2, t_2|\delta W_{21}|q'_1, t_1\rangle$$

* We note that supersymmetric invariance of Lagrangians is modulo a divergence, which in the context of Schwinger's action principle, is relevant for quantisation.

where by (1)

$$\delta W_{21} = F_2 - F_1. \tag{2}$$

We note that, in the above, F_1 and F_2 are in fact infinitesimal. Schwinger's action principle (Schwinger 1951, 1953) identifies δW_{21} with the variation of action W_{21} which is given as

$$W_{21} = \int_{t_1}^{t_2} L(q, \dot{q})\,\mathrm{d}t. \tag{3}$$

In the above, $L(q, \dot{q})$ is the Lagrangian operator containing only up to first order time derivatives of the generalised coordinates*. In (3) the variations $\delta t_1, \delta t_2$ and $\delta q_\alpha(t)$ for $t_1 \leqslant t \leqslant t_2$ are all permitted (Schwinger 1951, 1953). One then obtains from (3)

$$\begin{aligned}\delta W_{21} &= \int_{t_1+\delta t_1}^{t_2+\delta t_2} (L+\delta L)\,\mathrm{d}t - \int_{t_1}^{t_2} L\,\mathrm{d}t \\ &= L(t_2)\delta t_2 - L(t_1)\delta t_1 + \delta q_\alpha(t_2)\frac{\partial L}{\partial \dot{q}_\alpha(t_2)} - \delta q_\alpha(t_1)\frac{\partial L}{\partial \dot{q}_\alpha(t_1)} \\ &\quad + \int_{t_1}^{t_2} \delta q_\alpha \left(\frac{\partial L}{\partial q_\alpha} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_\alpha}\right)\mathrm{d}t.\end{aligned}$$

Comparing with (2), we then obtain

$$F_2 = L\delta t_2 + \delta q_\alpha \frac{\partial L}{\partial \dot{q}_\alpha}, \tag{4}$$

at $t = t_2$ and similarly for F_1, as well as the equation of motion

$$\frac{\partial L}{\partial q_\alpha} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_\alpha} = 0. \tag{5}$$

The *modifications we introduce* here in action principle (Schwinger 1951, 1953) is to explicitly include the *constraints from the equation of motion* (5) *in the expressions for the generators,* F_2 or F_1. We shall see that this enables us to deal with the auxiliary coordinates as well as fermionic coordinates without any ambiguity.

Let $\delta t = 0$ in (4) and, we consider the variation of states and operators simultaneously. Then (Schwinger 1951, 1953)

$$G + \delta G = \exp(iF)\,G\exp(-iF)$$

such that

$$[G, F] = i\delta G, \tag{6}$$

where, F is infinitesimal. If the Lagrangian is such that δq_α's in (4) have no additional constraints derived from equation of motion, then (6) yields the quantum conditions

$$\left[q_\alpha, \frac{\partial L}{\partial \dot{q}_\beta}\right] = i\delta_{\alpha\beta}, \tag{7}$$

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* Schwinger's action principle with higher order derivatives has been considered by Misra (1959) (also see Barut and Mullen 1962; Riewe and Green 1972).

for bosonic coordinates (Schwinger 1951, 1953) and

$$\left[q_\alpha, \frac{\partial L}{\partial \dot{q}_\beta} \right]_+ = -i\delta_{\alpha\beta}, \tag{8}$$

for fermionic coordinates. The difference in signs in (8) may be noted*. We may explicitly see the derivation of (8) when we see that from (6), for the fermionic variables,

$$\left[q_\alpha, \delta q_\beta \frac{\partial L}{\partial \dot{q}_\beta} \right] = [q_\alpha, \delta q_\beta]_+ \frac{\partial L}{\partial \dot{q}_\beta} - \delta q_\beta \left[q_\alpha, \frac{\partial L}{\partial \dot{q}_\beta} \right]_+$$

$$= -\delta q_\beta \left[q_\alpha, \frac{\partial L}{\partial \dot{q}_\beta} \right]_+ = i\delta q_\alpha.$$

However, we shall see that often there are *constraints* on the variations δq_α's to be used in F arising from the equations of motion (5). In § 3, we shall illustrate this fact for the super-symmetric Lagrangian, where the increments of the fermionic coordinates get related from (5), as well as there are auxiliary coordinates for which the canonical conjugate momenta do not exist. In § 4 we shall further derive an intimate connection between Schwinger's action principle and the path integral method of Feynman by taking in (4) the variations $\delta q_\alpha = 0$. For the sake of completeness, however, we first note here a derivation of the Schrödinger equation from the action principle. We take in (4) the total variation $\Delta q_\alpha = \dot{q}_\alpha \delta t + \delta q_\alpha$ as zero. Then it yields,

$$F = \left(L - \dot{q}_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} \right) \delta t. \tag{9}$$

By (1) and (9) we also have

$$i \frac{\mathrm{d}}{\mathrm{d}t} |q', t\rangle_s = H |q', t\rangle_s \tag{10}$$

where

$$H = \dot{q}_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} - L. \tag{11}$$

Thus, by choosing the intrinsic variations δq_α to make q_α independent of time, we have obtained the Schrödinger equation and the Hamiltonian as outputs from the action principle, as was expected.

3. Supersymmetric Lagrangian

We shall now demonstrate the viability of the present form of the action principle through an example. We take the supersymmetric Lagrangian (Salomonson and Van

* It appears peculiar that (8) for fermions has never been noted earlier. We have followed here the differentiation rules of Berezin (1966) and Martin (1959), and bosons and fermions have been defined through commutativity or anti-commutativity.

Holten 1982; Witten 1981; Misra and Pattnaik 1983),

$$L = \tfrac{1}{2}\dot{q}^2 + \tfrac{1}{2}d^2 + \tfrac{i}{2}(\psi^\dagger\dot{\psi} - \dot{\psi}^\dagger\psi) - dv' - \frac{i}{2}d\dot{q}$$
$$+ \frac{i}{2}\dot{q}d - \tfrac{1}{2}(\psi\psi^\dagger - \psi^\dagger\psi)v'', \tag{12}$$

where q, ψ and d represent the bosonic, fermionic and auxiliary coordinates respectively. q and d are real and ψ is complex. In (12), $v = v(q)$ and the dashes denote the derivatives. Applying the action principle we get from (5) the equations of motion

$$d = v'(q), \tag{13}$$

$$i\dot{\psi} + v''\psi = 0, \tag{14}$$

$$i\dot{\psi}^\dagger - v''\psi^\dagger = 0, \tag{15}$$

and

$$\ddot{q} + dv'' + \tfrac{1}{2}(\psi\psi^\dagger - \psi^\dagger\psi)v''' = 0. \tag{16}$$

We note that the derivations of the equations of motion (13) to (16) or (5) needs only the commutativity (or anticommutativity for the fermions) of the *increments* δq_α with q_β. We shall now proceed to obtain the quantum algebra for the coordinates and their derivatives through (6). We thus evaluate $F(t)$ of (4). With $\delta t = 0$, this becomes from (12)

$$F(t) = \delta q\dot{q} - \frac{i}{2}\delta\psi\psi^\dagger - \frac{i}{2}\delta\psi^\dagger\psi. \tag{17}$$

We now have $\delta d = v''(q)\,\delta q$ from (13). Also from (14) and (15), $\mathrm{d}/\mathrm{d}t\,(\psi^\dagger\psi) = 0$, such that $\psi^\dagger\psi$ is constant. This equation leads to the constraint for the variation of the fermionic coordinates as

$$\delta\psi\psi^\dagger + \psi\delta\psi^\dagger = 0. \tag{18}$$

Thus $\delta\psi$ and $\delta\psi^\dagger$ cannot be treated as independent in F. Also, with (18) *e.g.* $i\psi\delta\psi^\dagger$ is hermitian. The two degrees of freedom present now are δq, and $\delta\psi$ or $\delta\psi^\dagger$. Then from (6) we obtain that

$$[q, \dot{q}] = i, \tag{19}$$

$$[q, \psi^\dagger] = 0, \tag{20}$$

$$[q, \psi] = 0 \tag{21}$$

$$[\psi, \psi^\dagger]_+ = 1 \tag{22}$$

and

$$[d, \dot{q}] = i\delta d. \tag{23}$$

Equation (23) is a *consistency* relation which follows from (13) and (19). Here, we are unable for the present to get the relation $\psi^2 = 0$; however from equations of motion we can verify that if ψ^2 is zero at any time t, it continues to be zero. This follows since we get from (14) that $\mathrm{d}/\mathrm{d}t\,\psi^2 = 2iv''\psi^2$. $\psi^2 = 0$ can also be explicitly obtained if we consider the supercharge (Misra and Pattnaik 1983). From (11) and (12) we also get

$$H = \tfrac{1}{2}\dot{q}^2 + \tfrac{1}{2}v'^2 + \tfrac{1}{2}(\psi\psi^\dagger - \psi^\dagger\psi)v'', \tag{24}$$

which is the starting point of (Salomonson and Van Holten 1982; Witten 1981) for considering supersymmetric quantum mechanics.

4. Feynman picture and path integral quantisation

We had seen that the total variation $\Delta q_\alpha = 0$ yields the Schrödinger equation (10), along with the Hamiltonian as the time evolution operator for the states. This was the Schrödinger picture (Dirac 1935). It is obvious that in (4) choice of the intrinsic variation such that $F = 0$ will yield the Heisenberg picture. We now take the intrinsic variation $\delta q_\alpha = 0$, and call this the Feynman picture, since we shall see that it generates the path integral quantisation of Feynman. We note that $\delta q_\alpha = 0$ yields $\Delta q_\alpha = \dot{q}_\alpha \delta t$. Here q_α must change according to the equations of motion. Also from (1) and (4)

$$|q'(t+\delta t), t+\delta t\rangle = \exp(iL\delta t)\,|q'(t), t\rangle. \tag{25}$$

Thus here the Lagrangian becomes the time evolution operator for the states. Then, as usual, the transition amplitude $\langle q', t|q'_0, t_0\rangle$ with $t_0 < t_1 < \ldots < t_n = t$ is given as

$$\langle q', t|q'_0, t_0\rangle = \int \prod_{i=0}^{n-1} (\mathrm{d}q'_i A_i)$$
$$\exp\left[-i\sum_{i=0}^{n-1} S(q'_{i+1}, t_{i+1}; q'_i, t_i)\right], \tag{26}$$

where S is the "elementary" action in the interval (t_i, t_{i+1}). Clearly, while evaluating the matrix element corresponding to S in (26) through (25), the ambiguities inherently present in its definition (Feynman and Hibbs 1965) will become exhibited. The constants A_i's in the present case can be determined from the condition that*

$$\int \langle q'_{i+1}, t_{i+1}|q'_i, t_i\rangle \,\mathrm{d}q'_i\, \langle q'_i, t_i|q'_{i-1}, t_{i-1}\rangle$$
$$= \langle q'_{i+1}, t_{i+1}|q'_{i-1}, t_{i-1}\rangle. \tag{27}$$

Obviously in (26) $\prod_i A_i \mathrm{d}q'_i = Dq'$ such that we obtain the familiar path integral of Feynman, and the "volume" element is already properly normalised. The derivation of Feynman's path integral from Schwinger's action principle was possible since, on taking the intrinsic variations δq_α as zero, the *Lagrangian* becomes the time evolution operator for the states**

One further remark appears to be worthwhile. When $\delta t = 0$, we obtain from (4) that $F = 0$ in the *Feynman* picture, since $\delta q_\alpha = 0$. This implies that the coordinates will be classical variables and (6) *does not* really lead to a quantisation condition. We may regard this as a basis for taking classical functions for the purpose of path integrals, as is conventionally done.

In the present note we thus find that the modified version of the action principle is equally applicable in ordinary space. It is further significant that the *same principle* leads

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* The normalisation constants which can be evaluated directly is a new feature. Example for $\frac{1}{2}\dot{q}^2$ in the Lagrangian L, a straight-forward calculation will show that $A_i = (2\pi)\,(\delta t_i)^{-1/2}$ in the limit of small δt_i.

** The converse problem of obtaining Schwinger's operator version of quantisation from Feynman path integral quantisation is considered in Yourgrau and Mandelstam (1968), and also by Riewe and Green (1972).

to canonical quantisation of bosonic and fermionic variables along with auxiliary coordinates as well as to the Feynman's path integral quantisation in § 4. Here we have dealt with quantum mechanics, but the problem in field theory is similar, and the generalisation is straightforward.

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