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Two-mode para-Bose number states

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Abstract. Two-mode para-Bose number states are discussed. The two-mode system has been chosen as it is a representative of the multi-mode system. Salient properties like normalization, orthogonality and degeneracy of these states have also been discussed.

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1. Introduction

Since the introduction of generalized commutation relations (Green 1953)

$$[\tfrac{1}{2}\{a_i, a_j\}, a_k] = 0, \tag{1}$$

$$[\tfrac{1}{2}\{a_i, a_j^+\}, a_k^+] = \delta_{ik} a_j^+ \tag{2}$$

and their complex conjugates for the particles not obeying the symmetrization postulate, considerable work has been done to find representations of para-Bose and para-Fermi operators. The main problem confronted while considering multi-mode para-Bose system is the non-commuting nature of the operators belonging to different modes. As a result additional indices are required for defining a basis spanning Fock-space of a para-Bose system. We consider in this paper a two-mode para-Bose system as it is a representative of the multi-mode system. We shall restrict our discussion to number states only.

In section 2 the number states are defined and the necessity of using an additional operator (parameter) is elucidated. The orthogonality and normalization of these states are discussed in § 3. The matrix elements of the number operator and the free Hamiltonian are obtained and the degeneracy of these states is discussed in § 4.

2. Number states

While discussing the number states of a two mode system, we shall first review briefly the normal two-mode Bose oscillator. For the normal case the convenient choice of the basis is the state $|n_1, n_2\rangle$ defined by

$$|n_1, n_2\rangle = a_1^{+n_1} a_2^{+n_2} |0\rangle, \tag{3}$$

where $|0\rangle$ is the unique ground state. Obviously, n_1 and n_2 represent the number of excitations of modes 1 and 2 respectively. An alternative basis is obtained by the states

$$|l,m\rangle = (\tfrac{1}{2}\{a_1^+, a_2\})^m a_2^{+l}|0\rangle \; (l > m). \tag{4}$$

Here m and $(l-m)$ are the number of excitations of modes 1 and 2 respectively. The state $|l, m\rangle$ with $l < m$ vanishes.

In the case of a para-Bose system, operators belonging to the different modes do not commute, *i.e.*

$$[a_i^+, a_j^+] \neq 0, \tag{5a}$$

$$[a_i, a_j^+] \neq 0 \quad (i \neq j). \tag{5b}$$

We shall see below that the number of excitations of modes 1 and 2 does not uniquely define a given state and one needs a third parameter to label the basis. A basis with only two parameters can span only a part of the Hilbert-Fock space of a two-mode para-Bose system. A convenient choice of the basis, the number states (Alabiso *et al* 1969) is;

$$|n,l,m\rangle = K^{+m} a_2^{+l} J^{+n}|0\rangle, \quad (l > m) \tag{6}$$

where

$$J^+ = \tfrac{1}{2}[a_1^+, a_2^+], \tag{7a}$$

$$K^+ = \tfrac{1}{2}\{a_1^+, a_2\}, \tag{7b}$$

and $|0\rangle$ is a unique vacuum state such that

$$\{a_i, a_j^+\}|0\rangle = \delta_{ij} L|0\rangle. \tag{8}$$

Here L is an integer (Greenberg and Messiah 1965) and has values greater than one for para-bosons. For bosons $L = 1$, L is termed as the order of para-statistics. It is readily seen from (6) that the number of excitations of the first and second modes are $n + m$ and $l + n - m$ respectively. The states $|n,l,m\rangle$ are termed two-mode para-Bose number states. We shall later show that it is an eigen state of the number operator and the free Hamiltonian of the system.

3. Orthogonality and normalization of number states

The number states $|n,l,m\rangle$ as defined by (6) are not normalized. We shall now discuss the orthogonality and normalization of these states. For this purpose we shall make use of the following relations

$$[K^+, a_2^+] = a_1^+, \tag{9a}$$

$$[K^+, J^+] = \{a_2^+, J^+\} = 0. \tag{9b}$$

Now we can obtain using (6) and (9) the following relations,

$$K|n,l,m\rangle = m(l-m+1)|n,l,m-1\rangle, \tag{10}$$

$$J|n,0,0\rangle = \frac{\{n+1+(L-2)\Lambda_n\}!}{\{n-1+(L-2)\Lambda_n\}!}\,|n-1,0,0\rangle, \tag{11}$$

and

$$a_2|n,l,0\rangle = \frac{l+\Lambda_l\{n+l+1+(L-2)\Lambda_{n+l}\}}{(l+1)}|n,l-1,0\rangle$$

$$+(-1)^{l+1}\frac{\{n+(L-2)\Lambda_n\}}{(l+1)}|n-1,l+1,1\rangle, \tag{12}$$

where Λ_k is the projection to odd k values *i.e.*

$$\Lambda_k = \tfrac{1}{2}\{1-(-1)^k\} = 0 \text{ if } k \text{ is even},$$
$$\Lambda_k = \tfrac{1}{2}\{1-(-1)^k\} = 1 \text{ if } k \text{ is odd}. \tag{13}$$

Successively applying the operator K to the basis $|n,l,m\rangle$ for m' times, we obtain from (10),

$$K^{m'}|n,l,m\rangle = \frac{m!\,(l-m+m')!}{(m-m')!\,(l-m)!}|n,l,m-m'\rangle. \tag{14}$$

Now consider the matrix element,

$$\langle n',l',0|K^{m'}|n,l,m\rangle = \frac{\mathrm{m}!(l-m+m')!}{(m-m')!\,(l-m)!}\langle n',l',0|n,l,m-m'\rangle. \tag{15}$$

By defining the basis $|n,l,m\rangle$ (equation (6)) we have

$$\langle n',l',0|K^{m'} = \langle n',l',m'|.$$

Therefore (15) can be rewritten as

$$\langle n',l',m'|n,l,m\rangle = \frac{m!\,(l-m+m')!}{(m-m')!\,(l-m)!}\langle n',l',0|n,l,m-m'\rangle. \tag{16}$$

For $m < m'$ the right side of (16) vanishes. On taking the complex conjugate of (16) and interchanging the roles of m and m' it readily follows that the scalar product;

$$\langle n',l',m'|n,l,m\rangle = 0, \tag{17}$$

for $m > m'$ as well. For $m = m'$ we obtain

$$\langle n',l',m|n,l,m\rangle = \frac{m!\,l!}{(l-m)!}\langle n',l',0|n,l,0\rangle. \tag{18}$$

From (16)–(18) it can be concluded that

$$\langle n',l',m'|n,l,m\rangle = \delta_{mm'}\frac{l!\,m!}{(l-m)!}\langle n',l',0|n,l,0\rangle. \tag{19}$$

Now consider the scalar product $\langle n',l',0|n,l,0\rangle$. By definition

$$\langle n',l',0|n,l,0\rangle = \langle n',l'-1,0|a_2|n,l,0\rangle. \tag{20}$$

From (12) and (19) it follows

$$\langle n',l',0|n,l,0\rangle = \frac{l+\Lambda_l}{(l+1)}\{n+l+1+(L-2)\Lambda_{n+l}\}\ \langle n',l'-1,0|n,l-1,0\rangle. \tag{21}$$

A repeated iteration of this procedure clearly indicates that $\langle n',l',0|n,l,0\rangle$ vanishes unless $l = l'$. It is so because if $l' > l$,

$$\langle n',l'-l,0|n,0,0\rangle = \langle n',l'-l-1,0|a_2|n,0,0\rangle = 0,$$

and for $l' < l$,

$$\langle n', 0, 0 | n, l-l', 0 \rangle = \langle n', 0, 0 | a_2^+ | n, l-l'-1, 0 \rangle = 0.$$

Hence we may write

$$\langle n', l', 0 | n, l, 0 \rangle = \delta_{ll'} \frac{(l-\Lambda_l)!!\,(n+l+1-\Lambda_{n+l})!!\,(n+l+L-2+\Lambda_{n+l})!!}{(l+1-\Lambda_l)!!\,(n+1-\Lambda_n)!!\,(n+L-2+\Lambda_n)!!} \langle n', 0, 0 | n, 0, 0 \rangle. \tag{22}$$

Now put $l = m = 0$ in (6) and consider the scalar product

$$\langle n', 0, 0 | n, 0, 0 \rangle = \langle n'-1, 0, 0 | J | n, 0, 0 \rangle, \tag{23}$$

Now using (11) we have;

$$\langle n', 0, 0 | n, 0, 0 \rangle = \frac{\{n+1+(L-2)\Lambda_n\}!}{\{n-1+(L-2)\Lambda_n\}!} \langle n'-1, 0, 0 | n-1, 0, 0 \rangle. \tag{24}$$

A repeated iteration of this process clearly indicates that $\langle n', 0, 0 | n, 0, 0 \rangle$ vanishes unless $n = n'$. As for $n' > n$,

$$\langle n'-n, 0, 0 | 0 \rangle = \langle n'-n-1, 0, 0 | J | 0 \rangle = 0,$$

and if $n' < n$ then,

$$\langle 0 | n-n', 0, 0 \rangle = \langle 0 | J^+ | n-n'-1, 0, 0 \rangle = 0.$$

Therefore,

$$\langle n', 0, 0 | n, 0, 0 \rangle = \delta_{nn'} \prod_{k=1}^{n} \left[\frac{\{k+1+(L-2)\Lambda_k\}!}{\{k-1+(L-2)\Lambda_k\}!} \right] \langle 0 | 0 \rangle. \tag{25}$$

Assuming that the vacuum state $|0\rangle$ is normalized *i.e.* $\langle 0 | 0 \rangle = 1$, (25) can be rewritten as

$$\langle n', 0, 0 | n, 0, 0 \rangle = \frac{(n+1-\Lambda_n)!\,(n+L-2+\Lambda_n)!}{(L-2)!} \delta_{nn'}. \tag{26}$$

From (19), (22) and (26) we have,

$$\langle n', l', m' | n, l, m \rangle = \delta_{mm'} \delta_{ll'} \delta_{nn'} N_{nlm}.$$

Here N_{nlm}, the normalization factor, is given by,

$$N_{nlm} = \frac{l!m!\,(l-\Lambda_l)!!\,(n+l+1-\Lambda_{n+l})!!\,(n+l+L-2+\Lambda_{n+l})!!}{(l+1-\Lambda_l)!!\,(l-m)!} \frac{(n-\Lambda_n)!!\,(n+L-3+\Lambda_n)!!}{(L-2)!}. \tag{27}$$

Here Λ_k is the projection to the odd k values (cf. equation (13)). The normalized number state is defined as

$$|n, l, m\rangle = \frac{1}{(N_{nlm})^{1/2}} K^{+m} a_2^{+l} J^{+n} |0\rangle. \tag{28}$$

For normalized number states $|n, l, m\rangle$ we have

$$\langle n', l', m' | n, l, m \rangle = \delta_{nn'} \delta_{ll'} \delta_{mm'}. \tag{29}$$

We shall henceforth use only the normalized number states defined by (28). We have thus established that the number states $|n, l, m\rangle$ are orthogonal and normalized.

4. Matrix elements of number operator and degeneracy of number states

We shall now obtain the matrix elements of the number operator and the free Hamiltonian and then discuss the degeneracy of number states. The number operator for two mode para-Bose system is defined as

$$N = N_1 + N_2, \tag{30a}$$

where

$$N = \tfrac{1}{2}(\{a_1, a_1^+\} - L), \tag{30b}$$

and

$$N_2 = \tfrac{1}{2}(\{a_2, a_2^+\} - L). \tag{30c}$$

From commutation relations

$$[a_2, \tfrac{1}{2}\{a_1, a_2^+\}] = a_1, \tag{31a}$$

$$[a_2, \tfrac{1}{2}\{a_1^+, a_2\}] = 0, \tag{31b}$$

and (12) and (27) we obtain,

$$a_1|n, l, m\rangle = \left[\frac{m\{n+l+1+(L-2)\Lambda_{n+l}\}}{(l+1-\Lambda_l)}\right]^{1/2}|n, l-1, m-1\rangle + (-1)^l\left[\frac{(l-m+1)\{n+(L-2)\Lambda_n\}}{(l+1+\Lambda_l)}\right]^{1/2} |n-1, l+1, m\rangle, \tag{32a}$$

$$a_1^+|n, l, m\rangle = \left[\frac{(m+1)\{n+l+2+(L-2)\Lambda_{n+l+1}\}}{(l+1+\Lambda_l)}\right]^{1/2}|n, l+1, m+1\rangle + (-1)^{l+1}\left[\frac{(l-m)\{n+1+(L-2)\Lambda_{n+1}\}}{(l+1-\Lambda_l)}\right]^{1/2} |n+1, l-1, m\rangle, \tag{32b}$$

$$a_2|n, l, m\rangle = \left[\frac{(l-m)\{n+l+1+(L-2)\Lambda_{n+l}\}}{(l+1-\Lambda_l)}\right]^{1/2}|n, l-1, m\rangle + (-1)^{l+1}\left[\frac{(m+1)\{n+(L-2)\Lambda_n\}}{(l+1+\Lambda_l)}\right]^{1/2} |n-1, l+1, m+1\rangle, \tag{32c}$$

$$a_2^+|n, l, m\rangle = \left[\frac{(l-m+1)\{n+l+2+(L-2)\Lambda_{n+l+1}\}}{(l+1+\Lambda_l)}\right]^{1/2}|n, l+1, m\rangle + (-1)^l\left[\frac{m\{n+1+(L-2)\Lambda_{n+1}\}}{(l+1-\Lambda_l)}\right]^{1/2} |n+1, l-1, m-1\rangle. \tag{32d}$$

Equations (32a)–(32d) can be used for specifying completely the number state representation of two-mode para-Bose system. We obtain from these equations

$$N_1|n,l,m\rangle = \tfrac{1}{2}(\{a_1, a_1^+\} - L)|n,l,m\rangle = (n+m)|n,l,m\rangle, \quad (33a)$$

$$N_2|n,l,m\rangle = \tfrac{1}{2}(\{a_2, a_2^+\} - L)|n,l,m\rangle = (n+l-m)|n,l,m\rangle, \quad (33b)$$

and for the Number operator N,

$$N|n,l,m\rangle = (N_1 + N_2)|n,l,m\rangle = (2n+l)|nlm\rangle. \quad (33c)$$

It readily follows that $|n,l,m\rangle$ is also an eigenstate of free Hamiltonian $H = H_1 + H_2 = \frac{1}{2}(\{a_1, a_1^+\} + \{a_2, a_2^+\})$ with eigenvalue $(2n + l + L)$. The eigenvalue equation is given by

$$H|n,l,m\rangle = (2n + l + L)|n,l,m\rangle. \quad (34)$$

We have thus established that $|n,l,m\rangle$ is an eigenstate of energy operator H with the lowest energy eigenvalue being L, the order of para-statistics. For a para-boson system we observe that the lowest energy eigenvalue is related to the order of para-statistics while for normal bosons it is a constant as $L = 1$.

It follows from (33) that there are several states $|n,l,m\rangle$ with the same given eigenvalues of N_1 and N_2 for a given order of statistics L. This degeneracy of the number state $|n,l,m\rangle$ with given eigenvalues of N_1 and N_2 is another salient feature of two mode para-Bose system, besides the presence of anti-symmetrical states ($J^+ = \frac{1}{2}[a_1^+, a_2^+] \neq 0$). The degeneracy of the basis $|n,l,m\rangle$ is also obvious from the presence of three running indices n, l, m. The state $|n,l,m\rangle$ has $n_1 = n+m$ and $n_2 = n+l-m$ excitations of the first and second modes respectively. The order of degeneracy of the state $|n,l,m\rangle$ with fixed excitation $N = n_1 + n_2$ is given by

$$D = \tfrac{1}{4}(N + 2 - \Lambda_N)(N + 2 + \Lambda_N). \quad (35)$$

On the other hand if we fix values of n_1 and n_2 respectively then the order of degeneracy becomes

$$D = n' + l, \quad (36)$$

where n' is the smaller of the two integers n_1 and n_2, *i.e.*

$$n' = \min(n_1, n_2).$$

5. Conclusion

The para-Bose number states for the two-mode system discussed in this paper have two salient features, namely, presence of anti-symmetrical states and degeneracy. The treatment of the problem is complex as additional indices are needed for defining the number states. It is not possible in the case of para-Bose number states to straightaway generalize a one- or two-mode system to an n mode system. The number states for a three mode system will require six indices and one of the convenient choices (Saxena 1983) is;

$$(\tfrac{1}{2}\{a_1^+, a_2\})^m (\tfrac{1}{2}\{a_3^+, a_2\})^p (a_2^+)^l J_{31}^{+q} J_{23}^{+r} J_{12}^{+n}|0\rangle = |nlmpqr\rangle,$$

where $J_{ij}^+ = \frac{1}{2}[a_i^+, a_j^+]$. For n-mode para-Bose number states it can be readily verified that $\frac{n}{2}(n+1)$ operators will be required. The para-Bose number states can be used as a basis for a representation of the $Sp_{2n,R}$ group.

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