Pramāṇa, Vol. 24, No. 3, March 1985, pp. 485–497. © Printed in India.

Four component electromagnetic fields and electrodynamics

M T TELI
Department of Physics, Marathwada University, Aurangabad 431 004, India

MS received 10 August 1984; revised 17 November 1984

Abstract. Dirac-Maxwell equations with magnetic monopoles are generalized to electromagnetic fields by introducing fourth components to the fields and their solutions are obtained. The formalism is presented into tensor, dyonic as well as quaternionic forms and conservation theorems for the field energy and momenta are obtained involving the new contribution from the mutual interaction of the fields and currents. The generation of the standard modes TE, TM and TEM of EM waves is also obtained in the formalism.

Keywords. Dirac-Maxwell equations; monopoles; quantum electrodynamics; dyons; quaternions; conservation theorems; TE, TM, TEM-modes.

PACS No. 03·50

1. Introduction

The study of classical (Maxwellian) electrodynamics (MED) and quantum electrodynamics (QED) reveals the following points:

(i) MED directly deals with the measurable electric **E** and magnetic **H** fields.

(ii) **E** and **H** satisfy second order differential equation (wave equation).

(iii) **E**, **H** and propagation vector **k** form an orthogonal triad in empty space (*i.e.* absence of source).

(iv)
$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t} \quad \text{and} \quad \mathbf{H} = \nabla \times \mathbf{A}. \tag{1}$$

are the solutions of the following pair of Maxwell's equations ($c = 1$ everywhere)

$$\nabla \cdot \mathbf{H} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t} \tag{2}$$

(v) These solutions are supposed to satisfy the other pair

$$\nabla \cdot \mathbf{E} = \rho, \ \nabla \times \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t} \tag{3}$$

i.e. **A** and φ satisfy

$$\left.\begin{aligned}\Box\mathbf{A}-\nabla\left(\nabla\cdot\mathbf{A}+\frac{\partial\varphi}{\partial t}\right)&=-\mathbf{j}\\ \Box\varphi+\frac{\partial}{\partial t}\left(\nabla\cdot\mathbf{A}+\frac{\partial\varphi}{\partial t}\right)&=-\rho\\ \Box=\nabla^2-\frac{\partial^2}{\partial t^2}&\end{aligned}\right\}\tag{4}$$

(vi) Equation (4) can be simplified in two ways.

(a) Lorentz condition

$$\nabla\cdot\mathbf{A}+\frac{\partial\varphi}{\partial t}=0.\tag{5}$$

The potentials $A_\mu=(\mathbf{A},i\varphi)$ (Lorentzian potentials) obeying this condition satisfy the wave equation

$$\Box A_\mu=-j_\mu\tag{6}$$

This is referred to as the second order form of Maxwell's equations.

(b) Coulomb condition (gauge)

$$\nabla\cdot\mathbf{A}=0\tag{7}$$

Such (Coulombian) potentials satisfy the equations

$$\left.\begin{aligned}&\Box\mathbf{A}-\frac{\partial}{\partial t}(\nabla\varphi)=-\mathbf{j}.\\ \text{and}\quad&\\ &\nabla^2\varphi=-\rho\end{aligned}\right\}\tag{8}$$

The advantage of this gauge is that the scalar potential φ given by Laplace's equation vanishes in empty space and the electric field **E** becomes parallel to **A**.

(vii) The transverse nature of EM fields is clearly evident in Maxwell's equations in empty space irrespective of any condition on the potentials A_μ. In other words Lorentzian condition or any other condition on A_μ is not required to make **E**, **H** and **k** to form an orthogonal triad.

(viii) QED on the other hand deals with the potentials A_μ and not directly with **E** and **H**. Equation (6) is treated as the electromagnetic field equation in QED and two types of Lagrangian densities are constructed from A_μ.

$$\mathscr{L}_1=-\frac{\partial A_\mu}{\partial x_\nu}\frac{\partial A_\mu}{\partial x_\nu}\tag{9}$$

and

$$\mathscr{L}_2=-f_{\mu\nu}f^{\mu\nu}\tag{10}$$

where

$$f_{\mu\nu}=\frac{\partial A_\mu}{\partial x_\nu}-\frac{\partial A_\nu}{\partial x_\mu}\tag{10a}$$

$\mathscr{L}_1$ and $\mathscr{L}_2$ give the same field equation (6) provided A_μ in $\mathscr{L}_2$ is subjected to Lorentz condition (5).

These points bring out the following important conclusions:

(a) Maxwell's equations directly dealing with **E** and **H** themselves decide about the transverse nature of polarization of EM waves and the assistance of any conditions on A_μ is not needed.
(b) QED with the $\mathscr{L}_1$ gives (6) without the Lorentz condition.
(c) MED becomes QED when $\mathscr{L}_2$ is adapted since $f_{\mu\nu}$ gives the solution of Maxwell's equations with A_μ subjected to the Lorentz condition.

From these conclusions we draw the possibility of the existence of two types of potentials A_μ both of which satisfy (6):

(i) Type one: A_μ without Lorentz condition

(non-Lorentzian $A_\mu = NL - A_\mu$)

(ii) Type two: A_μ obeying Lorentz condition

(Lorentzian $A_\mu = L - A_\mu$)

Thus for $NL - A_\mu$ the Lagrangian density will be given by $\mathscr{L}_1$ (equation (9)) while $L - A_\mu$ requires the Lagrangian density $\mathscr{L}_2$ (equation (10)). So far $NL - A_\mu$ remained ignored since they cannot provide solutions to Maxwell's equations without Lorentz gauge.

The above analysis remains valid even if the existence of magnetic monopoles is added to Maxwell's equations. There now exist two magnetic potentials $NL - B_\mu$ and $L - B_\mu$. These provide the following solutions (Ferrari 1978; Cabibbo and Ferrari 1969; Epistein 1967).

$$\left.\begin{aligned} \mathbf{E} &= -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t} - \nabla \times \mathbf{B} \\ \mathbf{H} &= \nabla \times \mathbf{A} - \frac{\partial \mathbf{B}}{\partial t} - \nabla B_0 \end{aligned}\right\} \tag{11}$$

of the Dirac-Maxwell (DM) equations

$$\left.\begin{aligned} &\nabla\cdot\mathbf{E} = \rho^e,\ \nabla\cdot\mathbf{H} = \rho^m \\ &\nabla\times\mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t} - \mathbf{j}^m,\ \nabla\times\mathbf{H} = \frac{\partial \mathbf{E}}{\partial t} + \mathbf{j}^e \end{aligned}\right\} \tag{12}$$

Here A_μ is the electric four potential arising from electric charge density ρ^e and $B_\mu = (\mathbf{B}, iB_0)$ is the magnetic four potential arising from the magnetic charge density ρ^m, $\mathbf{j}^e$ and $\mathbf{j}^m$ are respectively the electric and magnetic current densities.

If the existence of $NL - (A_\mu, B_\mu)$ is admitted then we have to make the formulation of the EM field equations directly dealing with **E** and **H** like the DM equations. This is done in § 2. Our formalism shows that $NL - A_\mu$ and $NL - B_\mu$ generate four component electric and magnetic fields $E_\mu = (\mathbf{E}, iE_0)$ and $H_\mu = (\mathbf{H}, iH_0)$ which satisfy the same DM equations (12) with additional terms for E_0 and H_0 having the same solutions (11) for **E** and **H**. There exist two types of scalar components corresponding to each of the E_0 and

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H_0 which are respectively given by

$$E_0 = \pm \frac{\partial A_\mu}{\partial x_\mu} \quad \text{and} \quad H_0 = \pm \frac{\partial B_\mu}{\partial x_\mu} \tag{13}$$

Thus E_0 is determined purely by A_μ and so it is the electric scalar field and this approves $NL - A_\mu$ as the electric four potential. Similarly H_0 is the magnetic scalar field and implies $NL - B_\mu$ to be magnetic four potential. When E_0 and H_0 vanish, A_μ and B_μ both satisfy Lorentz condition and become $L - A_\mu$ and $L - B_\mu$ thereby reproducing the standard DM equations. This new formalism can, therefore, be treated as the generalization of the DM-equations.

Section 3 deals with the tensor formulation of the generalized DM theory. The GDM formulation has been found to be very convenient to put it into dyonic form in a simple and elegant way. This is given in §4. Likewise, this formalism has been easily represented in the quaternionic form in §5. The investigation of the energies and momenta possessed by the GDM fields and of their conservation theorems is made in §6. There exist three kinds of energy-momenta possessed by the GDM fields. Two of these give the energy momenta contributed by the fields E_μ and H_μ individually while the third one corresponds to the energy-momentum contributed because of the mutual interaction of E_μ and H_μ. Such a mutual interaction is absent in the standard DM-theory in empty space. The generation of various modes of EM waves such as TE, TM and TEM is given in §7 and important conclusions are given in §8.

2. Four component electromagnetic field equations

The total number of unknown components of $E_\mu = (\mathbf{E}, iE_0)$ and $H_\mu = (\mathbf{H}, iH_0)$ is eight and so we need eight equations to determine them. Maxwell's equations (and also DM-equations with monopoles) provide eight equations. These, however, do not involve E_0 and H_0. Hence their generalization by introducing the terms for E_0 and H_0 will give the proper set of differential equations for E_μ and H_μ. Taking $E_\mu = (\mathbf{E}, i\lambda_1 E_0)$ and $H_\mu = (\mathbf{H}, i\lambda_2 H_0)$, $\lambda_1, \lambda_2 = \pm 1$ and $c = 1$ we generalize the DM-equations (Ferrari 1978) as follows.

$$\left.\begin{aligned} &\nabla \times \mathbf{H} - \frac{\partial \mathbf{E}}{\partial t} - \lambda_1 \nabla E_0 = \mathbf{j}^e \\ &\nabla \times \mathbf{E} + \frac{\partial \mathbf{H}}{\partial t} + \lambda_2 \nabla H_0 = -\mathbf{j}^m \end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned} &\nabla \cdot \mathbf{E} + \lambda_1 \frac{\partial E_0}{\partial t} = \rho^e \\ &\nabla \cdot \mathbf{H} + \lambda_2 \frac{\partial H_0}{\partial t} = \rho^m \end{aligned}\right\} \tag{15}$$

These equations (GDM-equations) now involve all the quantities in the four vector form: $\partial_\mu = \left(\nabla, -i\frac{\partial}{\partial t}\right)$, electric ($e$) and magnetic ($m$) currents by $j_\mu^{e,m} = (\mathbf{j}^{e,m}, i\rho^{e,m})$ and fields E_μ and H_μ. This was not so with the Maxwell and the DM equations. Equations (14) and (15) thus possess the dimensional symmetry in fields, space-time and current densities.

2.1 *Solutions*

If $A_\mu^{e,m} = (\mathbf{A}^{e,m}, i\varphi^{e,m})$ are the electric and magnetic four potentials then (14) and (15) possess the following solutions

$$\left.\begin{aligned} \mathbf{H} &= \nabla \times \mathbf{A}^e - \frac{\partial \mathbf{A}^m}{\partial t} - \nabla \varphi^m \\ \mathbf{E} &= -\nabla \times \mathbf{A}^m - \frac{\partial \mathbf{A}^e}{\partial t} - \nabla \varphi^e \end{aligned}\right\} \tag{16}$$

$$H_0 = \lambda_2 \partial_\mu A_\mu^m, \; E_0 = \lambda_1 \partial_\mu A_\mu^e \tag{17}$$

Equations (16) are the solutions of DM-equations (Ferrari 1978; Cabibbo and Ferrari 1962; Epistein 1967) while (17) is the additional one.

When E_0 and H_0 are neglected, (14) and (15) reduce to the standard DM-equations and hence they give the proper generalization.

2.2 *Second order differential equations*

The four potentials $A_\mu^{e,m}$ satisfy the wave equations (6). This follows from (14)–(17).

$$\Box A_\mu^{e,m} = -j_\mu^{e,m} \tag{18}$$

It is remarkable to note here that to satisfy equation (18) $A_\mu^{e,m}$ are not required to obey any conditions such as Lorentz conditions or Coulomb conditions. Hence they are free from any conditions. From (14) and (15) we obtain the following equations satisfied by E_μ and H_μ

$$\left.\begin{aligned} \Box \mathbf{H} &= -\nabla \times \mathbf{j}^e + \frac{\partial \mathbf{j}^m}{\partial t} + \nabla \rho^m \\ \Box \mathbf{E} &= \nabla \times \mathbf{j}^m + \frac{\partial \mathbf{j}^e}{\partial t} + \nabla \rho^e \end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned} \Box E_0 &= -\lambda_1 \partial_\mu j_\mu^e \\ \Box H_0 &= -\lambda_2 \partial_\mu j_\mu^m \end{aligned}\right\} \tag{20}$$

The occurrence of E_0 and H_0 made the Lorentz and Coulomb conditions disappear. Similarly they seem to make the equations of continuity (conservation of charges) disappear unless they have the plane wave solutions.

$$\left.\begin{aligned} E_0 &= E(0) \exp(ik_\mu x_\mu) \\ H_0 &= H(0) \exp(ik_\mu x_\mu) \end{aligned}\right\} \tag{21}$$

$E(0)$ and $H(0)$ are the constant amplitudes.

3. Tensor form of GDM equations

We can very easily express the GDM equations (14) and (15) and their solutions in the tensor or covariant form as follows.

Field equations:

$$j_{\mu\nu} = \partial_\mu H_\nu - \partial_\nu H_\mu + i\varepsilon_{\mu\nu\alpha\beta}\partial_\alpha E_\beta \tag{22}$$

$$j_k^e = j_{ij}, \quad j_k^m = -ij_{4k} \tag{23}$$

$$\rho^e = \partial_\mu E_\mu, \quad \rho^m = \partial_\mu H_\mu \tag{24}$$

Solutions:

$$F_{\mu\nu} = \partial_\mu A_\nu^e - \partial_\nu A_\mu^e + i\varepsilon_{\mu\nu\alpha\beta}\partial_\alpha A_\beta^m \tag{25}$$

$$H_k = F_{ij}, \quad E_k = -iF_{4k} \tag{26}$$

$$E_0 = \lambda_1 \partial_\mu A_\mu^e, \quad H_0 = \lambda_2 \partial_\mu A_\mu^m \tag{27}$$

Wave equations:

$$\Box F_{\mu\nu} = -(\partial_\mu j_\nu^e - \partial_\nu j_\mu^e + i\varepsilon_{\mu\nu\alpha\beta}\partial_\alpha j_\beta^m) \tag{28}$$

$$\Box E_0 = -\lambda_1 \partial_\mu j_\mu^e, \quad \Box H_0 = -\lambda_2 \partial_\mu j_\mu^m \tag{29}$$

where $\varepsilon_{\mu\nu\alpha\beta}$ is the fully antisymmetric symbol (Levi-Civita density) and $\mu, \nu, \alpha, \beta = 1, 2, 3, 4$.

We can introduce the comma notation for differentiation *e.g.*

$$\partial_\mu H_\nu = H_{\nu,\mu} \text{ etc.} \tag{30}$$

and rewrite equations (22) to (29) as follows

$$j_{\mu\nu} = -(H_{\mu,\nu} - H_{\nu,\mu} + i\varepsilon_{\mu\nu\alpha\beta}E_{\alpha,\beta}) \tag{31}$$

$$\rho^e = E_{\mu,\mu}, \quad \rho^m = H_{\mu,\mu} \tag{32}$$

$$F_{\mu\nu} = -(A_{\mu,\nu}^e - A_{\nu,\mu}^e + i\varepsilon_{\mu\nu\alpha\beta}A_{\alpha,\beta}^m) \tag{33}$$

$$E_0 = \lambda_1 A_{\mu,\mu}^e, \quad H_0 = \lambda_2 A_{\mu,\mu}^m \tag{34}$$

$$\Box F_{\mu\nu} = j_{\mu,\nu}^e - j_{\nu,\mu}^e + i\varepsilon_{\mu\nu\alpha\beta}j_{\alpha,\beta}^m \tag{35}$$

$$\Box E_0 = -\lambda_1 j_{\mu,\mu}^e, \quad \Box H_0 = -\lambda_2 j_{\mu,\mu}^m \tag{36}$$

Equations (25) or (33) are the same as the Cabibbo-Ferrari tensor (Ferrari 1978; Cabibbo and Ferrari 1967; Mignani and Recami 1975, 1976) which can also be compared with the Mignani-Recami tensor (1975, 1976), who considered tachyon charges as magnetic charges.

4. Dyonic form of GDM equations

The GDM equations (14) to (17) can be converted into a set of two equations for complex EM fields. In the same way other sets can be converted. For this purpose we define the following dyonic complex quantities similar to Rajput *et al* (1982, 1983).

Dyonic electromagnetic field four vector

$$f_\mu^d = H_\mu + iE_\mu \tag{37}$$

Dyon four current:

$$j_\mu^d = j_\mu^e - ij_\mu^m \tag{38}$$

Dyon four potential:

$$V^d_\mu = A^e_\mu - iA^m_\mu \tag{39}$$

where

$$f_\mu = (\mathbf{f}, if_0),\ f_0 = \lambda_2 H_0 + i\lambda_1 E_0 \tag{40}$$

Then from equations (14) to (29) we get the following dyonic equations

$$j^d_k = \partial_i f^d_j - \partial_j f^d_i - \partial_4 f^d_k + \partial_k f^d_4 \tag{41}$$

or

$$j^d_k = -(f^d_{i,j} - f^d_{j,i} + f^d_{k,4} - f^d_{4,k}) \tag{41a}$$

$$j^d_4 = \partial_\mu f^d_\mu = f^d_{\mu,\mu} \tag{42}$$

$$f^d_k = \partial_i V^d_j - \partial_j V^d_i + \partial_4 V^d_k - \partial_k V^d_4 \tag{43}$$

or

$$f^d_k = -(V^d_{i,j} - V^d_{j,i} + V^d_{4,k} - V^d_{k,4}) \tag{43a}$$

$$f^d_4 = -\partial_\mu V^d_\mu = -V^d_{\mu,\mu} \tag{44}$$

$$\Box f^d_k = -(\partial_i j^d_j - \partial_j j^d_i) - \partial_4 j^d_k + \partial_k j^d_4 \tag{45}$$

or

$$\Box f^d_k = j^d_{i,j} - j^d_{j,i} - j^d_{k,4} + j^d_{4,k} \tag{45a}$$

$$\Box f^d_4 = \partial_\mu j^d_\mu = j^d_{\mu,\mu} \tag{46}$$

In the dyonic formulation of the DM-electrodynamics Rajput *et al* (1982, 1983) consider $\boldsymbol{\psi} = \mathbf{E} - i\mathbf{H}$ which can be related to our field dyon vector $\mathbf{f}^d$ as

$$\mathbf{f}^d = i\boldsymbol{\psi} \tag{47}$$

All other dyons are identical. The scalar field dyon f^d_4 is additional in our formalism because of the generalization of $\mathbf{E} \to E_\mu$ and $\mathbf{H} \to H_\mu$. The advantage of this additional dyon component is that the dyonic equations (41) to (46) are expressible in the very simple and elegant quaternionic form which we give in the following section.

5. Dyonic quaternions and quaternionic form of dyon equations

We define the following quaternion for a dyon q^d_μ:

$$q = q_\mu e_\mu = (\mathbf{q}^d, q^d_4) \tag{48}$$

We may differ in this definition in one way or the other from the definitions used by Singh (1981, 1982), Rajput *et al* (1982, 1983), Dattoli and Mignani (1978), Synge (1972) and Imaeda (1976).

The quaternion units e_μ possess the following algebra

$$\left.\begin{aligned} e_\mu e_\nu + e_\nu e_\mu &= 2\delta_{\mu\nu} e_4 \\ e_i e_j &= \delta_{ij} + \varepsilon_{ijk} e_k \\ e_4 e_i &= -e_i e_4 = e_i \end{aligned}\right\} \tag{49}$$

The conjugate quaternion is defined as

$$\bar{q} = (\mathbf{q}, -q_4) \tag{50}$$

The complex conjugate of q is defined as

$$q^* = (\mathbf{q}^*, iq_0^*) \tag{51}$$

and the hermitian conjugate is defined as

$$q^\dagger = (\mathbf{q}^*, q_4^*) = (\mathbf{q}^*, -iq_0^*) \tag{52}$$

The product of two quaternions a and b is

$$a\ b = (\mathbf{a} \times \mathbf{b} + a_4\mathbf{b} - \mathbf{a}b_4, a_\mu b_\mu) \tag{53}$$

Now we have the following quaternions in our formalism

$$\left.\begin{aligned} \partial &= (\nabla, \partial_4), \quad f = (\mathbf{f}^d, f_4^d) \\ J &= (\mathbf{j}^d, j_4^d), \quad V = (\mathbf{V}^d, V_4^d) \end{aligned}\right\} \tag{54}$$

We then have the simplest quaternionic form of equations (41) to (46) as

field equations:

$$\bar{\partial}\bar{f} = J \tag{55}$$

solutions:

$$\bar{f} = \partial V \tag{56}$$

Second order differential equations:

$$\left.\begin{aligned} \bar{\partial}(\partial V) &= -(\partial\partial)V = J \\ \partial(\bar{\partial}\bar{f}) &= -(\partial\partial)\bar{f} = \partial J \end{aligned}\right\} \tag{57}$$

Since there are two types of E_μ and H_μ for $\lambda_1, \lambda_2 = \pm 1$, we have in our formalism four types of field quaternions satisfying (55) to (57). (Hereafter we shall drop the superscript d in the dyon quantities).

$$f_+ = (\mathbf{f}, if_{0+}) \tag{58}$$

$$f_- = (\mathbf{f}, if_{0-}) \tag{59}$$

$$\bar{f}_+ = (\mathbf{f}, -if_{0+}) \tag{60}$$

and

$$\bar{f}_- = (\mathbf{f}, -if_{0-}) \tag{61}$$

where

$$f_{0\pm} = H_0 \pm iE_0 \tag{62}$$

6. Energy-momenta of the fields f_μ and their conservation theorems

We consider the dyonic equations of f_μ to evaluate the energy and momentum densities associated with the GDM-fields. We construct four equations from (37) and (38) as follows:

(i) Scalar product of (37) with $\mathbf{f}^*$
(ii) Scalar product of the complex conjugated equation (41) with $\mathbf{f}$
(iii) Multiplication of (42) with f_4^* and
(iv) Multiplication of the complex conjugated equation (42) with f_4.

When these four equations thus obtained are combined together we get the following conservation theorem for field energy and momentum (= energy divided by c)

$$\nabla \cdot \mathbf{S}_1 + \frac{\partial}{\partial t} S_{10} = F_1(j_\mu^e, j_\mu^m, E_\mu, H_\mu) \tag{63}$$

where $\mathbf{S}_1$ is the momentum density vector, S_{10} is the energy density and F_1 is a function of the fields and currents:

$$\mathbf{S}_1 = \frac{i}{2}(\mathbf{f} \times \mathbf{f}^* + \mathbf{f} f_4^* - \mathbf{f}^* f_4) \tag{64}$$

$$S_{10} = \tfrac{1}{2}(\mathbf{f} \cdot \mathbf{f}^* + f_0 f_0^*) \tag{65}$$

$$F_1 = \frac{i}{2}(\mathbf{j}^* \cdot \mathbf{f} - \mathbf{j} \cdot f^* + j_0 f_0^* - j_0^* f_0) \tag{66}$$

Expressing the quantities in (64) to (66) in terms of their real and imaginary parts we get

$$\left.\begin{aligned} \mathbf{S}_1 &= \mathbf{E} \times \mathbf{H} + \lambda_1 E_0 \mathbf{E} + \lambda_2 H_0 \mathbf{H} \\ S_{10} &= \tfrac{1}{2}(\mathbf{E}^2 + \mathbf{H}^2 + E_0^2 + H_0^2) \\ F_1 &= -(j_\mu^e E_\mu + j_\mu^m H_\mu) \end{aligned}\right\} \tag{67}$$

Thus $\mathbf{S}_1$ in (67) is the generalization of the standard Poynting vector and so are the other quantities.

We still obtain two more energy momentum densities. Taking scalar product of (41) with $\mathbf{f}$ and multiplying (42) by f_4 and combining these two resulting equations we get the following conservation theorem.

$$\nabla \cdot (i f_0 \, \mathbf{f}) + \frac{\partial}{\partial t}\left[\frac{i}{2}(\mathbf{f}^2 + f_0^2)\right] = j_\mu f_\mu \tag{68}$$

Equating real and imaginary parts we get two conservation theorems.

$$\nabla \cdot \mathbf{S}_2 + \frac{\partial}{\partial t} S_{20} = F_2 \tag{69}$$

$$\nabla \cdot \mathbf{S}_3 + \frac{\partial}{\partial t} S_{30} = F_3 \tag{70}$$

where

$$\left.\begin{aligned} \mathbf{S}_2 &= \lambda_2 H_0 \mathbf{H} - \lambda_1 E_0 \mathbf{E} \\ S_{20} &= \tfrac{1}{2}(\mathbf{H}^2 - \mathbf{E}^2 + H_0^2 - E_0^2) \\ F_2 &= j_\mu^e E_\mu - j_\mu^m H_\mu \end{aligned}\right\} \tag{71}$$

$$\left.\begin{aligned} \mathbf{S}_3 &= \lambda_1 E_0 \mathbf{H} + \lambda_2 H_0 \mathbf{E} \\ S_{30} &= \mathbf{E} \cdot \mathbf{H} + \lambda_1 \lambda_2 E_0 H_0 \\ F_3 &= -(j_\mu^e H_\mu + j_\mu^m E_\mu) \end{aligned}\right\} \tag{72}$$

Inspection of (67), (71) and (72) reveals the following points.

(i) E_μ and H_μ have self energies and momenta. $\mathbf{S}_1$ and S_{10} give the sum total of these while $\mathbf{S}_2$ and S_{20} give their differences. Thus the self energies and momenta of E_μ and H_μ

are given by

$$\left.\begin{aligned} \mathbf{S}_E &= \tfrac{1}{2}(\mathbf{S}_1 - \mathbf{S}_2) = \tfrac{1}{2}\mathbf{E}\times\mathbf{H} + \lambda_1 E_0 \mathbf{E} \\ S_{0E} &= \tfrac{1}{2}(S_{10} - S_{20}) = \tfrac{1}{2}(\mathbf{E}^2 + E_0^2) \\ F_E &= \tfrac{1}{2}(\mathrm{F}_1 - F_2) = -j_\mu^e E_\mu \end{aligned}\right\} \tag{73}$$

$$\left.\begin{aligned} \mathbf{S}_H &= \tfrac{1}{2}(\mathbf{S}_1 + \mathbf{S}_2) = \tfrac{1}{2}\mathbf{E}\times\mathbf{H} + \lambda_2 H_0 \mathbf{H} \\ S_{0H} &= \tfrac{1}{2}(S_{10} + S_{20}) = \tfrac{1}{2}(\mathbf{H}^2 + H_0^2) \\ F_H &= \tfrac{1}{2}(F_1 + F_2) = -j_\mu^m H_\mu \end{aligned}\right\} \tag{74}$$

(ii) The quantities $\mathbf{S}_3$, S_{30} and F_3 (equation (72)) arise because of the mutual interaction of the fields E_μ and H_μ and the currents. Such an interaction does not occur in the standard DM-theory in empty space. This is the important achievement of our GDM theory. Thus the interaction energy-momentum and the function F are given by

$$\left.\begin{aligned} &\mathbf{S}_{\text{Int}} = \mathbf{S}_3, \quad S_{0\text{Int}} = S_{30}, \\ &F_{\text{Int}} = F_3 \end{aligned}\right\} \tag{75}$$

(iii) The electromagnetic quanta described by E_μ and H_μ thus receive contribution to their energy and momentum from the individual quantities of the fields and from their mutual interaction. The net momentum energy density and the function F for such quanta can therefore be considered to be given by

$$\left.\begin{aligned} \mathbf{S}_T &= \mathbf{S}_E + \mathbf{S}_H + \mathbf{S}_{\text{Int}} \\ S_{0T} &= S_{0E} + S_{0H} + S_{0\text{Int}} \\ F_T &= F_E + F_H + F_{\text{Int}} \end{aligned}\right\} \tag{76}$$

Substituting (72), (73) and (74) into (76) we get

$$\left.\begin{aligned} \mathbf{S}_T &= \mathbf{E}\times\mathbf{H} + (\lambda_1 E_0 + \lambda_2 H_0)(\mathbf{E} + \mathbf{H}) \\ S_{0T} &= \tfrac{1}{2}[(\mathbf{E}+\mathbf{H})^2 + (\lambda_1 E_0 + \lambda_2 H_0)^2] \\ F_T &= -(j_\mu^e + j_\mu^m)(E_\mu + H_\mu) \end{aligned}\right\} \tag{77}$$

The quantities in (77) are equivalent to those contributed by the total four vectors of fields and currents defined as

$$\left.\begin{aligned} M_\mu &= E_\mu + H_\mu \\ J_\mu &= j_\mu^e + j_\mu^m \end{aligned}\right\} \tag{78}$$

Then (77) can be written as

$$\left.\begin{aligned} \mathbf{S}_T &= \mathbf{E}\times\mathbf{M} + M_0 \mathbf{M} \\ S_{0T} &= \tfrac{1}{2}(\mathbf{M}^2 + M_0^2) \\ F_T &= -J_\mu M_\mu \end{aligned}\right\} \tag{79}$$

where

$$M_0 = \lambda_1 E_0 + \lambda_2 H_0 \tag{79a}$$

The conservation theorem for $\mathbf{S}_T$ and S_{0T} is then

$$\nabla\cdot(\mathbf{E}\times\mathbf{M} + M_0\mathbf{M}) + \frac{\partial}{\partial t}[\tfrac{1}{2}(\mathbf{M}^2 + M_0^2)] + J_\mu M_\mu = 0 \tag{80}$$

The four vectorial addition of fields and currents as defined in (78) could be allowed because of the possible mutual interaction of the fields and currents.

7. Generation of TEM, TE and TM waves

Here we generate the standard modes (TEM, TE and TM) of EM waves from the GDM waves. For this purpose we consider (14) and (15) in empty space (*i.e.* $j^e_\mu = 0, j^m_\mu = 0$) with the plane wave solutions

$$E_\mu = e_\mu E(0)\exp(ik_\mu x_\mu),\ H_\mu = h_\mu H(0)\exp(ik_\mu x_\mu) \tag{81}$$

where e_μ and h_μ are the polarization four vectors of E_μ and H_μ respectively and $k_\mu = (\mathbf{k}, ik_0)$ is their propogation four vector. The GDM equations then become

$$\mathbf{k}\times\mathbf{H}+k_0\mathbf{E}-\lambda_1\mathbf{k}E_0 = 0 \tag{82}$$

$$\mathbf{k}\times\mathbf{E}-k_0\mathbf{H}+\lambda_2\mathbf{k}H_0 = 0 \tag{83}$$

$$\mathbf{k}\cdot\mathbf{E}-\lambda_1 k_0 E_0 = 0 \tag{84}$$

$$\mathbf{k}\cdot\mathbf{H}-\lambda_2 k_0 H_0 = 0 \tag{85}$$

In TEM mode **E**, **H** and **k** form an orthogonal triad. This mode can satisfy equations (82) to (85) if E_0 and H_0 are both zero. Such modes can be constructed from the field quaternions f_+ and $\bar{f}_+$ or from f_- and $\bar{f}_-$ equations (58) to (62) as follows:

$$f_{\text{TEM}} = \tfrac{1}{2}(f_+ + \bar{f}_+) = \tfrac{1}{2}(f_- + \bar{f}_-) \tag{86}$$

This is equivalent to putting

$$\left.\begin{aligned} & f_4 = -\partial_\mu V_\mu = 0 \\ \textit{i.e.}\quad & E_0 = \lambda_1\partial_\mu A_\mu = 0 \\ \text{and}\quad & H_0 = \lambda_2\partial_\mu B_\mu = 0 \end{aligned}\right\} \tag{87}$$

which means imposing of the Lorentz conditions on both A_μ and B_μ. The GDM theory is thus able to explain the need of the Lorentz conditions for the TEM modes. Such need is not explicit in the DM equations is empty space.

In TE mode **E** is perpendicular to the plane of **k** and **H** *i.e.* $\mathbf{E}\cdot\mathbf{H} = 0$ and $\mathbf{E}\cdot\mathbf{k} = 0$ but $\mathbf{H}\cdot\mathbf{k} \neq 0$. Such a mode satisfies the set of equations (82) to (85) if $E_0 = 0$. This can be equivalently generated from f_+ and f_- as follows

$$f_{\text{TE}} = \tfrac{1}{2}(f_+ + f_-) \tag{88}$$

This is equivalent to putting $E_0 = 0$ or imposing the Lorentz condition on A_μ only. Likewise TM modes require Lorentz condition on B_μ and they can be constructed from f_+ and $\bar{f}_-$.

$$f_{\text{TM}} = \tfrac{1}{2}(f_+ + \bar{f}_-) \tag{89}$$

8. Conclusions

The above formalism of the four component electrodynamics shows that the occurrence of the fourth scalar components of the EM fields removes the difficulties of presenting field equations into simple dyonic and quaternionic forms. The potentials involved in the field solutions of this theory are not required to obey the Lorentz condition or any other conditions. Since $\partial_\mu A_\mu$ and $\partial_\mu B_\mu$ are invariant under Lorentz transformations the scalar components E_0 and H_0 are Lorentz invariant quantities. The EM waves generated by the four component fields E_μ and H_μ have transverse as well as longitudinal components of polarization even in the empty space.

The interesting feature of this formalism is that the energy momentum of the EM wave receives contribution from the mutual interaction of the associated fields in addition to their individual contribution. The fourth scalar components of the fields are therefore responsible for exploring the mutual interaction of the fields associated with EM-waves. This is the reason why in the standard DM theory in empty space the mutual interaction of the fields remained absent.

When the scalar components E_0 and H_0 are accepted to exist the polarization and spin of EM waves are to be reviewed. The idea of generating the various modes from the four component fields may throw light on the constitution of the photon itself. These reports will appear separately.

DM theory reproduces in the absence of charges transverse E and H fields which are well experienced. Our theory, however, produces four component fields propagating with the velocity of light in the absence of charges and the fourth components produce longitudinal polarization. Such a field may be referred to as the tetral field T (or tetron—when quantized). Photons are usually observed to possess low energy of the order of few MeV and below. One has to see whether very high energy photons if produced by DM-theory remain transversely polarized; if not then tetrons can be supposed to exist at very high energy.

While searching for the answer to this question the author came across Ohmura's (1956) old letters to the editor who for analysing the stability of the electron also introduced fourth components to the E and H fields. He interpreted that these components provided strong cohesive forces so as to maintain the stability of the electron charge distributed over its finite surface. He has also claimed that these longitudinal components are emitted only by electrons possessing very high energy—more than 100 MeV.

One characteristic of the tetrons which Ohmura missed is that they possess spin $\frac{1}{2}$ like electrons and neutrinos (proof elsewhere). Further, though they are massless they are not neutrinos since the latter are two component fields. As such the tetrons are completely different from photons and neutrinos. Their role can be assumed to provide short range and strong cohesive forces for binding particles and nuclei. *e.g.* we can take $p = \pi^+T$, $n = \pi^0 T$, $d = np = \pi^0\pi^+T_2$, $\alpha = nnpp = \pi^0_2\pi^+_2 T_4$ etc. Thus protons are π^+ mesons dressed with the tetron field T, neutrons are π^0 mesons dressed with T, etc. Likewise nuclei are the assemblies of π^+ and π^0 mesons embedded in the T-field. This model does not need the exchange of pions between neutrons and protons as per Yukawa for producing nuclear forces since the latter are provided by the T-field. The T-field can have various strengths depending upon the nuclear systems. In the unstable sytems it can be rather weaker to give allowance for decay and fission.

The last point now is: why such tetrons are not yet observed at all? The reason may be that if some part of the T-field becomes free it immediately annihilates with its antifield $(\mathbf{f}, -f_4)$ thereby producing a photon (TEM-spin 1). Quantum field theoretical studies of tetrons can develop properly their role in particle physics.

Acknowledgement

The author appreciates the referee's suggestion for incorporating the analysis of the observability of the longitudinal parts of the E and H fields.

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