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On the classical solution of SO(3) gauge theory and 't Hooft-Polyakov monopole

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Abstract. It is shown that the ansatz for the asymptotic ($r \to \infty$) gauge fields used by 't Hooft in the study of monopoles in SO(3) electroweak theory is not unique.

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Classical solutions for the Yang-Mills theory with and without Higg's scalars have been studied vigorously in recent years. The first static solution to the SU(2) Yang-Mills theory without Higg's scalars has been obtained by Wu and Yang (1969). Their solution reads as

$$A_o^a = 0 \,\forall\, a$$
$$A_i^a = \varepsilon_{ain} \frac{x_n}{r^2}, \; r \to \infty. \tag{1}$$

This has been successfully used by 't Hooft (1974) and Polyakov (1974) in their pioneering work on magnetic monopoles in SO(3) electroweak gauge theory.

Consider the Lagrangian density for the SO(3) Georgi-Glashow model of electroweak gauge theory;

$$\mathscr{L} = -\tfrac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a - \tfrac{1}{2}(D_\mu Q_a)^2 - \frac{\mu^2}{2} Q_a^2 - \frac{\lambda}{8}(Q_a^2)^2, \tag{2}$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \varepsilon_{abc} A_\mu^b A_\nu^c,$$
$$D_\mu^{ab} Q_b \doteq \partial_\mu Q_a + \varepsilon^{acb} A_\mu^c Q_b\,. \tag{3}$$

The equations of motion for $\mathscr{L}$ are satisfied asymptotically with the Wu-Yang (1969) form for A_μ^a and with $(X_a/r)\,F(r)$ for Q_a the Higg's field. $F(r)$ is subjected to the boundary condition $F(r) \to F$ as $r \to \infty$. F has to be $(-2\mu^2/\lambda)^{1/2}$ to satisfy the field equations. In fact 't Hooft (1974) considered $A_i^a = \varepsilon_{ian}\, x_n\, W(r)$ and the asymptotic solutions to the equations of motion give $W(r) \to 1/r^2$ as $r \to \infty$. The physical fields, obtained from the electromagnetic tensor $G_{\mu\nu}$ (corresponding to the invariant subgroup $U(1)$, equation 2·17 of 't Hooft 1974 then exhibit the magnetic monopole configuration.

It is natural to ask whether the asymptotic solutions for A_i^a and Q_a considered by 't Hooft (1974) are unique. We recall that Witten (1977) has considered a general ansatz for self-dual fields in SU(2) gauge field theory. This paper examines the above question by considering a slightly modified form of Witten's (1977) ansatz to study the Lagrangian (2). We show that the ansatz of 't Hooft (1974) is not unique. Our result is important since the new ansatz gives finite energy and reproduces all the physical and topological properties of the 't Hooft-Polyakov monopole.

The proposed ansatz is

$$A_0^a = 0 \forall a$$
$$A_i^a = \varepsilon_{ain}\frac{x_n}{r}W(r) + \frac{x_a x_i}{r^2}\phi_1(r) + \left(\delta_{ai} - \frac{x_a x_i}{r^2}\right)\frac{\phi_2(r)}{r}$$
$$Q_a = x_a f(r) \tag{4}$$

with the boundary condition $W(r) \to 1/r$ as $r \to \infty$, ϕ_1 (r), $\phi_2(r)$ and $f(r)$ are arbitrary to start with.

The ansatz proposed by us in (4) is the most general static one. This is shown below. Let $A_i = A_i^a \tau^a$. The gauge field (matrix) A_i can be represented as a vector (A_1, A_2, A_3), in the combined internal SO(3) and Euclidean space $\mathbf{R}^3 \times \mathbf{R}^3$. The available vectors from the Euclidean and internal spaces are $\mathbf{r}$ and $\boldsymbol{\tau}$. Then there are only three non-trivial linearly independent vectors in the product space, which are $\boldsymbol{\tau}$, $\mathbf{r} \times \boldsymbol{\tau}$ and $(\mathbf{r} \cdot \boldsymbol{\tau})\mathbf{r}$. The ansatz for A_i^a in (4) exhausts all of them. We shall return to the discussion of A_i^a with respect to parity shortly. Similarly the only scalar, in the adjoint or regular representation of SO(3), available is $(\mathbf{r} \cdot \boldsymbol{\tau})$ and this has been chosen for Q in (4). The arbitrary functions $\phi_1(r)$, $\phi_2(r)$ and $f(r)$ are to be determined from the field equations at $r \to \infty$. It may be noted that we have imposed the boundary condition on $W(r)$ in (4). This is to ensure the asymptotic form of Wu and Yang (1969) in the first term for A_i^a so that the other terms can be considered as modifications on the *asymptotic form* of Wu and Yang (1969).

The field equations emerging from the Lagrangian density (2) are

$$D_\mu^{ab} F_{\mu\nu}^{\ b} + \varepsilon^{abc} Q_c (D_\nu^{bd} Q_d) = 0, \tag{5a}$$

$$D_\mu^{ab}(D_\mu^{bc} Q_c) - \mu^2 Q_a - \frac{\lambda}{2}(Q_b a_b) Q_a = 0. \tag{5b}$$

Substituting the ansatz (4) in (5a) and (5b) we obtain, at $r \to \infty$,

$$\left(\delta_{aj} - \frac{x_a x_j}{r^2}\right)\left\{\frac{1}{r}\phi_2'' - \frac{1}{r^2}\phi_1^2\phi_2 + \frac{\phi_2}{r^3} - \frac{\phi_2^3}{r^3} - r\phi_2 f^3\right\}$$
$$- 2\varepsilon_{ajb}\frac{x_b}{r^2}\phi_1\phi_2' - \varepsilon_{ajb}\frac{x_b}{r^2}\phi_2\phi_1' - \frac{2\phi_1\phi_2^2}{r^4}x_a x_j = 0. \tag{6a}$$

$$f'' + \frac{4}{r}f' + \frac{2}{r^2}f - \frac{2}{r^2}\phi_2^2 f - \mu^2 f - \frac{\lambda}{2}r^2 f^3 = 0, \tag{6b}$$

where prime denotes differentiation with respect to r.

A close examination of (6a) reveals that there are only two non-trivial possibilities; either $\phi_1 = 0$ or $\phi_2 = 0$.

Case 1: $\phi_2(r) = 0; \phi_1(r) \neq 0$ as $r \to \infty$.

Equation (6a) is trivially satisfied. If we now impose the boundary condition that $f(r) \to F/r$ as $r \to \infty$ then (6b) is satisfied if $F = (-2\mu^2/\lambda)^{1/2}$. ϕ_1 is arbitrary at this stage.

Thus there is a class of analytic asymptotic solutions to the equations of motion with arbitrary $\phi_1(r)$. The form $F = (-2\mu^2/\lambda)^{1/2}$ suggests immediate relevance to the study of 't Hooft (1974) and Polyakov (1974) to which we shall return shortly.

Case 2: $\phi_1(r) = 0$; $\phi_2(r) \neq 0$ as $r \to \infty$.
Equation (6a) assumes

$$\phi_2'' + \frac{1}{r^2}\phi_2(1-\phi_2^2) - r^2\phi_2 f = 0.$$

A simple analytic solution consistent with (6b) is obtained for $\mu^2 = 0$. This is $f(r) = D/r^2$ and $\phi_2(r) = C$ as $r \to \infty$. The constants C and D satisfy $C^2 + D^2 = 1$ and $\lambda = -4C^2/D^2$. Since $\mu^2 = 0$, this class of solutions are not relevant to our purpose. They may be of some relevance to massless but self-interacting scalar fields and gauge fields.

Returning to case 1, the asymptotic solution gives

$$A_i^a = \varepsilon_{ain}\frac{x_n}{r^2} + \frac{x_a x_i}{r^2}\phi_1(r), \tag{7}$$

far from the origin. This result for A_i^a is to be contrasted with the original ansatz of 't Hooft (1974) which happens to have the polar property, *viz* it changes sign under space inversion. Our result does not have this property. However the two forms for A_i^a are gauge equivalent, if $\phi_1(r)$ falls off faster than $1/r$ as $r \to \infty$. Thus, a form which does not have the polar property can be gauge transformed to a form having the polar property, for a special choice of $\phi_1(x)$. Hence the determination of polar property by simply looking at its space-inversion behaviour is not tenable in a non-abelian theory. This is one of the striking differences between non-abelian and abelian gauge fields. We hasten to add that if the arbitrary function $\phi_1(r)$ is chosen such that it does not fall off faster than $1/r$ as $r \to \infty$, then the two forms are gauge-inequivalent. We will demonstrate this at the end. It is to be noted that although our A_i^a involves $\phi_1(r)$, the field strength $F_{\mu\nu}^a$ does not involve $\phi_1(r)$. We also recover $D_i^{ab}Q_b = 0$. Since the Lagrangian (2) involves A_i^a only through $F_{\mu\nu}^a$ and $D_\mu^{ab}Q_b$, we find that the expression for the energy is independent of $\phi_1(r)$. It is found to be identical to that of 't Hooft (1974). Hence the energy is finite, as in the case of 't Hooft (1974).

The physical electromagnetic tensor corresponding to the unbroken U(1) group (spontaneous symmetry breaking of SO(3) $\to$ U(1) occurs since $F = (-2\mu^2/\lambda)^{1/2} \neq 0$) is

$$G_{ij} = -\frac{1}{r^3}\varepsilon_{ijn}x_n,$$

with a radial magnetic field at $r \to \infty$, $B_a = x_a/r^3$ with a total flux 4π.

Thus the Lagrangian (2) admits another solution far from the origin, for which the energy is finite, reproducing all the properties of 't Hooft (1974) monopole. Therefore it appears that the ansatz of 't Hooft (1974) is not unique.

Now a few observations are in order. First of all, the solution given by (7) and the 't Hooft's ansatz give the same field strength. That is the two different fields give the same $F_{\mu\nu}^a$. One may hope to get some boundary condition on ϕ_1 by looking at the finiteness of the energy. Since the expression $\int d^3x\, \mathscr{L}$ does not involve ϕ_1, the boundary conditions at the origin or at infinity for ϕ_1 cannot be obtained from the energy considerations.

Since the new ansatz gives the same $F^a_{\mu\nu}$, it may appear that the additional term could be a gauge artifact. If so, then follows the uniqueness of 't Hooft's (1974) ansatz. However, this is not so. As previously noted, only for ϕ_1 (r) falling off faster than $1/r$ as $r \to \infty$, the two potentials are gauge equivalent. Since ϕ_1 (r) is completely arbitrary, it can be such that it does not fall off faster than $1/r$ as $r \to \infty$. It is easy to show that the two forms are related by an SU (2) transformation with the local parameter θ^a of the above as

$$\theta^a = -\frac{x_a}{r}\int^r \phi_1(r')\mathrm{d}r'.$$

The gauge transformation is non-singular if $\phi_1(r)$ falls off faster than $1/r$ as $r \to \infty$. Then the two forms are gauge-equivalent. If $\phi_1(r)$ is so chosen on account of its arbitrariness that it *does not fall off faster than* $1/r$ *at infinity*, then the *gauge transformation* is *singular* and the two forms are gauge-*inequivalent*. The extra term $(x_a x_i/r^2)$ $\phi_1(r)$ cannot be gauged away. The two forms yield identical monopole configuration. Secondly a situation similar to ours, namely two gauge inequivalent potentials giving the same $F^a_{\mu\nu}$, has been noted by Wu and Yang (1975) in their study of SU(2) Yang-Mills theory without Higgs scalars. In fact our situation is similar to their second ansatz but applied to spontaneously broken SO(3) gauge field theory.

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