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Fields with vanishing colour-currents

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Abstract. We consider spinor, scalar and vector fields with colour degrees of freedom and find the classical solutions when the constraint of vanishing colour currents is imposed. We find that there are no non-trivial c-number solutions for spinor fields transforming as a triplet under SU(3), although solutions exist for scalar and vector fields. We also show that the colour current of spinor fields coupled to an instanton is zero.

Keywords. Quark confinement; vanishing colour currents; instanton solutions.

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1. Introduction

Some time ago, Amati and Testa (1974) proposed a model of quark confinement in which one has coloured quarks which satisfy the constraint of vanishing colour currents. This model was interpreted by them as the strong coupling limit of quantum chromodynamics (QCD). Using functional methods it has been shown that this model is in fact equivalent to QCD and the equivalence was interpreted as due to the compositeness of the gluons (Rajasekaran and Srinivasan 1978a).

The model of colour confinement (for colour SU(2)) is defined by the equations:

$$(i\gamma^{\mu}\partial_{\mu}-m)\psi = 0, \tag{1}$$

$$\bar{\psi}\gamma_{\mu}\tau^{i}\psi = 0, \quad i = 1, 2, 3 \tag{2}$$

Here ψ is the quark field which is a doublet under the colour SU(2) group and τ^i are the Pauli matrices. Interesting classical solutions of this system have been obtained (Rajasekaran and Srinivasan 1978b). The plane wave, positive energy solutions are given by:

$$\psi_1 = \begin{pmatrix} \xi \\ \dfrac{\sigma\cdot\mathbf{p}}{p_0+m}\xi \end{pmatrix}\exp(-ip\cdot x), \qquad \psi_2 = \begin{pmatrix} \eta \\ \dfrac{\sigma\cdot\mathbf{p}}{p_0+m}\eta \end{pmatrix}\exp(-ip\cdot x) \tag{3}$$

Here ψ_1 and ψ_2 are the isospin up and down components of ψ, and ξ and η are two-component spinors which are chosen to be mutually orthogonal and to have equal normalization

$$\xi^{\dagger}\xi = \eta^{\dagger}\eta, \quad \xi^{\dagger}\eta = \eta^{\dagger}\xi = 0. \tag{4}$$

Negative energy solutions can be constructed similarly. The original eight-fold multiplicity of the plane-wave solutions of (1) is reduced to a four-fold multiplicity because of the constraint of vanishing colour densities (2).

In this paper, we first extend the work to scalar and vector fields transforming as doublets or triplets under colour SU(2) and find the solutions of the free-field equations when we impose the constraint of vanishing colour densities. We find that the amplitudes for different colour components in a multiplet are related for plane wave solutions. We next take up the case of SU(3) which is more interesting physically. We find that there is no non-trivial solution to the model with Dirac fields transforming as a triplet under SU(3). However, non-trivial solutions exist for scalar and vector fields. These results are expected to play an important role in the quantized version of the models.

We have one more interesting observation to make. The solutions of the Dirac equation for massless quarks in the presence of instanton field configurations have been obtained in some other context for SU(2) ('t Hooft 1976a). We find that for these solutions too, the quark-part of the colour current vanishes.

2. Colour SU(2)

The colour-confinement solutions for the case of Dirac fields transforming as a SU(2) doublet have been already given elsewhere (Rajasekaran and Srinivasan 1978b) and also briefly described in the introduction. Here we shall restrict ourselves to the cases of scalar and vector fields.

2.1 *Scalar fields*

The currents J_i^μ for a colour multiplet of scalar fields are given by

$$J_i^\mu = (T_i)_{jk}\phi_j^* \overleftrightarrow{\partial}_\mu \phi_k, \tag{5}$$

where T_i are the generators in the representation to which ϕ_i belong. So, we have to solve for

$$(\Box + m^2)\phi_i = 0, \tag{6}$$

and

$$(T_i)_{jk}\phi_j^* \overleftrightarrow{\partial}_\mu \phi_k = 0. \tag{7}$$

2.2 *Scalar doublet*

For the doublet, $T_i \equiv \sigma_i$ where σ_i are the Pauli matrices. Equation (7) would imply

$$\left.\begin{aligned} \phi_1^* \overleftrightarrow{\partial}_\mu \phi_1 &= \phi_2^* \overleftrightarrow{\partial}_\mu \phi_2; \\ \phi_1^* \overleftrightarrow{\partial}_\mu \phi_2 &= \phi_2^* \overleftrightarrow{\partial}_\mu \phi_1 = 0. \end{aligned}\right\} \tag{8}$$

It is easily seen that if one of the components is zero and the other is real, (8) is satisfied. There are other solutions. Consider plane wave solutions of (6)

$$\phi_i = a_i \exp(ik \cdot x) + b_i \exp(-ik \cdot x), \; (k^2 = m^2). \tag{9}$$

It can be easily verified that (8) will imply the following relations:

$$|a_1|^2 - |b_1|^2 = |a_2|^2 - b_2|^2, \qquad a_1^* a_2 = b_1^* b_2. \tag{10}$$

The solution of these equations is given by

$$a_1/b_1 = a_2/b_2 = \exp i\chi \tag{11}$$

where χ is an arbitrary phase factor.

2.3 *Scalar triplet*

For the triplet,

$$(T_i)_{jk} = -i\,\varepsilon_{ijk},$$

and we shall take the fields to be real. Then (8) will imply that:

$$\partial_\mu \boldsymbol{\phi} \times \boldsymbol{\phi} = 0, \tag{12}$$

where the cross product is in the colour space. Consider the plane-wave solutions,

$$\boldsymbol{\phi} = \mathbf{A} \exp(ik \cdot x) + \mathbf{A}^* \exp(-ik \cdot x). \tag{13}$$

From (12) it follows that

$$\mathbf{A} \times \mathbf{A}^* = 0, \tag{14}$$

that is, the amplitude of positive and negative frequency components should be parallel in the colour space.

2.4 *Vector fields*

The expression for the colour current for a complex multiplet $A_{\mu i}$ of vector fields is given by:

$$J_i^\mu = -(T_i)_{jk}[(\partial^\mu A_j^{\nu*} - \partial^\nu A_j^{\mu*})A_{\nu k} - (\partial^\mu A_k^\nu - \partial^\nu A_k^\mu)A_{\nu j}^*]. \tag{15}$$

The vector fields satisfy

$$(\Box + m^2)A_i^\mu = 0, \quad \partial_\mu A_i^\mu = 0. \tag{16}$$

The constraint equation is

$$(T_i)_{jk}[(\partial^\mu A_j^{\nu*} - \partial^\nu A_j^{\mu*})A_{\nu k} - (\partial^\mu A_k^\nu - \partial^\nu A_k^\mu)A_{\nu j}^*] = 0. \tag{17}$$

Consider the plane wave solutions of (16):

$$A_{\mu i} = \varepsilon_{\mu i}^+ \exp ik \cdot x + \varepsilon_{\mu i}^- \exp(-ik \cdot x). \tag{18}$$

We will now take the case of the doublet and triplet representations separately.

2.5 *Vector doublet*

In this case, (17) will imply the following relations between the polarization vectors:

$$\varepsilon_{\mu 1}^{+*}\varepsilon_{\mu 1}^{+} - \varepsilon_{\mu 1}^{-*}\varepsilon_{\mu 1}^{-} = \varepsilon_{\mu 2}^{+*}\varepsilon_{\mu 2}^{+} - \varepsilon_{\mu 2}^{-*}\varepsilon_{\mu 2}^{-}, \qquad \varepsilon_{\mu 1}^{+*}\varepsilon_{\mu 2}^{+} = \varepsilon_{\mu 1}^{-*}\varepsilon_{\mu 2}^{-}. \tag{19}$$

These conditions are similar to (10) for the scalar fields. A particular solution would be:

$$\varepsilon^{+}_{\mu i} = \exp(i\phi)\,\varepsilon^{-}_{\mu i}. \tag{20}$$

2.6 *Vector triplet*

We shall take the fields to be real for this representation. Therefore

$$\varepsilon^{-}_{\mu} = \varepsilon^{+*}_{\mu} \equiv \varepsilon^{*}_{\mu}.$$

Then the constraints will imply that

$$\boldsymbol{\varepsilon}_{\mu} \times \boldsymbol{\varepsilon}^{*}_{\mu} = 0, \tag{21}$$

where $\boldsymbol{\varepsilon}$ indicates a vector in colour space. That is, the positive and negative frequency components of the polarization should be parallel in colour space, after taking the scalar product in space-time components.

3. Colour SU(3)

3.1 *Dirac triplet*

This is the physically important case of quarks. The constraints of vanishing colour currents are

$$\bar{\psi}\gamma_{\mu}\lambda_{a}\psi = 0, \quad a = 1 \ldots . 8, \tag{22}$$

where λ_a are the 3×3 matrices corresponding to the generators of SU(3) and ψ is a colour triplet. We will show that, in contrast to the SU(2) case, there are no non-trivial c-number solutions to these 8 equations.

We introduce the notation

$$\gamma^{0}\gamma^{\mu}\psi_{j} \equiv \psi^{\mu}_{j}, \quad j = 1, 2, 3, \tag{23}$$

where j refers to the colour index and consider the following set of 12 four-component spinors:

$$\begin{matrix} \psi^{0}_{1} & \psi^{1}_{1} & \psi^{2}_{1} & \psi^{3}_{1}, \\ \psi^{0}_{2} & \psi^{1}_{2} & \psi^{2}_{2} & \psi^{3}_{2}, \\ \psi^{0}_{3} & \psi^{1}_{3} & \psi^{2}_{3} & \psi^{3}_{3}. \end{matrix}$$

Using the explicit form of the λ_a matrices and the general properties of the Dirac γ_μ matrices, it is easily shown that the constraints in (22) take the following form:

$$\psi^{\mu\dagger}_{i}\psi^{\mu}_{j} = r\delta_{ij} \qquad (\mu \text{ not summed}), \tag{24a}$$

$$\psi^{0\dagger}_{i}\psi^{k}_{j} = s_{k}\delta_{ij} \qquad (k = 1, 2, 3), \tag{24b}$$

where r and s_k are arbitrary numbers.

Our aim is to find a set of 12 four-component objects ψ^{μ}_{i} satisfying the orthogonality-cum-normalization conditions in (24). It turns out that these conditions are too strong; there is not enough room in the four-dimensional linear vector space to admit 12 vectors with these properties. We proceed to demonstrate this.

Without loss of generality, we may choose the normalization constant r in (24) to be unity. Since all the 12 vectors have equal norm according to (24a), it is clear that none of them can be taken to be the null vector. Since (24a) also implies that ψ_1^0, ψ_2^0 and ψ_3^0 are mutually orthogonal, we may choose these three as basis vectors. Let ϕ be the fourth basis vector which is orthogonal to these three. In terms of these four basis vectors, we can now write down, the remaining 9 vectors ψ_i^k. Since ψ_1^k is orthogonal to ψ_2^0 and ψ_3^0 according to (24b), its most general form is

$$\psi_1^k = a_k\psi_1^0 + x_k\phi. \tag{25a}$$

Similarly, we have

$$\psi_2^k = b_k\psi_2^0 + y_k\phi, \tag{25b}$$

$$\psi_3^k = c_k\psi_3^0 + z_k\phi, \tag{25c}$$

where a_k, b_k, c_k, x_k, y_k and z_k are some numbers. But, according to (24a)

$$\psi_i^{k\dagger}\psi_j^k = 0 \qquad \text{for} \quad i \neq j \quad (k \text{ not summed}). \tag{26}$$

This requires

$$x_k^* y_k = y_k^* z_k = z_k^* x_k = 0 \quad (k \text{ not summed}), \tag{27}$$

which implies that for a given k, at least two of the three numbers (x_k, y_k, z_k) should be zero. Let us choose for instance

$$\left.\begin{array}{lll} x_1 \neq 0, & y_1 = 0, & z_1 = 0 \\ x_2 = 0, & y_2 \neq 0, & z_2 = 0 \\ x_3 = 0, & y_3 = 0, & z_3 \neq 0. \end{array}\right\} \tag{28}$$

Then using (25)

$$\left.\begin{array}{l} \psi_2^1 = b_1\psi_2^0 \\ \psi_2^3 = b_3\psi_2^0. \end{array}\right\} \tag{29}$$

In view of the definitions of ψ_i^k given in (23), (29) implies that ψ_2^0 is a simultaneous eigenfunction of the Dirac matrices $\gamma^0\gamma^1$ and $\gamma^0\gamma^3$ which contradicts the anti-commutation relations of these matrices. Any other choice of (x_k, y_k, z_k) consistent with (27) leads to a similar result. Hence it follows that there are no non-trivial c-number solutions[§] to (22).

The above argument can be easily extended to any SU(n) for $n > 3$. So, c-number spinor functions with vanishing current densities do not exist for SU(n) ($n \geqslant 3$).

The non-existence of solutions except for SU(2) has a simple physical interpretation. For the SU(2) case, the spin-up and spin-down Dirac wave functions turned out to be the colour isospin up and down components required for the SU(2) doublet (Rajesekaran and Srinivasan 1978b). No more kinematic degree of freedom is available for a spin 1/2 field to accommodate the larger number of colour-components necessary for SU(n) ($n \geqslant 3$).

3.2 *Scalar triplet*

For plane wave solutions

$$\phi_i = a_i \exp ik\cdot x + b_i \exp(-ik\cdot x), \tag{30}$$

[§] However, it is possible to find solutions in terms of Grassmann numbers.

the constraint (7) will yield the relations

$$\begin{aligned} |a_1|^2-|b_1|^2 &= |a_2|^2-|b_2|^2 = |a_3|^2-|b_3|^2 \\ a_1^* a_2 &= b_1^* b_2, \\ a_1^* a_3 &= b_1^* b_3, \\ a_2^* a_3 &= b_2^* b_3, \end{aligned} \tag{31}$$

The solution of these relations is given by

$$a_1/b_1 = a_2/b_2 = a_3/b_3 = \exp i\chi, \tag{32}$$

where χ is an arbitrary phase factor. This solution is very similar to the SU(2) doublet solution.

3.3 *Vector triplet*

In this case also, the solutions are a straightforward extension of the SU(2) doublet case. For plane wave solutions

$$A_{\mu i} = \varepsilon_{\mu i}^{+} \exp ik\cdot x + \varepsilon_{\mu i}^{-} \exp(-ik\cdot x), \tag{33}$$

the polarization vectors satisfy the relations

$$\begin{aligned} \varepsilon_{\mu 1}^{+*}\varepsilon_{\mu 1}^{+} - \varepsilon_{\mu 1}^{-*}\varepsilon_{\mu 1}^{-} &= \varepsilon_{\mu 2}^{+*}\varepsilon_{\mu 2}^{+} - \varepsilon_{\mu 2}^{-*}\varepsilon_{\mu 2}^{-} \\ &= \varepsilon_{\mu 3}^{+*}\varepsilon_{\mu 3}^{+} - \varepsilon_{\mu 3}^{-*}\varepsilon_{\mu 3}^{-}. \end{aligned} \tag{34}$$

$$\varepsilon_{\mu i}^{+*}\varepsilon_{\mu j}^{+} - \varepsilon_{\mu i}^{-*}\varepsilon_{\mu j}^{-} = 0, \quad i \neq j.$$

As in the SU(2) doublet case, a particular solution would be

$$\varepsilon_{\mu i}^{+} = \exp i\phi\, \varepsilon_{\mu i}^{-}. \tag{35}$$

4. Solutions in the presence of an instanton

Consider a doublet of Dirac fields coupled to SU(2) colour gauge fields A_μ^a in Euclidean space-time. The equations of motion are

$$[i\gamma_\mu(\partial_\mu - igA_\mu) - m]\psi = 0, \tag{36}$$

and

$$D_\mu F_{\mu\nu}^a = -\bar{\psi}\gamma_\nu \frac{\tau^a}{2}\psi. \tag{37}$$

Consider the instanton solution of SU(2) gauge theory in Euclidean space-time*

$$A_\mu^a = \frac{4}{g}\frac{\bar{\eta}_{a\mu\nu}x_\nu\rho^2}{(x^2+\rho^2)x^2}, \tag{38}$$

where $\bar{\eta}_{a\mu\nu}$ is a fully antisymmetric object defined by

$$\bar{\eta}_{aij} = \varepsilon_{aij}, \quad \bar{\eta}_{a0i} = -\delta_{ai},$$

* Our notation follows that of Carlitz (1978), with the rotation parameter Ω set equal to zero.

and ρ is a parameter associated with the size of the instanton; here the instanton is located at the origin. Note that A_μ is a gauge transform of the usual instanton solution (Belavin *et al* 1975)

$$A_\mu^a = \frac{2}{g}\frac{\eta_{a\mu\nu}x_\nu}{x^2+\rho^2}.$$

It has been shown ('t Hooft 1976a) that the solution of the Dirac equation in the presence of the instanton field is given by

$$\psi = \frac{\sqrt{2}\rho x_\mu \gamma_\mu \chi}{\pi(x^2)^{1/2}(x^2+\rho^2)^{3/2}} \tag{39}$$

with

$$\chi\bar{\chi} = \frac{1}{2}(1+\gamma_5)\left[\frac{1}{4}+\frac{i}{16}\eta_{\mu\nu a}\tau^a\gamma_\mu\gamma_\nu\right], \tag{40}$$

Consider the colour current of the Dirac field

$$\tfrac{1}{2}\bar{\psi}\gamma_\mu\tau\psi = \tfrac{1}{2}\,\mathrm{Tr}\,[\tau\psi\bar{\psi}\gamma_\mu], \tag{41}$$

where the trace is over both internal and Dirac components.

Now, from (39),

$$\mathrm{Tr}\,[\tau\psi\bar{\psi}\gamma_\mu] = \frac{2\rho^2 x_\lambda x_\rho}{\pi x^2(x^2+\rho^2)^3}\,\mathrm{Tr}\,[\tau\gamma_\lambda\chi\bar{\chi}\gamma_0\gamma_\rho\gamma_0\gamma_\mu]. \tag{42}$$

From (40) the right side of this equation vanishes as it involves the trace of an odd number of Dirac matrices.

Hence

$$\bar{\psi}\gamma_\mu\tau\psi = 0 \tag{43}$$

Thus we have shown that the colour-current of spinor fields coupled to the instanton is zero. Further, (43) implies that (37) is also satisfied, as the instantons are solutions of

$$D_\mu F_{\mu\nu}^a = 0.$$

Therefore the fields in (38) and (39) simultaneously solve (36) and (37).

To conclude, when the earlier authors solved the Dirac equation in the instanton background field, they did not refer to the equation of motion of the gauge field with the source term. We have shown that this equation of motions is, in fact, satisfied.

For a colour multiplet of scalar fields, the equation of motion is

$$\mathscr{D}_\mu\mathscr{D}_\mu\Phi = 0, \tag{44}$$

where

$$\mathscr{D}_\mu \equiv (\partial_\mu - igT^a A_\mu^a),$$

T^a being the generators in the representation of the scalars. But (43) has only a trivial solution, when A_μ^a is an instanton field configuration ('t Hooft 1976a).

5. Discussion

We have looked for the classical solutions of free field equations for SU(2) and SU(3) colour multiplets of spinor, scalar and vector fields when the additional constraint of vanishing colour currents is imposed. We have found that for the physically interesting case of a spinor triplet of SU(3) there are no non-trivial c-number solutions. This result may have important implications for the Amati-Testa model of quark confinement. There are non-trivial solutions for other cases, and these are exhibited. We have also considered the solution for a massless Dirac field in the presence of an instanton field and have shown that the colour current vanishes in this case. It would be interesting to study the significance of these results in the quantized version of the models considered above.

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References

Amati D and Testa M 1974 *Phys. Lett.* **B48** 227
Belavin A, Polyakov A, Schwartz A and Tyupkin Y 1975 *Phys. Lett.* **B59** 85
Carlitz R D 1978 *Phys. Rev.* **D17** 3225
Rajasekaran G and Srinivasan V 1978a *Pramana* **10** 33
Rajasekaran G and Srinivasan V 1978b *Pramana* **11** 739
't Hooft G 1976a *Phys. Rev. Lett.* **37** 8
't Hooft G 1976b *Phys. Rev.* **D14** 3432