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Self-energy effect in a relativistic bound state problem

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Abstract. A modification of the Wick-Cutkosky equation for the relativistic bound state of two scalar particles interacting through the exchange of a massless scalar field within the ladder approximation has been considered by incorporating the self-energy diagrams in the integral kernel. An exact analytical solution of the equation is obtained at vanishing total energy and it is shown that the self-energy effects generally diminish the eigenvalues in agreement with the findings of Li *et al*, who, however solved the equation numerically for the case of massive scalar exchange. An additional feature of the modified equation is that it preserves the 0(5) symmetry at zero total energy as was first noted by Cutkosky for the scalar bound state equation without self-energy effects.

Keywords. Bethe-Salpeter equation; relativistic bound state problem; self-energy effects.

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1. Introduction

The covariant wave equation originally formulated by Bethe and Salpeter (1951) more than thirty years ago still remains the main tool to study the bound state problem in the context of quantum field theory. In momentum space it is given by an integral equation with integral kernel determined by four-point irreducible Feynman diagrams as well as by self-energy diagrams. In the absence of a complete knowledge of all these, the ladder approximation is generally used. In this context the studies of Wick (1954) and Cutkosky (1954) for a scalar Bethe-Salpeter equation (BS equation) have revealed many interesting consequences of a relativistic bound state problem. Later Blanckenbecler and Sugar (1966) have pointed out the effect of unitarity condition on the BS equation and in particular Levine and Wright (1967) have shown that the unitarity relation namely $\sigma_{\text{total}} \geqslant \sigma_{\text{inelastic}}$ can be violated in the inhomogeneous BS equation describing scattering processes if only the ladder diagrams are taken into the integral kernel. Inclusion of self-energy diagrams, do, however, remove this difficulty. Recently Li *et al* (1980) have considered the influence of the self-energy diagrams on the solutions of the bound state equation involving scalar particles. Briefly, they considered a scalar BS equation with the interaction chosen as $\mathscr{H}_I(x) = g\phi_1^*(x)\phi_1(x)\phi_2(x)$, where ϕ_1 is a complex scalar field with mass m and ϕ_2 is a neutral scalar field with mass μ. In the second-order the integral kernel of the equation has the form

$$\frac{i\lambda}{\pi^2}\frac{1}{(p-p')^2+\mu^2} - \lambda h(p)(p_1^2+m^2)(p_2^2+m^2)\delta^4(p-p'), \tag{1}$$

where $h(p)$ stands for the self-energy contribution to the kernel and p_1 and p_2 are respectively $p+\frac{1}{2}P$ and $-p+\frac{1}{2}P$. P_μ is the centre-of-mass four-momentum of the composite system. It is well known that for nonvanishing μ the scalar BS equation, even with the first term of the kernel given by (1), cannot be solved analytically. Li *et al* (1980) adopted a comprehensive numerical study of the problem with the integral kernel given in (1). Their calculations show that self-energy diagrams in general diminish the eigenvalues and also alter the wave functions slightly.

In this note we point out that with a little modification of the kernel, exact analytic solution of the problem exists for some given values of total energy and that the properties of the eigenvalues with respect to the coupling parameter can be easily determined. Li *et al* (1980) have discussed in detail through their numerical analysis the eigenvalues of the coupling parameter for various non-vanishing values of μ and for a given set of values of the energy parameter $\eta = E/2m$ where E is the fourth component of P_μ. In order to obtain analytic solutions we modify the kernel as

$$\frac{i\lambda}{\pi^2}\frac{1}{(p-p')^2} - \lambda h(p)(p_1^2+m^2)(p_2^2+m^2)\delta^4(p-p'). \tag{2}$$

We thus consider the situation when only massless scalar particles are exchanged, however, the mass of the neutral scalar contributing to the self-energy expression $h(p)$ need not be zero and can be taken at will. The equation we thus consider is a simple generalization of the original Wick (1954)-Cutkosky (1954) equation together with an additional term in the kernel due to self-energy contribution to second order in coupling strength $g(\lambda \approx g^2)$.

2. Separability of the modified Wick's equation

The modified Wick's equation which we like to solve reads

$$(p_1^2+m^2)(p_2^2+m^2)\chi(p) = -\frac{i\lambda}{\pi^2}\int\frac{\chi(p')\,\mathrm{d}^4p'}{(p-p')^2} + \lambda h(p)(p_1^2+m^2)(p_2^2+m^2)\chi(p), \tag{3}$$

when both momentum p and p' are in Euclidean space, and the self-energy function $h(p)$ is given by

$$h(p) = \int_{(m+\mu)^2}^{\infty}\frac{(p_1^2+m^2)[\sigma^2-(m+\mu)^2]^{1/2}[\sigma^2-(m-\mu)^2]^{1/2}}{\sigma^2(\sigma^2-m^2)^2(p_1^2+\sigma^2)}\,\mathrm{d}\sigma^2 + (p_1 \rightarrow p_2) \tag{4}$$

We rewrite (3) as

$$\psi(p) = \frac{-i\lambda}{\pi^2}\int\frac{\mathrm{d}^4p'\psi(p')}{(p-p')^2(p_1'^2+m^2)(p_2'^2+m^2)[1-\lambda h(p')]} \tag{5}$$

where p_1' and p_2' are respectively equal to $p'+(E/2)$ and $-p'+(E/2)$ and $\psi(p) = [1-\lambda h(p)]\,(p_1^2+m^2)\,(p_2^2+m^2)\chi(p)$. Hereafter, we will assume that the 3-vector part of P_μ is a null-vector for simplicity. Equation (5) can also be written as a differential equation *i.e.*

$$\Box\,\psi(p) = -4\lambda\frac{\psi(p)}{(p_1^2+m^2)(p_2^2+m^2)}\cdot\frac{1}{1-\lambda h(p)}, \tag{6}$$

and

$$\Box = (\partial^2/\partial p_4^2) + (\partial^2/\partial \mathbf{p}^2), \tag{7}$$

We now show that (6) is separable in terms of new (Green 1957; Biswas 1967) variables defined by

$$p_1 = p_s \sin\theta \cos\phi,$$
$$p_2 = p_s \sin\theta \sin\phi, \qquad (8)$$
$$p_3 = p_s \cos\theta,$$

and further, $p_s = C \sin\eta/(\cosh\alpha - \cos\eta)$ and $p_4 = C \sinh\alpha/(\cosh\alpha - \cos\eta)$. The whole energy-momentum plane is contained in the region $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$ and $-\infty \leqslant \alpha \leqslant \infty$ and the parameter C is given by

$$C^2 = (E/2)^2 + m^2 \qquad (9)$$

In terms of these new variables, the self-energy function $h(p)$ becomes a function of the variable α only. For instance, introducing a quantity c' with

$$C'^2 = (E/2)^2 + \sigma^2. \qquad (10)$$

The function $h(p)$ can be written as,

$$h(p) = 2\int_{(m+\mu)^2}^{\infty} \frac{[\sigma^2-(m+\mu)^2]^{1/2}[\sigma^2-(m-\mu)^2]^{1/2}}{\sigma^2(\sigma^2-m^2)^2}\frac{C}{C'}\left[\frac{CC' - E^2/4\tanh^2\alpha}{C'^2 - E^2/4\tanh^2\alpha}\right]\mathrm{d}\sigma^2$$
$$= \Delta(m, \mu, \tanh\alpha) \qquad (11)$$

In the new variables defined by (8) separable solutions are effected by writing $\psi(p)$ as

$$\psi(p) = f(\alpha) g(\eta) Y_{lm}(\theta, \phi)/p_s. \qquad (13)$$

Substituting (13) in (6) we obtain after separation of variables the following two equation for $g(\eta)$ and $f(\alpha)$ namely,

$$\frac{\mathrm{d}^2 g}{\mathrm{d}\eta^2} + \left\{n^2 - \frac{l(l+1)}{\sin^2\eta}\right\}g = 0, \qquad (14)$$

and

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\alpha^2} - \left\{n^2 - \frac{\lambda}{C^2\cosh^2\alpha - \frac{E^2}{4}\sinh^2\alpha}\cdot\frac{1}{1-\lambda\Delta(m, \mu, \tanh\alpha)}\right\}f = 0, \qquad (15)$$

where n is a separation constant.

We thus see that the modified Wick-Cutkosky equation which includes the effect of self-energy insertions in the integral kernel of the BS equation still admits separable solution. In §3 we discuss the eigen solution of the problem and concentrate on the particular case $E = 0$ which affords a simple exact solution of (15) determining the eigenvalues.

3. Solution at $E = 0$

At $E = 0$ we note from (11) that $\Delta(m, \mu, \tanh\alpha)$ assumes a particularly simple form: Here $\Delta(m, \mu, \tanh\alpha)$ becomes independent of α and $\Delta(m, \mu, \tanh\alpha)$ reduces to a constant depending on m and μ only which we denote by Δ, *i.e.*

$$\Delta(m, \mu, \tanh\alpha)_{E=0} = \Delta, \tag{16}$$

and (15) then reduces to

$$\frac{d^2 f(\alpha)}{d\alpha^2} - \left\{ n^2 - \frac{\lambda/(1-\lambda\Delta)}{m^2\cosh^2\alpha} \right\} f(\alpha) = 0. \tag{17}$$

The boundary conditions on $g(\eta)$ and $f(\alpha)$ in the new coordinates are obtained by substituting them in the original integral (9) which is satisfied if $g(0) = g(\pi)$ and $f(\alpha)$ vanishes at large α and is an even function of α. Further f and g must be finite within the given ranges as mentioned at the end of (8).

The acceptable solutions of (14) consistent with the above mentioned boundary conditions are the well-known Gegenbauer polynomials,

$$g(\eta) = \text{constant} \times \sin^{l+1}(\eta) \cdot C_{n-l-1}^{l+1}(\cos\eta). \tag{18}$$

Equation (17) determines the energy eigenvalues of the bound states. Here, for a given E, the equation, however, determines the spectrum of values of λ. If we transform (17) in terms of a new variable χ where $\cos\chi = \tanh\alpha$, the solution of (17) can be easily obtained as

$$f = \text{constant} \times \sin^n(\chi)\, C_{N-n}^{n+1/2}(\cos\chi), \tag{19}$$

where the integer N is related to the eigenvalue λ by

$$\frac{\lambda}{m^2(1-\Delta\lambda)} = N(N+1). \tag{20}$$

This result is to be compared with the eigen spectrum of λ when the effect of self-energy is not present, which is obtained from (20) by setting $\Delta = 0$ on the left-hand side. The relevant relation is

$$\lambda/m^2 = N(N+1). \tag{21}$$

Thus when $E = 0$ and the scalar particles interact through ladder diagrams *via* the exchange of massless scalar particle the self-energy diagrams for $\Delta > 0$ will diminish the eigenvalue. This result is in agreement with the finding of Li *et al* (1980), who, however, solved the problem numerically for a massive scalar field exchange. The eigenfunctions in the present case when we consider the massless scalar particle exchange remain identical both for equations with or without the inclusion of self-energy effects. Li *et al* (1980) found a slight change in the nature of the eigenfunctions in these two cases.

4. Other remarks

If we collect the solutions at $E = 0$ of $g(\eta)$ and $f(\alpha)$ from (18) and (19) we see from (13) that the Bethe-Salpeter wave-function ψ takes the form

$$\psi \approx Y_{Nnlm} \tag{22}$$

where Y_{Nnlm} are the spherical harmonics in the 5-dimension euclidean space and are given by

$$Y_{Nnlm} = (\sin\chi)^n C_{N-n}^{n+1/2}(\cos\chi)(\sin\eta)^{l+1} C_{n-l-1}^{l+1}(\cos\eta)\cdot Y_{lm}(\theta,\phi), \tag{23}$$

where $N > n > l+1$, $l > |m|$ and the degeneracy of Y_{Nnlm} is $\frac{1}{6}N(N+1)(2N+1)$. In other words the BS equation even in the presence of self-energy effects preserves the 0(5) symmetry (Biswas 1967) originally discovered by Cutkosky (1954) when the self-energy effect was not included.

At $E \neq 0$ the energy-eigenvalue equation (17) can be recast in the following form

$$(1-Z^2)\frac{d^2G}{dZ^2} + 2(n-1)Z\frac{dG}{dZ} + \left\{\frac{(\lambda/m^2)}{(1-E^2+E^2Z^2)}\frac{1}{(1-\lambda\Delta(m,\mu,Z^2))}\right\}G = n(n-1)G. \tag{24}$$

Equation (20) reduces to Cutkosky's (1954) equation when Δ is set to zero. The equation cannot be solved exactly except at special limits. Both the eigenfunctions and eigenvalues will be largely altered due to the presence of the factor $[1-\lambda\Delta(Z^2)]^{-1}$ in the third term which makes the equation more singular than the standard Huen's equation (Erdelyi 1942, 1944).

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