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Quantum supersymmetric generalisation of Bogomolnyi bounds

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Abstract. We review here classical Bogomolnyi bounds, and their generalisation to supersymmetric quantum field theories by Witten and Olive. We also summarise some recent work by several people on whether such bounds are saturated in the quantised theory.

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1. Introduction

Bogomolnyi published, in 1976, a very interesting paper (Bogomolnyi 1976) on some aspects of classical topological soliton solutions of several relativistic field theories. His procedure was generic in nature and covered such examples of solitons* as Kinks in two (space-time) dimensions, vortices and flux lines in 3 dimensions as well as the famous monopole solution in 4 dimensions. The thrust of his results was two-fold. Firstly he showed that the energy of any finite-energy field configuration in these systems was bounded from below by its topological index, apart from a known numerical factor. Secondly, on requiring that this bound is saturated, one attains first order partial differential equations—far simpler to solve than the parent field equations which are second order non-linear partial differential equations. The familiar solitons like the Kink, the flux lines and the monopole can be obtained as solutions of these first-order equations. (It must be mentioned that the basic trick used by Bogomolnyi had been employed slightly earlier, in the context of instantons, by Belavin *et al* 1975).

Bogomolnyi's work was at the classical level. Meanwhile, methods have been developed during the past decade for "quantising" solitons, to yield quantum soliton particles in the corresponding quantum field theories. A detailed discussion of the quantisation of solitons is given in the book by the present author (Rajaraman 1982). One finds that these quantum solitons possess, in a suitably generalised sense, many of the remarkable properties that the classical solitons did. One can ask therefore whether

* Strictly speaking, most of these solutions should be called solitary waves, rather than solitons. But here we follow the practice in most of the recent literature of using the term solitons to also cover solitary waves, *ie* localised, non-dispersive solutions. Note however, that contrary to the incorrect feeling that still persists in some circles, solitary waves are not limited to $(1+1)$ dimensions. Especially after the entry of particle physicists in this field, many solitary wave solutions have been found in higher dimensions as well. Thus, interesting gauge-theoretic solutions such as vortices in $(2+1)$ dimensions and the 'tHooft-Polyakov monopole in $(3+1)$ dimensions are legitimate solitary waves.

Bogomolnyi-type bounds can also be obtained for quantum solitons, relating their exact quantum masses to their topological quantum numbers, and whether these bounds are saturated.

In an elegant and compact paper, Witten and Olive (1978) showed that for quantum supersymmetric extensions of some of the models considered by Bogomolnyi, an exact bound can be obtained for the masses of the quantum soliton states. In fact, Witten and Olive pointed out that the existence of the soliton sector affects the very algebra of supersymmetry (SUSY), introducing a central charge in the algebra in that sector. This central charge is intimately related to the topological charge of the soliton and, after some manipulations, yields a lower bound to the masses of all the states in the soliton sector. Having obtained the bounds, Witten and Olive also offered some speculations on whether quantum solitons actually *saturate* these bounds, as in fact they do at the classical approximation.

The question of whether these bounds are saturated at the quantum level is an important one. For, if they are, they would give us some handle on the exact quantum mechanical masses of these soliton particles. One must remember that calculating the mass-spectrum is one of the primary goals of particle physics. Yet hitherto no one has been able to calculate exactly, the mass of any massive bound state in any non-trivial 4-dimensional field theory! We emphasize that these quantum bounds apply not just to solitons in two space-time dimensions like the Kink or the sine-Gordon soliton, but also to some solitons in realistic four-dimensions as well, such as the (supersymmetric) monopole.

Recent investigations by several workers have pursued further the question of whether the Witten-Olive bounds are saturated and the related questions of whether the soliton mass and its topological index each receive quantum corrections in these SUSY models. The detailed results, as we will see, vary with the models.

In this article, we review the developments mentioned above. This is strictly a review—prepared specifically for the special volume in honour of Dr Raja Ramanna. There are no new results in this article, but only a synthesis of results already contained in the references cited. Also, in order to keep the length of this article within its prescribed bounds, we have to assume that the reader is familiar with the overall background in which our specialised topic is imbedded, which includes the basics of gauge theory, supersymmetry, classical and quantum solitons, and the 't Hooft-Polyakov monopole.

2. Classical Bogomolnyi bounds

Bogomolnyi inequalities at the classical level are well-known. Our chief concern here is their quantum generalisation. Nevertheless, for the sake of completeness, let us quickly sketch the classical bounds and their derivation. We will use as illustrations $(1+1)$-dimensional scalar field theories as well as the $(3+1)$ dimensional Georgi-Glashow model which yields that famous soliton, the 't Hooft-Polyakov monopole (Georgi and Glashow 1972; 't Hooft 1974; Polyakov 1974).

Let us begin with the simple problem of a real scalar field theory in $(1+1)$ dimensions, with any potential which has degenerate minima so that topological solitons can arise. The Lagrangian is

$$L = \int \mathrm{d}x \left[\tfrac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \tfrac{1}{2} S^2(\phi)\right], \tag{1}$$

where without loss of generality, we can take the potential to be positive and to vanish at its absolute minima. We write this potential in the form $\frac{1}{2}S^2(\phi)$ for the later convenience when we supersymmetrise the system in §3.

The classical energy functional associated with this Lagrangian is

$$E = \int \mathrm{d}x \left[\tfrac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \tfrac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + \tfrac{1}{2}S^2(\phi)\right]. \tag{2}$$

Clearly, all finite energy configurations must approach one of the zeroes of $S(\phi)$ at the spatial extremities $x = -\infty$ and $x = +\infty$. We are interested in a *lower* bound to the energy, and therefore need to consider only static configurations, where the (positive) kinetic energy is absent. The energy of static (time independent) fields $\phi(x)$ is

$$\begin{aligned} E &= \int_{-\infty}^{\infty} \mathrm{d}x \left[\tfrac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 + \tfrac{1}{2}S^2(\phi)\right] \\ &= \int_{-\infty}^{\infty} \mathrm{d}x\, \tfrac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x} \pm S(\phi)\right)^2 \mp \int_{-\infty}^{\infty} \mathrm{d}x\, S(\phi)\frac{\mathrm{d}\phi}{\mathrm{d}x} \\ &\geqslant \left| \int_{-\infty}^{\infty} \mathrm{d}x\, S(\phi)\frac{\mathrm{d}\phi}{\mathrm{d}x} \right| \\ &\equiv 2\left[D(\phi(x))\right]_{x=-\infty}^{x=\infty}, \end{aligned} \tag{3}$$

where $$D(\phi) \equiv \int \mathrm{d}\phi\, S(\phi). \tag{4}$$

This is the Bogomolnyi bound for this system and sets a lower bound on the energy of any field configuration. The bound is *saturated*, i.e. the equality in (3) is obeyed provided (i) ϕ is static, and (ii) it obeys

$$\mathrm{d}\phi/\mathrm{d}x = \pm S(\phi) \tag{5}$$

Meanwhile, the field equation arising from (1), for static configurations, is

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = S(\phi)\frac{\mathrm{d}S}{\mathrm{d}\phi} \tag{6}$$

The first integral of this field equation is just the saturation condition (5). Hence any *static* classical solution, including the topological soliton, will satisfy (5) and therefore saturate the bound (3). Thus we know, without explicitly finding the soliton solution, that its energy will be just $D(\phi_2) - D(\phi_1)$ where ϕ_2, ϕ_1 are the boundary values of the soliton at $x = \pm\infty$ respectively. Clearly this bound, depending as it does only on boundary values of the field, is a topological index of the soliton.

As an illustration, consider the kink solution of the double-well potential, where

$$S(\phi) = \sqrt{\frac{\lambda}{2}}\left(\phi^2 - \frac{\mu^2}{\lambda}\right).$$

Then

$$D(\phi) = \sqrt{\frac{\lambda}{2}}\left(\frac{\phi^3}{3} - \frac{\mu^2}{\lambda}\phi\right).$$

The kink solution goes from $\phi_1 = -\mu/\sqrt{\lambda}$ at $x = -\infty$ to $\phi_2 = +\mu/\sqrt{\lambda}$ at $x = \infty$. Hence the bound is

$$D(\phi_2) - D(\phi_1) = \frac{2\sqrt{2}}{3}\frac{\mu^3}{\lambda}. \tag{7}$$

Meanwhile, on explicitly solving the field equation, the kink solution has the well-known form

$$\phi_{\text{kink}} = \frac{\mu}{\sqrt{\lambda}} \tanh \frac{\mu x}{\sqrt{2}},$$

with a classical energy (the classical kink mass) equal to

$$M^{(0)}_{\text{kink}} = \frac{2\sqrt{2}}{3}\frac{\mu^3}{\lambda}. \tag{8}$$

This exactly equals the bound (7), as expected.

As a second example, let us turn to the Georgi-Glashow model, which is a non-abelian (3 + 1) dimensional gauge theory, with a triplet of scalar fields ϕ^a coupled to a triplet of SU(2) gauge fields A^a. The Lagrangian density is

$$\mathscr{L} = \tfrac{1}{2}(D_\mu\phi)^a (D^\mu\phi)^a - \frac{\lambda}{4}(\phi^a\phi^a - C^2)^2 - \tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \tag{9}$$

with μ, $\nu = 0, 1, 2, 3$; $a = 1, 2, 3$, $(D_\mu\phi)^a \equiv \partial_\mu\phi^a + g\varepsilon_{abc}A^b_\mu\phi^c$

and $$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon_{abc}A^b_\mu A^c_\nu. \tag{10}$$

Although this system is more complicated than (1), Bogomolnyi showed that a bound can be obtained in a similar way. Consider again the energy functional for static configurations, in the $A^a_0 = 0$ gauge.

$$E = \int d^3x \left[\tfrac{1}{4}F^a_{ij}F^a_{ij} + \tfrac{1}{2}D_i\phi^a D_i\phi^a + \frac{\lambda}{4}(\phi^a\phi^a - C^2)^2 \right] \tag{11}$$

with $i, j = 1, 2, 3$. Finiteness of energy requires, as $|\mathbf{x}| \to \infty$,

$$\phi^a\phi^a \to C^2, \tag{12a}$$

and $$F^a_{ij},\ D_i\phi^a \to 0. \tag{12b}$$

Each field configuration permitted by the boundary condition (12a) corresponds to a mapping of a two-sphere S_2 into another S_2. Since the second homotopy group $\pi_2(S_2) = Z$, such configurations fall into homotopy classes characterised by an integer-valued index, given by

$$N = \frac{1}{8\pi}\int_{S_2} d\sigma_i \varepsilon_{ijk}\varepsilon^{abc}(\hat{\phi}^a)(\partial_j\hat{\phi}^b)(\partial_k\hat{\phi}^c), \tag{13}$$

where $\hat{\phi}^a = \phi^a/|\phi| = \phi^a/C$. Recall also the result that a configuration with homotopy index N carries magnetic charge $m = N/g$. For a detailed discussion of these well known results, see Rajaraman (1982). The energy in (11) can be written, after a little algebra, in the form

$$\begin{aligned} E &= \frac{4\pi C}{g}N + \int d^3x \left[\tfrac{1}{4}(F^a_{ij} - \varepsilon_{ijk}D_k\phi^a)^2 + \frac{\lambda}{4}(\phi^a\phi^a - C^2)^2 \right] \\ &\geqslant \frac{4\pi C}{g}N = 4\pi C m \end{aligned} \tag{14}$$

where m is the magnetic charge of the configuration. This is Bogomolnyi's inequality for the Georgi-Glashow model. In the so-called Prasad-Sommerfield limit (Prasad and Sommerfield 1975), where $\lambda \to 0$ with C fixed, the bound (14) is saturated, provided

$$F^a_{ij} = \varepsilon_{ijk}(D_k\phi)^a. \tag{15}$$

A solution of (15), by virtue of minimising the static energy in any given N-sector, will also be a classical solution of the field equations. In fact, the well-known Prasad-Sommerfield solution for the monopole

$$\phi^a = Cr^a\left[\coth(rgC) - \frac{1}{rgC}\right]; \quad A^a_0 = 0; \quad \text{and}$$

$$A^a_i = \frac{1}{gr}\varepsilon_{aib}\hat{r}^b\left(1 - \frac{rgC}{\sinh rgC}\right) \tag{16a}$$

satisfies (15) and lies in the $N = 1$ sector. Correspondingly, its classical mass,

$$M^{(0)}_{\text{mono}} = 4\pi C/g, \tag{16b}$$

saturates the bound (14). Note that (15) is a first-order equation, like (7), although the parent field equations derived from (10) would be of second order.

Similar results can also be obtained for the Nielsen-Oleson vortex lines, the CP_n and non-linear O_3 models in $(2+1)$ dimensions. Note that the discussion in this section, based on Bogomolnyi's (1976) paper, has been entirely at the classical level.

3. The Witten-Olive bounds

Witten and Olive (1978) considered the quantised and supersymmetric extension of the models discussed in the previous section. First take the two-dimensional example and consider the Lagrangian density

$$\mathscr{L} = \tfrac{1}{2}[(\partial_\mu\phi)^2 - S^2(\phi) + \bar{\psi}(i\partial_\mu\gamma^\mu - S'(\phi))\psi], \tag{17}$$

where ψ is a Majorana spinor in two dimensions. The two γ-matrices could be taken as, say $\gamma^0 = \sigma_2$ and $\gamma^1 = -i\sigma_1$. $S(\phi)$ is any function of the real scalar field ϕ which permits topological solitons. It must have degenerate minima. This Lagrangian is supersymmetric under the transformation

$$\phi \to \phi + \bar{\xi}\psi; \quad \psi \to \psi + (-i\gamma_\mu\partial^\mu\phi - S(\phi))\xi; \quad \bar{\psi} \to \bar{\psi} + \bar{\xi}(i\gamma_\mu\partial^\mu\phi - S) \tag{18}$$

where $\xi, \bar{\xi}$ are Grassmann numbers. For $S = (\lambda/2)^{1/2}(\phi^2 - C^2)$, the system (17) is the supersymmetric extension of the double well system (1).

The supercharge which generates the transformations (18) is also a Majorana spinor Q given by

$$Q = \int dx(\gamma^\mu\partial_\mu\phi + iS)\gamma^0\psi. \tag{19}$$

In our representation of γ-matrices, charge conjugation is just complex conjugation. Hence the Majorana spinors Q and ψ have only real components:

$$Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}; \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}; \tag{20}$$

with
$$Q_1 = \int dx\,[(\partial_0\phi + \partial_1\phi)\psi_1 - S\psi_2],$$
$$Q_2 = \int dx\,[(\partial_0\phi - \partial_1\phi)\psi_2 + S\psi_1]. \tag{21}$$

Using the canonical commutation rules for the fields ϕ and ψ it is straightforward to check that

$$(Q_1)^2 = H + P, \tag{22a}$$

$$(Q_2)^2 = H - P, \tag{22b}$$

and
$$\{Q_1, Q_2\}_+ = 2\int dx\, S(\phi)\frac{d\phi}{dx} = 2[D(\phi)]_{\phi(x=-\infty)}^{\phi(x=\infty)} \equiv T, \tag{22c}$$

where H and P are the total energy and momentum operators respectively, for the system (17), and $D(\phi) \equiv \int d\phi\, S(\phi)$. Equations (22a, b) can be recognised as part of the "usual" SUSY algebra in 2 dimensions. But in the "usual" algebra, the anticommutator in (22c) would have been taken to vanish, and indeed it does vanish in the *vacuum* sector, where $\phi(x)$ takes the same value at $x = -\infty$ and $x = +\infty$, and so will $D(\phi(x))$.

But, in the soliton sector, where $\phi(\infty) \neq \phi(-\infty)$, $D(\phi(\infty))$ *need not* equal $D(\phi(-\infty))$. In fact, for systems we are interested in, *ie* those which support static topological solitons, $D(\phi(\infty))$ *will not* equal $D(\phi(-\infty))$. To see this, note that a static topological soliton will go from some $\phi = \phi_1$ at $x = -\infty$ to some other $\phi = \phi_2$ at $x = \infty$, where ϕ_1 and ϕ_2 are two *neighbouring* distinct zeroes of $S(\phi)$. Therefore ϕ_1 and ϕ_2 will also be two *neighbouring* extremes of $D(\phi)$, one of them a minimum and the other the next maximum. Clearly $D(\phi_1)$ cannot equal $D(\phi_2)$. Thus, in the sector of states based on the topological soliton, the operator T is non-vanishing, and forms (as can be verified) a central charge in the SUSY algebra. (See, however, remarks by Schonfeld (1979) on the validity of (22) in the face of boundary conditions.)

The quantum version of the bound (3) follows immediately from the algebra (22). We have

$$\begin{aligned} H &= \tfrac{1}{2}(Q_1^2 + Q_2^2) \\ &= \tfrac{1}{2}[(Q_1 \pm Q_2)^2 \mp \{Q_1, Q_2\}_+] \\ &= \tfrac{1}{2}[(Q_1 \pm Q_2)^2 \mp T] \\ &\geqslant \tfrac{1}{2}|T|. \end{aligned} \tag{23}$$

This operator inequality is the quantum generalisation of (3), obtained by Witten and Olive for the SUSY system (17). The last step in (23) holds because $(Q_1 \pm Q_2)^2$ is a real non-negative operator. Equation (23) is an operator inequality, *ie* it holds for the expectation values of both sides taken between any state of the quantum field theory associated with (17). In the vacuum sector $\langle T \rangle = 0$ for all states, so that (23) is trivial. But in the soliton sector it is not. The quantum soliton-particle at rest is the lowest energy state in this sector. Let us denote it by $|\text{sol}\rangle$. Then the full quantum mass of the soliton, M_{sol}, obeys

$$\begin{aligned} M_{\text{sol}} &= \langle \text{sol}|H|\text{sol}\rangle \\ &\geqslant \tfrac{1}{2}|\langle \text{sol}|T|\text{sol}\rangle|. \end{aligned} \tag{24}$$

Note that (24) is still in the form of an inequality. We will return later to the question of whether the equality holds.

A similar bound can be obtained for the quantum supersymmetric monopole by considering a SUSY extension of the Georgi-Glashow model (9–10). The Lagrangian, in $(3+1)$ dimensions, is

$$\mathscr{L} = -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \tfrac{1}{2}i\bar{\psi}^a_j\gamma_\mu(D^\mu\psi)^a_j + \tfrac{1}{2}(D_\mu\phi_j)^a(D_\mu\phi_j)^a + \frac{g^2}{2}\mathrm{Tr}([\phi_1,\phi_2])^2 + \frac{ig}{2}\varepsilon_{jk}\mathrm{Tr}\{[\bar{\psi}^j,\psi^k]\phi_1 + [\bar{\psi}^j,\gamma_5\psi^k]\phi_2\} \tag{25}$$

where $\psi^a_j, j = 1, 2; a = 1, 2, 3$ form a pair of isotriplet Majorana spinors, ϕ^a_1 is a scalar and ϕ^a_2 a pseudoscalar isotriplet, and $F^a_{\mu\nu}$ the gauge field tensor of the SU(2) gauge group. We have also used the popular notation of representing isotriplets ϕ^a_j and ψ^a_j by matrix valued fields $\phi_j \equiv (\tau^a/2)\phi^a_j$ and $\psi_j = (\tau^a/2)\psi^a_j$ respectively in the last two trace terms in (25). The model (25) enjoys $N = 2$ supersymmetry whose generators are

$$Q_i = \mathrm{Tr}\int \mathrm{d}^3x\,\{\sigma^{\mu\nu}F_{\mu\nu}\gamma^0\psi_i + \varepsilon_{ij}(\not{D}\phi_1) + (\not{D}\phi_2)(\gamma_5)\gamma^0\psi^i + g[\phi_1,\phi_2]\gamma^0\gamma_5\psi_i\}, \tag{26}$$

where $i = 1, 2$ and the spinorial indices of Q and ψ are suppressed. When the anticommutators of the Q_i are evaluated using the canonical commutation rules of the fields, one obtains

$$\{Q_i, \bar{Q}_j\}_+ = \delta_{ij}\gamma^\mu P_\mu + \varepsilon_{ij}(U + \gamma_5 V), \tag{27}$$

where

$$U \equiv \int \mathrm{d}^3x\,\partial_i(\phi^a_1 F^a_{oi} + \tfrac{1}{2}\varepsilon_{ijk}\phi^a_2 F^a_{jk}), \tag{28a}$$

$$V \equiv \int \mathrm{d}^3x\,\partial_i(\phi^a_2 F^a_{oi} + \tfrac{1}{2}\varepsilon_{ijk}\phi^a_1 F^a_{jk}). \tag{28b}$$

Note that the boson potential in (25) has the positive form $\mathrm{Tr}([\phi_1,\phi_2])^2$. When $\phi_1 = \phi_2 = 0$, this vanishes. Correspondingly, one choice of the vacuum is the symmetric one, with $\langle\phi_1\rangle_{\mathrm{vac}} = \langle\phi_2\rangle_{\mathrm{vac}} = 0$. However, $\mathrm{Tr}([\phi_1,\phi_2])^2$ also vanishes if the matrices ϕ_1 and ϕ_2 commute, *ie* if the iso-vectors ϕ^a_1 and ϕ^a_2 are parallel in internal space, regardless of the magnitudes $|\phi^a_1|$ and $|\phi^a_2|$. This choice $\langle\phi^a_1\rangle_{\mathrm{vac}} \propto \langle\phi^a_2\rangle_{\mathrm{vac}}$, with either non-zero, corresponds to spontaneous breaking of the SU(2) gauge symmetry. Now, consider the charges U and V in (28). Both are integrals of divergences and can be written as surface integrals at infinity. In the symmetric case, $\langle\phi^a_1\rangle_{\mathrm{vac}} = \langle\phi^a_2\rangle_{\mathrm{vac}} = 0$, both ϕ^a_1 and ϕ^a_2 will vanish at spatial infinity for any physical state, and hence $U = V = 0$. Then (27) reduces to the "usual" SUSY algebra. It is only when $\langle\phi^a_1\rangle_{\mathrm{vac}}$ or $\langle\phi^a_2\rangle_{\mathrm{vac}}$ is non-zero that the central charges U and V come into play.

The quantum Bogomolnyi-type bound follows easily from (27). A simple trick, used by Witten and Olive is to exploit the fact that the system (25) is also chirally invariant. Under a chiral rotation, the fields ϕ^a_1 and ϕ^a_2 rotate into one another, and therefore so do U and V. Perform a rotation such that $\langle V\rangle = 0$. Take the expectation value of (27) between any energy-momentum eigenstate (whether solitonic or otherwise) at rest.

Then (27) gives

$$\langle \{Q_i, Q_j\}_+ \rangle = \delta_{ij} M + \varepsilon_{ij} \gamma^0 \langle U \rangle, \tag{29}$$

where M is the invariant mass of the state. Note that Q_i (or Q_j) is also a spinor in Dirac space, with spinor index α (or β) which we had suppressed so far, in addition to the index i (or j) which takes values 1, 2. Considered as an 8×8 matrix, $\langle \{Q_{i\alpha}, Q_{j\beta}\}_+ \rangle$ is a positive matrix. It will have real non-negative eigenvalues. Therefore so will the matrix $\delta_{ij} M + \varepsilon_{ij} \gamma^0 \langle U \rangle$, thanks to (29). But $\varepsilon_{ij} \gamma^0 \langle U \rangle$ clearly has eigenvalues $\pm \langle U \rangle$. Hence,

$$M \geqslant |\langle U \rangle|. \tag{30}$$

This result was obtained after setting $\langle V \rangle = 0$ through a chiral rotation. The general result is clearly a chirally invariant generalisation of (30), *viz.*

$$M^2 \geqslant \langle U \rangle^2 + \langle V \rangle^2. \tag{31}$$

This is the Witten-Olive bound for the SUSY Georgi-Glashow model, for the mass of any state, in terms of the expectation values of U and V in that state. For, the case $\langle \phi_1^a \rangle_{\text{vac}} = \langle \phi_2^a \rangle_{\text{vac}} = 0$, which corresponds to full unbroken SU(2) gauge symmetry, $\langle U \rangle = \langle V \rangle = 0$ for any state. Then the bound (31) is trivial. But, for all symmetry-broken cases, with either $\langle \phi_1^a \rangle_{\text{vac}}$ or $\langle \phi_2^a \rangle_{\text{vac}}$ non-zero, the bound (31) is non-trivial. In such cases, it is applicable not only in the solitonic sectors, but also in the vacuum sector. That is, (31) can be used to get a lower bound on the exact quantum masses of the monopoles and dyons, as well as the vector bosons and the Higgs bosons of this SUSY theory. Thus, this result of Witten and Olive is a truly remarkable one.

4. Saturation of the bounds

The quantum Witten-Olive bounds (23) and (31) reduce, in the appropriate classical limit, to the Bogomolnyi bounds discussed in §2, and are, furthermore, saturated classically by soliton solutions. For the (1 + 1) dimensional scalar field case, it is obvious that the quantum bound (23), where the operator T is given in (22c), reduces to just the Bogomolnyi bound (3) in the classical limit. This classical bound, as we showed in §2, is saturated by the soliton. For the (3 + 1) dimensional gauge theory, the relationship of the classical bound (14) to the quantum bound (31) may be made more transparent as follows. Using chiral symmetry freedom, we can consider the case where only $\langle \phi_1^a \rangle_{\text{vac}} \neq 0$, with $\langle \phi_2^a \rangle_{\text{vac}} = 0$. Let the modulus $|\langle \phi_1^a \rangle_{\text{vac}}| = C$, with the internal space direction of $\langle \phi_1^a \rangle_{\text{vac}}$ arbitrary. For all finite energy systems, topological or otherwise, $|\phi_1^a|$ must then tend to C and ϕ_2^a must tend to zero as $|\mathbf{x}| \to \infty$. The gauge group SU(2) is broken down to $U(1)$ by such a vacuum. The associated electromagnetic field $F_{\mu\nu}^{e.m.}$ has the gauge invariant form ('t Hooft 1974)

$$\begin{aligned} F_{\mu\nu}^{\text{e.m.}} &= \frac{1}{C} \phi_1^a F_{\mu\nu}^a - \frac{1}{gC^3} \varepsilon_{abc} \phi_1^a (D_\mu \phi_1)^b (D_\nu \phi_1)^c \\ &\underset{x \to \infty}{\to} \frac{1}{C} \phi_1^a F_{\mu\nu}^a \end{aligned} \tag{32}$$

since finiteness of energy requires that $(D_\mu \phi_1)^a \to 0$ faster than $1/|\mathbf{x}|^{3/2}$ as $|\mathbf{x}| \to \infty$. Now, consider the charges U and V defined in (28). Since $\phi_2^a \to 0$ as $x \to \infty$, we have, using Gauss' theorem,

$$U = \oint_S \mathrm{d}\sigma_i\,(\phi_1^a F_{oi}^a) = C \oint_S \mathrm{d}\sigma_i F_{oi}^{\text{e.m.}} = 4\pi C q, \tag{33a}$$

$$V = \oint_S \mathrm{d}\sigma_i \tfrac{1}{2}(\phi_1^a \varepsilon_{ijk} F_{jk}^a) = C \oint_S \mathrm{d}\sigma_i \tfrac{1}{2}\varepsilon_{ijk} F_{jk}^{\text{e.m.}} = 4\pi C m, \tag{33b}$$

where q and m are just the total electric and magnetic charge operators. Hence the bound (31) can be written as

$$M^2 \geqslant (4\pi C)^2\,[\langle q \rangle^2 + \langle m \rangle^2], \tag{34}$$

where $\langle q \rangle$ and $\langle m \rangle$ are the expectation values of the total electric and magnetic charges of the state in question. Thus this bound applies non-trivially to particles which carry either electric or magnetic charge, or both. Equation (34) is still the same quantum bound as (31), but rewritten in terms of more familiar charges. In the classical limit, when applied to static topological configurations in the $A_o = 0$ gauge (which carry zero electric charge) it reduces to precisely the classical Bogomolnyi bound (14). As pointed out in §2, the exact single-monopole solution, in the Prasad-Sommerfield limit, saturates this classical bound. For states which carry no magnetic charge (these will be non-solitonic states, in the $N = 0$ homotopy sector) the bound in (34) is still useful, giving

$$M \geqslant 4\pi C|\langle q \rangle|. \tag{35}$$

The familiar "quanta" of the fields, such as the Higgs boson, the massive Vector (W) boson and the photon of this theory come in this category. Notice again that at the classical ("tree") level, the W boson has electric charge $q/4\pi$ and mass gC, thus saturating the bound classically. The photon is a special case. Thanks to unbroken $U(1)$ gauge invariance, the photon ($q = 0$) has exactly zero-mass quantum theoretically, and will infact saturate the exact quantum bound (35). For the other particles, *ie* the monopole, the W-boson, and the Higgs boson of this model, as well as for the solitons of the two-dimensional model, while precise quantum *bounds* exist in the form of (24) and (34), their *saturation* at the *quantum* level calls for further discussion. The rest of this section is devoted to this question.

Let us begin with solitons of $(1 + 1)$ dimensional scalar field theories, whose quantum bound is given in (24). There is no rigorous closed result available, as far as we know, establishing that the quantum soliton must saturate the bound. What most people have attempted is to evaluate separately the quantum corrections to both sides of the bound, *ie* to the soliton mass and to its topological index T. Then one can see if these corrections equal one another. Of course this is not a completely definitive way of answering the question. Quantum corrections are generally calculated using a semi-classical loop expansion, in powers of $\hbar$. In practice such a calculation can be done only up to some given order, usually to order $\hbar$. If the two sides of (24) agree upto $O(\hbar)$, that by itself does not guarantee that they will agree to all orders in $\hbar$. (See however an ingenious and indirect argument by Witten and Olive, based on counting of states, suggesting that the monopole does exactly saturate the quantum bound. See also Imbimbo and Mukhi (1984a) for the 2-dimensional soliton). Of course, if the two sides of (24) do not equal each other upto order $\hbar$, then we can be sure that the bound is *not* saturated.

Let us first consider calculations of the $O(\hbar)$ corrections to the soliton mass. The general principles of soliton quantisation, particularly in $(1 + 1)$ dimensions, are by now well known (for reviews, see Rajaraman 1975; Jackiw 1977; Coleman 1977) so that we need not present the details of this calculation here. But there are some special features which arises in SUSY models that one should be careful about. The soliton's quantum mass to $O(\hbar)$ is given by

$$M_{\text{sol}}^{(1)} = M_{\text{sol}}^{(0)} + \tfrac{1}{2}\hbar\left(\sum_{w_n > 0} w_n\right) - \tfrac{1}{2}\hbar\left(\sum_{e_n > 0} e_n\right) + M_{\text{c.t.}}, \tag{36}$$

where w_n^2 and e_n are respectively the eigenvalues of the boson fluctuation equation and the fermion Dirac equation in the background of the soliton. $M_{\text{c.t.}}$ is the contribution of the $O(\hbar)$ renormalisation counter term. For the system (17), the eigenvalue equations determining w_n^2 and e_n are

$$\left(-\frac{d^2}{dx^2} + (S')^2 + SS''\right)\eta_n(x) = w_n^2\eta_n(x) \tag{37}$$

$$\left(-i\alpha\frac{d}{dx} + \beta S'\right)\psi_n(x) = e_n\psi_n(x) \tag{38}$$

where $S' \equiv \left[\frac{d}{d\phi}S(\phi)\right]_{\phi_{\text{sol}}(x)}$ and $S'' \equiv \left[\frac{d^2S}{d\phi^2}\right]_{\phi_{\text{sol}}(x)}$

D'Adda and Di Vecchia (1978) made the following important observation regarding (37) and (38). Let us write the two-component spinor ψ_n as

$$\begin{pmatrix}\psi_n^{(+)}\\ \psi_n^{(-)}\end{pmatrix}.$$

It will be convenient to use the representation $\beta = \sigma_2, \alpha = -\sigma_1$. Then, upon squaring the Dirac Hamiltonian in (38), we get

$$(i\sigma_1 d_x + \sigma_2 S')^2\psi_n = \left(-d_x^2 + (S')^2 - \sigma_3\frac{d}{dx}S'\right)\psi_n = e_n^2\psi_n. \tag{39}$$

Since the soliton obeys $d\phi_{\text{sol}}/dx = -S(\phi_{\text{sol}})$, $dS'/dx = (S'')\,d\phi_{\text{sol}}/dx = -S''S$, (39) yields, for each component of ψ_n,

$$(-d_x^2 + (S')^2 + SS'')\psi_n^{(+)} = e_n^2\psi_n^{(+)}, \tag{40a}$$

$$(-d_x^2 + (S')^2 - SS'')\psi_n^{(-)} = e_n^2\psi_n^{(-)}. \tag{40b}$$

One can see that thanks to supersymmetry the upper component of the spinor, $\psi_n^{(+)}$, obeys the same eigenvalue equation as (37), obeyed by the boson fluctuations η_n. Thus any solution of (37) would also serve as a solution of (40a) with the same eigenvalue and vice versa. The associated $\psi_n^{(-)}$ could then be determined from the Dirac equation (38) *ie*

$$\psi_n^{(-)} = \frac{1}{e_n}(id_x + iS')\psi_n^{(+)}. \tag{41}$$

As evident from (41), this matching between boson and fermion fluctuations may not hold for zero-energy ($e_n = 0$) modes, but we need not worry about these for our purposes, since they make no contribution to the energy in (36).

Upto this point, these observations of D'Adda and Di Vechia are interesting and correct. However, from this they concluded that $\Sigma w_n - \Sigma e_n = 0$ and that the soliton receives no quantum corrections to its mass upto order ($\hbar$). This is not correct for two reasons, as pointed out by Schonfeld (1979) and by Kaul and Rajaraman (1983):

(i) The counter-term contribution $M_{c.t.}$ has been neglected. Normal ordering counter terms proportional to SS'' exist for these models consistent with supersymmetry and make a non-zero (and in fact divergent) contribution $M_{c.t}$

(ii) A more subtle point is that $(\Sigma w_n - \Sigma e_n)$ does *not* vanish even though, loosely speaking, every non-zero eigenvalue w of (37) also occurs as an eigenvalue of (38) and vice versa. The reason is that, apart from a few discrete levels, the spectra of both (37) and (38) are continuous, and the densities of continuum level in the two cases are different. The simplest way to understand this difference in density states is to put the system in a box of length L, and later take $L \to \infty$. Then the Boson eigenfunction $\eta_n(x)$ obeys boundary conditions

$$\eta_n(-L/2) = \eta_n(L/2), \tag{42a}$$

and

$$\frac{d\eta_n}{dx}(-L/2) = \frac{d\eta_n}{dx}(L/2), \tag{42b}$$

as appropriate to the second order differential operator in (37). However, the fermion eigenfunction $\psi_n(x)$ obeys the first order Dirac equation (38), and correspondingly, obeys first order boundary conditions, but, for *both* components:

$$\psi_n^{(+)}(-L/2) = \psi_n^{(+)}(L/2), \tag{43a}$$

and

$$\psi_n^{(-)}(-L/2) = \psi_n^{(-)}(L/2). \tag{43b}$$

The condition (43b) can be reduced, using (41), to a condition on $\psi_n^{(+)}$.

$$[(d_x + S')\psi_n^{(+)}(x)]_{-L/2} = [(d_x + S')\psi_n^{(+)}(x)]_{L/2}. \tag{43c}$$

We can see that although both $\eta_n(x)$ and $\psi_n^{(+)}(x)$ obey the same differential equations (37) and (40a), they must satisfy different boundary conditions. The condition (42a) is the same as (43a), but (42b) is not generally the same as (43c), especially for topological solitons for which $S' = (dS/d\phi)_{\phi_{sol}(x)}$ is not the same at $x = \pm L/2$, as $L \to \infty$. The eigenvalues of differential operators are specified both by the differential form of the operator as well as the boundary conditions. Therefore, the set of boson eigenvalues w_n^2 and fermion eigenvalues e_n^2 will in fact be different, for any finite L, however large. As $L \to \infty$, both spectra will merge into the same continuum, but with different spectral densities. The difference $\Sigma w_n - \Sigma e_n$ will not vanish in general.

Once we are aware of these pitfalls, the evaluation of the fermion and boson densities of states $\rho_F(E)$ and $\rho_B(E)$ using the correct boundary conditions (42–43), is straightforward. One finds that the fermion level density is in fact the average of the densities of the two equations (40a) and (40b), as one would expect from symmetry between the upper and lower components of the spinor ψ_n. Given the densities of states, the fluctuation energy $(\Sigma w_n - \Sigma e_n) = \int E\, dE(\rho_B(E) - \rho_F(E))$ can be calculated, added on to the counter term to obtain the quantum soliton mass (see Schonfeld 1979; Kaul and Rajaraman 1983; their results have been rederived, using more elegant methods and without recourse to boundary conditions, by Imbimbo and Mukhi 1984a).

To illustrate the results, consider again the example of the Kink of the SUSY double-well problem. For this system

$$S = \left(\frac{\lambda}{2}\right)^{1/2} (\phi^2 - \mu^2/\lambda), \tag{44a}$$

while the counter term added to S is

$$S_{\text{c.t.}} = -\hbar\left(\frac{\lambda}{2}\right)^{1/2}\left[\int_0^\infty \frac{dk}{2\pi}\frac{1}{(k^2+2\mu^2)^{1/2}} + K\right], \tag{44b}$$

where K is an arbitrary finite constant. On computing $\frac{1}{2}\Sigma w_n - \frac{1}{2}\Sigma e_n + M_{\text{c.t.}}$ one finds, for the $O(\hbar)$ quantum Kink mass,

$$M^{(1)}_{\text{kink}} = m_1^3/3\lambda + \frac{\hbar}{\sqrt{12}\pi} m_1, \tag{45}$$

where we have eliminated dependences on μ and K by using m_1 which is the *renormalised* mass, to $O(\hbar)$, of the boson in that model. In the classical limit, the second $O(\hbar)$ term in (45) is absent and m_1 can be replaced by the "tree" level boson mass $m_0 = (2)^{1/2}\mu$ in the double-well problem. Then (45) reduces to the classical kink mass $M^{(0)}_{\text{kink}}$ given in (8). However there is not much meaning in calculating the difference between $M^{(1)}_{\text{kink}}$ and $M^{(0)}_{\text{kink}}$ to see "how much" quantum correction the kink-mass acquires. In quantum field theory, thanks to ultraviolet divergences and their removal by subtraction schemes which are not unique, the relation between the same physical quantity evaluated to different orders in $\hbar$, contains arbitrariness. For instance, the one-loop renormalised boson mass m_1 can be evaluated in terms of the "bare" boson $m_0 = (2)^{1/2}\mu$ by standard perturbation techniques, but the relation between m_1 and m_0 is arbitrary upto a constant. Any attempt to write m_1 in terms m_0 and insert it into (45) so that it may be compared with the classical result (8), will be fraught with ambiguity (see Kaul and Rajaraman 1983 for more details). But (45) as it stands, is meaningful. It gives the one-loop quantum mass of the kink in terms of one-loop boson mass. One can also meaningfully compare (45) with $\frac{1}{2}|\langle \text{sol}|T|\text{sol}\rangle|$, also evaluated to one-loop level, to see if the bound (24) is saturated.

The one-loop correction to $\frac{1}{2}\langle \text{sol}|T|\text{sol}\rangle$ has been evaluated independently by Imbimbo and Mukhi (1984a) as well as by Chatterjee and Majumdar (1984). These one loop corrections come from two sources:

(i) Recall that $\frac{1}{2}T = [D(\phi(x))]^{\infty}_{-\infty}$. Classically, we evaluated this by inserting the classical function $\phi = \phi_{\text{sol}}(x)$. In quantum theory $\phi(x, t)$ is an operator, which we write in terms of the shifted field operator $\eta(x, t)$,

$$\phi(x, t) = \phi_{\text{sol}}(x) + \eta(x, t).$$

Then, at each point (x, t),

$$\langle \text{sol}|D(\phi)|\text{sol}\rangle = D(\phi_{\text{sol}}) + (\mathrm{d}D/\mathrm{d}\phi)\phi_{\text{sol}}\langle \text{sol}|\eta|\text{sol}\rangle + \tfrac{1}{2}(\mathrm{d}^2D/\mathrm{d}\phi^2)\phi_{\text{sol}}\langle \text{sol}|\eta^2|\text{sol}\rangle + \dots \tag{46}$$

Since T involves $D(\phi)$ only at $x = \pm\infty$, note that for the kink system (44)

$$D(\phi_{\text{sol}}) = \mp\frac{(2)^{1/2}}{3}\frac{\mu^3}{\lambda}; \quad \frac{\mathrm{d}D}{\mathrm{d}\phi}(\phi_{\text{sol}}) = 0; \quad \text{and}$$

$$\frac{\mathrm{d}^2D}{\mathrm{d}\phi^2}(\phi_{\text{sol}}) = \pm(2)^{1/2}, \quad \text{as } x \to \pm\infty. \tag{47}$$

Hence $\langle \text{sol}|D(\phi(x = \pm\infty, t))|\text{sol}\rangle$

$$= \pm\frac{(2)^{1/2}}{3}\frac{\mu^3}{\lambda} \pm \tfrac{1}{2}(2)^{1/2}\mu\langle \text{sol}|\eta^2(\pm\infty, t)|\text{sol}\rangle \tag{48}$$

The two-point function $\langle \text{sol}|\eta^2(x,t)|\text{sol}\rangle$ can be easily obtained, to leading order, by using the expansion in terms of the normal modes $\eta_n(x)$ in (37).

$$\eta(x,t) = \left[\sum_n \frac{a_n}{(2w_n)^{1/2}} e^{-iw_n t}.\eta_n(x) + \text{h.c.}\right]$$

with $$[a_n, a_m^+] = \hbar\delta_{n,m}. \tag{49}$$

This yields

$$\langle \text{sol}|\eta^2(\pm\infty, t)|\text{sol}\rangle = \hbar\int_0^\infty \frac{\mathrm{d}k}{2\pi}\frac{1}{(k^2+2\mu^2)^{1/2}} \tag{50}$$

The two-point function will be ultraviolet divergent, but this is cancelled by the counter term (44b) which leads to

$$D_{\text{c.t.}}(\phi_{\text{sol}}(x=\pm\infty)) = \left(\int \mathrm{d}\phi S_{\text{c.t.}}(\phi)\right)\phi_{\text{sol}}(\pm\infty)$$

$$= \mp\frac{\hbar\mu}{(2)^{1/2}}\left[\int_0^\infty \frac{\mathrm{d}k}{2\pi}\frac{1}{(k^2+2\mu^2)^{1/2}} + K\right]. \tag{51}$$

On adding (50) and (51), one gets a finite result, which when cast in terms of the renormalised boson mass m_1 yields

$$\tfrac{1}{2}|\langle \text{sol}|T|\text{sol}\rangle| = \frac{m_1^3}{3\lambda} + \frac{\hbar m_1}{(12)^{1/2}\pi} + O(\hbar^2) \tag{52}$$

in agreement with the quantum soliton mass in (45).

Thus we see that for a $(1+1)$-dimensional scalar field soliton like the kink, the Witten-Olive bound *is* saturated at the $O(\hbar)$ quantum level. In fact, a stronger result appears in the recent work by Yamagishi (1984), where the Witten-Olive inequality as well as its saturation in $O(\hbar)$ quantum theory are worked out at the level of densities—*ie* relating the expectation values of the Hamiltonian density and the density of the topological index T. That the bound holds to $O(\hbar)$ makes it plausible that it may hold to all orders in $\hbar$. An argument that it indeed does so hold, has been briefly advanced in the Imbimbo-Mukhi work.

Turning to the $(3+1)$ dimensional SUSY extension of the Georgi-Glashow model (25) an evaluation of the quantum corrections to the SUSY monopole's mass has recently been done by Kaul (1984). Although the underlying principles and pitfalls are similar to those in $(1+1)D$ models, the actual calculation is more difficult. One has to compute fluctuation – energies associated with the two species of isotriplet Fermi fields ψ_1^a and ψ_2^a, the two species of scalar fields ϕ_1^a and ϕ_2^a, the gauge field A_μ^a, and ghost fields, and that too in 3 space-dimensions. All this has been ably done by Kaul (1984). He finds that, once again, the boson and fermion fluctuation energies do not cancel, and leave behind an ultraviolet-divergent residue. This divergence, again, is cancelled by counter terms. Further, the resulting answer for the one-loop monopole mass has the same *form* as the classical mass (16b) provided one replaces the vacuum-expectation value C and the coupling constant g by their renormalised values. That is

$$M_{\text{mono}}^{(1)} = 4\pi\left(\frac{C}{g}\right)_{\text{ren}}. \tag{53}$$

Turning to the right side of the quantum bound (34), as applied to the monopole, it will become, after one-loop corrections, $4\pi(Cm)_{\rm ren}$. Assuming that as required by the exact Dirac condition, $m_{\rm ren} = 1/g_{\rm ren}$, the bound is saturated.

In the last stages of his proof of (53), Kaul uses the relation $g_{\rm ren} C_{\rm ren} = gC$, *ie* that this product gC receives no renormalisation correction in this model. The validity of this assumption as a gauge—and renormalisation-scheme-independent statement has been questioned in a very recent preprint by Imbimbo and Mukhi (1984b). This preprint reached us just when we were completing this review, and we have not had the opportunity to digest its contents. We will merely report that these authors also study the quantum mass corrections for the $N = 2$ SUSY monopole (as well as the $N = 4$ SUSY extension), by using trace theorems which are a generalisation of what they had employed in their earlier paper quoted above. Their differences with Kaul notwithstanding they too find that the quantum bound *is* saturated for the $N = 2$ SUSY monopole.

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