Pramāṇa, Vol. 23, No. 6, December 1984, pp. 809–814 © Printed in India.

A neutrino universe with viscosity

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MS received 17 May 1984; revised 13 August 1984

Abstract. A solution of the Einstein field equation corresponding to a distribution of fluid with equation of state $\rho = 3p$ but with a nonvanishing shear viscosity is presented. The solution is spherical symmetric and the flow lines are geodetic.

Keywords. Neutrino universe; cosmology; general relativity; viscosity

PACS No. 04·20 Jb

1. Introduction

With the spherically symmetric metric

$$ds^2 = \exp(\omega)\,dt^2 - \exp(\lambda)\,dr^2 - \exp(\mu)(d\theta^2 + \sin^2\theta\, d\phi^2)$$

where ω, λ, μ are functions of r and t, the condition that the flow lines, identified with the t-lines are geodetic requires ω to be a function of t alone. A scale transformation $t \to \int \exp(\omega/2)\,dt$ then reduces the metric to:

$$ds^2 = dt^2 - \exp(\lambda)\,dr^2 - \exp(\mu)(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{1}$$

With a nonvanishing shear viscosity, the energy momentum tensor is:

$$T^\mu_\nu = (\rho + p)v^\mu v_\nu - p\delta^\mu_\nu + 2\eta\sigma^\mu_\nu, \tag{2}$$

where $v^\mu = \delta^\mu_0$, η is the coefficient of viscosity which must be positive if the second law of thermodynamics is to be valid and σ^μ_ν is the shear tensor defined in the usual manner

$$\sigma_{\mu\nu} = \tfrac{1}{2}(v_{\mu;\nu} + v_{\nu;\mu}) - \frac{\theta}{3}(g_{\mu\nu} - v_\mu v_\nu) - \tfrac{1}{2}(\dot{v}_\mu v_\nu + v_\mu \dot{v}_\nu),$$

$$\theta = v^\mu_{;\mu},$$

$$\dot{v}_\mu = v_{\mu;\nu} v^\nu.$$

A simple calculation gives, for the shear components

$$\sigma^1_1 = -2\sigma^2_2 = -2\sigma^3_3 = \tfrac{1}{3}(\dot{\lambda} - \dot{\mu})$$
$$\sigma^2 = \tfrac{1}{2}\cdot\sigma^{\mu\nu}\sigma_{\mu\nu} = \tfrac{1}{12}(\dot{\lambda} - \dot{\mu})^2 \tag{3}$$

where we have numbered the coordinates t, r, θ, ϕ as 0, 1, 2, 3 respectively and the overhead dots indicate differentiation with respect to time.

2. Field equation and their integration

The non-trivial field equations are: (Tolman 1934)

$$8\pi T_0^0 = 8\pi\rho = \exp(-\mu) + \exp(-\lambda)\left[-\mu'' - \frac{3}{4}\mu'^2 + \frac{\mu'\lambda'}{2}\right] + \frac{\dot{\mu}^2}{4} + \frac{\dot{\mu}\dot{\lambda}}{2}, \quad (4)$$

$$8\pi T_1^1 = -8\pi p + 16\pi\eta\sigma_1^1 = \exp(-\mu) - \frac{\mu'^2}{4}\exp(-\lambda) + \ddot{\mu} + \frac{3}{4}\dot{\mu}^2, \quad (5)$$

$$8\pi T_2^2 = -8\pi p + 16\pi\eta\sigma_2^2 = \exp(-\lambda)\left[-\frac{\mu''}{2} - \frac{\mu'^2}{4} + \frac{\mu'\lambda'}{2}\right]$$

$$+\frac{\ddot{\mu}}{2} + \frac{\ddot{\lambda}}{2} + \frac{\dot{\mu}^2}{4} + \frac{\dot{\lambda}^2}{4} + \frac{\dot{\mu}\dot{\lambda}}{4}, \quad (6)$$

$$8\pi T_0^1 = 0 = \exp(-\lambda)\left[\dot{\mu}' + \frac{\mu'}{2}(\dot{\mu} - \dot{\lambda})\right], \quad (7)$$

where the primes denote differentiation with respect to r. Equation (7) shows that $\dot{\mu}'$ is nonvanishing if and only if the shear is nonvanishing and can be readily integrated to give

$$\mu'^2 \exp(\mu - \lambda) = g^2, \quad (8)$$

where g is an arbitrary function of r alone. Substituting (8) in (4) we get

$$8\pi\rho = (1 - g^2/4)\exp(-\mu) - \frac{gg'}{\mu'}\exp(-\mu) + \frac{3}{4}\dot{\mu}^2 + \frac{\dot{\mu}\dot{\mu}'}{\mu'}. \quad (9)$$

Also, eliminating the shear tensor components from (5) and (6) by using (8), we get

$$3(8\pi p) = -(1 - g^2/4)\exp(-\mu) + \frac{gg'}{\mu'}\exp(-\mu) - 3(\ddot{\mu} + \tfrac{3}{4}\dot{\mu}^2) - \frac{2\ddot{\mu}'}{\mu'} - \frac{3\dot{\mu}\dot{\mu}'}{\mu'}. \quad (10)$$

The equation of state $\rho = 3p$, now gives from (9) and (10):

$$2(1 - g^2/4) + (\ddot{\mu} + \dot{\mu}^2)\exp(\mu) = f(t)\exp(-\mu/2). \quad (10a)$$

As we shall see in §3, $f(t)$ must vanish if the solution is to be regular at the origin. Integrating twice, we then get

$$\exp(\mu) = (g^2/4 - 1)t^2 + ht + l, \quad (11)$$

where h and l are arbitary functions of r alone. Substituting (11) in (8), we get:

$$\exp(\lambda) = \frac{[(g^2/4 - 1)'t^2 + h't + l']^2}{g^2[(g^2/4 - 1)t^2 + ht + l]}. \quad (12)$$

3. Comparison with the isotropic case and regularity at the origin

Obviously if $(g^2/4 - 1)$, h and l are all constant multiples of a single function of r, $\dot{\mu} = \dot{\lambda}$ and the shear vanishes, one can then reduce the solution to the standard Friedmann form:

$$ds^2 = dt^2 - \frac{R^2}{[1+Kr^2/4]^2}[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)], \tag{12a}$$

(where $R = R(t)$ and $K = 0, +1, -1$) by a simple transformation of the radial variable.

In general also there is some restriction on the functions g, h and l given by the condition of regularity at the origin and the positive definiteness of the scalars ρ and η. Regularity at the origin requires that as $r \to 0$, $\exp(\lambda)$ should remain finite and $\exp(\mu)$ should be $0(r^2)$ with $\exp(\mu-\lambda) \to 1/r^2$. Plugging these conditions in (8) we get

$$\operatorname*{Lim}_{r\to 0} g^2 \to 4$$

Consistent with this, we shall take in what follows

$$g^2 = 4(1 - Zr^2), \tag{13}$$

where Z is a function of r which remains finite at $r = 0$ and is otherwise arbitrary. It is interesting to note that for the Friedmann universe

$$g^2 = 4\frac{(1-Kr^2/4)}{(1+Kr^2/4)}. \tag{14}$$

Obviously (13) becomes (14) with a suitable choice of Z. We write the expressions for $\rho(\equiv 3p)$, θ, σ^2 and η as

$$8\pi\rho = \frac{3}{4}\frac{(-2Zt+b)^2}{B^2} + \frac{Z}{B} + \frac{(Zr^2)'}{rD} + \frac{r}{2}\cdot\frac{(-2Zt+b)}{B^2}\cdot\frac{A}{D}, \tag{15}$$

$$\theta = \frac{3}{2}\frac{(-2Zt+b)}{B} + \frac{r}{2}\cdot\frac{A}{BD}; \quad \sigma^2 = \frac{r^2}{12}\cdot\frac{A^2}{B^2D^2}, \tag{16,17}$$

$$16\pi\eta = \frac{2}{r}\cdot\frac{ZD}{A} - \frac{(Zr^2)'}{r^2}\cdot\frac{B}{A} - \frac{\dot{A}}{A} - \frac{\dot{B}}{2B} + \frac{\dot{D}}{D} - \frac{r}{2}\frac{A}{BD}. \tag{18}$$

where $b = h/r^2$ and $c = l/r^2$ and

$$A = -(bZ' - b'Z)t^2 - 2(Z'c - Zc')t + (b'c - bc'), \tag{19}$$

$$B = -Zt^2 + bt + c, \tag{20}$$

$$D = -(Z + rZ'/2)t^2 + (b + rb'/2)t + (c + rc'/2). \tag{21}$$

The condition $\eta > 0$ for all r, t does not lead to any simple relation—indeed the expression for η seems too complicated to admit any simple physical interpretation of viscosity.

4. Singularities of the field

In the isotropic case one has a collapse of the volume and the density blows up at a finite time in the past or the future. In the present case, too, a collapse of the volume occurs at a finite time as is evident from the following consideration. The Raychaudhuri equation reads (Raychaudhuri 1979)

$$\theta_{,\alpha}v^\alpha + \frac{\theta^2}{3} + 2\sigma^2 = R_{\mu\nu}v^\mu v^\nu. \tag{22}$$

Using the field equation

$$R_{\mu\nu} = -8\pi(T_{\mu\nu} - \tfrac{1}{2}Tg_{\mu\nu}) \tag{22a}$$

we get

$$\theta_{,\alpha}v^{\alpha} + \frac{\theta^2}{3} = -4\pi(\rho+3p) - 2\sigma^2 \tag{23}$$

the viscosity term falling off because $\sigma_{\mu\nu}v^{\nu} = 0 = \sigma^{\nu}_{\mu}$

In our case

$$\theta = \dot{\mu} + \dot{\lambda}/2 \tag{24}$$

Thus (23) becomes

$$\frac{3\ddot{R}}{R} = -4\pi(\rho+3p) - 2\sigma^2 \tag{25}$$

where we have put

$$R^3 = \exp(\mu + \lambda/2) \tag{26}$$

Equations (25) and (26) show that with $\rho + 3p \geqslant 0$ the spatial volume as measured by $\exp(\mu + \lambda/2)$ would have no minimum and further would have a zero value at a finite time as in the isotropic models. However unlike in the isotropic case, $\exp(\mu)$ and $\exp(\lambda)$ may not simultaneously vanish—we may have the following cases according to the form of the function z, b, c.

(i) When at a point, B vanishes, D as defined by (21) does not vanish and remains finite so that $\exp(\lambda)$ blows up, and hence although the volume vanishes the radial dimension expands indefinitely. In this case ρ blows up.

(ii) D also vanishes along with B. We have a situation similar to that in an isotropic collapse. In particular this occurs at $r = 0$ for all values of Z, b, c.

(iii) D vanishes for some (r, t) but B does not. In this case also there is a collapse of the volume with infinite density but the directions orthogonal to r do not vanish.

5. Discussion of simple special solution

It is clear that the general solution we have so long discussed is rather complicated. It is interesting to consider the special solution $Z = 0$. The condition $Z = 0$ can be given a simple geometrical interpretation. One has for the three space scalar curvature R^* (see for example Ellis 1971)

$$\frac{R^*}{2} = 8\pi\rho + \sigma^2 - \theta^2/3 \tag{27}$$

Substituting the values of ρ, σ^2 and θ^2 from (9), (3) and (24) and using (8), this gives

$$\frac{R^*}{2} = r\left[Zr + \frac{2Z'r + 4Z}{\mu'}\right]\exp(\mu).$$

Thus R^* vanishes when Z vanishes. The other case where R^* vanishes is when $\mu' = -\dfrac{2Z'r+4Z}{Zr}$, this would mean $\dot{\mu}' = 0$ or that the shear would vanish and we have to use the Friedmann case. With $Z = 0$, (15)–(18) are considerably simplified

$$8\pi\rho = \frac{3}{4}\frac{b^2}{(bt+c)^2} + \frac{r}{2}\cdot\frac{b}{(bt+c)^2}\cdot\frac{(b'c-bc')}{[(b+rb'/2)t+(c+rc'/2)]} \tag{28}$$

$$\theta = \frac{3}{2}\frac{b}{(bt+c)} + \frac{r}{2}\cdot\frac{(b'c-bc')}{(bt+c)[(b+rb'/2)t+(c+rc'/2)]} \tag{29}$$

$$\sigma^2 = \frac{r^2}{12}\cdot\frac{(b'c-bc')^2}{(bt+c)^2[(b+rb'/2)t+(c+rc'/2)]^2} \tag{30}$$

$$32\pi\eta = \frac{b}{bt+c} \tag{31}$$

However all other equations except (31) remains too complicated for an understanding of the situation.

The singularities of the field are determined by the conditions (i) $bt+c = 0$ and (ii) $bt+c = \frac{r}{2}\cdot\frac{(b'c-bc')}{(b+rb'/2)}$. Of course the two coincide when $b'c = bc'$, but then the solution reduces to the Friedmann form.

When $bt+c = 0$

$$\rho \to -\frac{1}{4\pi}\cdot\frac{b^2}{(bt+c)^2} \to -\infty$$

In another case ρ tends to positive infinity and η tends to a finite value as the singular state is approached and is given by

$$32\pi\eta \to -\frac{2}{r}\frac{[1+(r/2)(\log b)']}{(c/b)'} \tag{32}$$

In this case $\exp(\lambda)$ vanishes while $\exp(\mu)$ remains finite. Thus the length in radial direction collapse with the circumferential dimension remain finite.

To have physically understandable solutions which are not very complicated we would have to choose special forms for the functions b and c. In this connection we first note that if there is to be no singularity at $r = 0$, h and l must vanish at least as fast as r^2. Secondly we can make a transformation of the radial coordinate in the form $r \to f(r)$ which would not effect the form of the line element or the comoving nature. Hence we may think of reducing b or c to unity. Thus we consider cases which appear to be specially simple.

With $b = 1$, $c = -r^2$, the expression for ρ leads to

$$8\pi\rho = \frac{1}{4(t-r^2)^2}\cdot\left(\frac{3t-2r^2}{t-2r^2}\right).$$

In view of the relation $p = \rho/3$, we interpret the solution as a neutrino universe with viscosity. The solution is obviously nonhomogeneous and is meaningful only for $t > 2r^2$ the analogue of the big bang singularity occurring at different t for different r (this is the case also for nonhomogeneous spherical symmetric dust universe (Landau and Lifshitz 1971)). At $t = 2r^2$ all the variables ρ, σ^2, θ blow up—indeed the ratio η/ρ also vanishes, which may seem rather odd. However owing to the fact that the shear vanishes much faster, the viscosity terms $\eta\sigma_{\mu\nu}$ never exceed the normal stress p, faster vanishing of $\sigma_{\mu\nu}$ leads to an isotropization as the expansion proceeds. We note further that θ as well as ρ remain positive for the entire domain of t and r satisfying the condition $t > 2r^2$.

Acknowledgement

Thanks are due to Prof. A K Raychaudhuri for guidance and to the CSIR for the award of a research grant. Thanks are due also to the referees for helpful comments.

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