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Nonlinear distribution function for a plasma obeying Vlasov-Maxwell system of equations

SAROJ K MAJUMDAR
Saha Institute of Nuclear Physics, Calcutta 700 009, India

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Abstract. The nonlinear distribution function of Allis, generalised to include the transverse electromagnetic waves in a plasma, is used to set up the coupled wave equations for the longitudinal and the transverse modes. These are solved, keeping terms up to the cubic order of nonlinearity, by using the method of multiple scales. The equations of wave modulation are derived, which are solved to discuss the nature of the modulational instability and solitary wave propagation. It is found that the solutions so obtained satisfy conditions which are very similar to the well known Lighthill criterion for stability, appropriately modified due to the coupling of the two modes. The role of the average constant current due to any flow of the resonant and trapped electrons in determining the stability, is also discussed.

Keywords. Nonlinear distribution function; coupled wave equations; dispersion functions; modulation equation; modulational instability; solitary wave.

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1. Introduction

In an earlier paper (Majumdar 1982), we have used the nonlinear distribution function of Allis (1968, 1969), to study the modulational stability of the solution of the Vlasov-Poisson's system of equations. In this paper, we extend that analysis to the case when a transverse electromagnetic field is also present, *i.e.*, to the solution of Vlasov-Maxwell's system of equations for a plasma.

It is known that in a nonlinear medium, the transverse and the longitudinal modes are always coupled. This coupling has been studied by Wang and Lojko (1963), Winkles and Eldridge (1972), Clemow (1975) and by several others. These studies are based on forms of the distribution functions which do not consider the effect of the particles trapped in the potential well as well as the resonant particles. Schamel (1972) developed a non-linear distribution function which includes the particle trapping and used it to investigate the modulational instability (Schamel 1975) and the effect of trapped particles in the formation of plasma waves and solitons, in which there is no role of the transverse field (Schamel 1979).

In this paper, we shall investigate the nonlinear coupling between the longitudinal and the transverse modes, by using a form of the nonlinear distribution function developed by us according to the prescription given by Allis (1969). The basic idea in estimating this nonlinear distribution function is that the longitudinal and the transverse modes propagate together with the same phase velocity, forming a single nonlinear mode. The effect of the wave field on the distribution function will be described in terms of a scalar and a vector potential, and we shall work in the Lorentz gauze so that space charge effect will explicitly appear in our model. We shall use this distribution function to calculate the non-linear expressions for the charge and the current densities which will be used in the Maxwell's equations to obtain the wave equations. The coupled wave equations will then be solved by the method of multiple scales to obtain the equations of wave modulations. Finally, these modulation equations will be analysed to obtain the stability conditions and the solitary wave formation in the plasma.

2. The nonlinear distribution functions

The electron distribution function is governed by the Vlasov equation,

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}=\frac{e}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f}{\partial \mathbf{v}}, \tag{1}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field of the wave in the plasma, and are described in terms of the scalar and the vector potential ϕ and $\mathbf{A}$:

$$\mathbf{E}=-\nabla\phi-\frac{\partial \mathbf{A}}{\partial t},$$

$$\mathbf{B}=\nabla\times\mathbf{A}. \tag{2}$$

The field quantities satisfy the Maxwell's equations,

$$\nabla\cdot\mathbf{E}=\frac{\rho}{\varepsilon_0},\quad \nabla\times\mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla\cdot\mathbf{B}=0,\qquad \nabla\times\mathbf{B}=\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}+\mu_0\mathbf{j}, \tag{3}$$

where ρ and $\mathbf{j}$ are the charge and the current densities. We assume that ϕ and $\mathbf{A}$ are connected by the Lorentz condition

$$\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\phi}{\partial t}=0. \tag{4}$$

We seek plane wave solutions of the set of equations (1) to (4) which propagate along the z-axis with phase velocity u, such that all the quantities are functions of the variable $(z-ut)$ only. Following the prescription of Allis (1969), we have constructed the non-linear distribution function which can be taken as the solution of the Vlasov equation (1). This distribution function is written in terms of the scalar and the vector potentials as (see appendix A):

$$f=n_0\left(\frac{\beta}{\pi}\right)^{3/2}\exp\left[-\beta(u+c_0)^2-\beta\left(\mathbf{v}_\perp-\frac{e}{m}\mathbf{A}_\perp\right)^2\right], \tag{5}$$

where $$c_0^2=w^2+v_\perp^2-\left(\mathbf{v}_\perp-\frac{e}{m}\mathbf{A}_\perp\right)^2-\frac{2e}{m}(\phi-uA_z), \tag{6}$$

$$\beta=\frac{m}{2kT},$$

and $$w=v_z-u,$$

is the z component of electron velocity in a frame moving with phase velocity u of the wave. The subscript $\perp$ denotes the component perpendicular to the z axis, and the other quantities have their usual meaning. The energy constant c_0 appearing in (5) plays a very important role in determining the nature of the nonlinear distribution function, and the trapping of particles in the wave potential. Let us consider the particle motion in the wave frame, and write

$$c_0^2 = w^2 - \psi, \tag{6a}$$

where $$\psi = \frac{2e}{m}(\phi - uA_z) + \frac{e^2 A^2_\perp}{m^2} - \frac{2e}{m}\mathbf{A}_\perp \cdot \mathbf{v}_\perp .$$

The phase orbits of the particles are shown in figure 1 for the simplified case of zero vector potential and zero transverse energy of the particles. We notice that $\frac{1}{2}mc_0^2 = W$is the total particle energy in the wave frame, and those particles, for which $-\psi_2 < c_0^2 < \psi_1$ is satisfied, will be trapped by the wave, ψ_1 and ψ_2 being the value of ψ at the top and the bottom of the wave. The phase orbits are surfaces of constant c_0. The surfaces with $c_0^2 > \psi_1$ represent open orbits, while $c_0^2 < \psi_1$ represents closed orbits, which are trapped particle orbits. The point or surface $c_0^2 = \psi_1$ thus represents a discontinuity which is really the seperatrix between the open and the closed orbits. Since by Liouville's theorem, the distribution function is a constant following a particular orbit, we see that f is oscillatory in the trapped region. The oscillation period for the inner orbits, near ψ_2 is given by

$$t_L = \frac{2\pi}{k}\left(\frac{m}{2e\phi}\right)^{1/2},$$

which steadily increases for the outer (but within the trapped region) orbits, and ultimately goes to infinity at the separatrix (see Davidson 1972). Phase-mixing then transfers the discontinuity at $w = 0$ to $c_0 = \pm(\psi_1)^{1/2}$, making the trapped distribution a function of c_0^2, rather than c_0. Thus only the symmetric part of f with respect to c_0 contributes to the trapped particle distribution.

The electron density and flow in the plasma can be obtained by calculating the

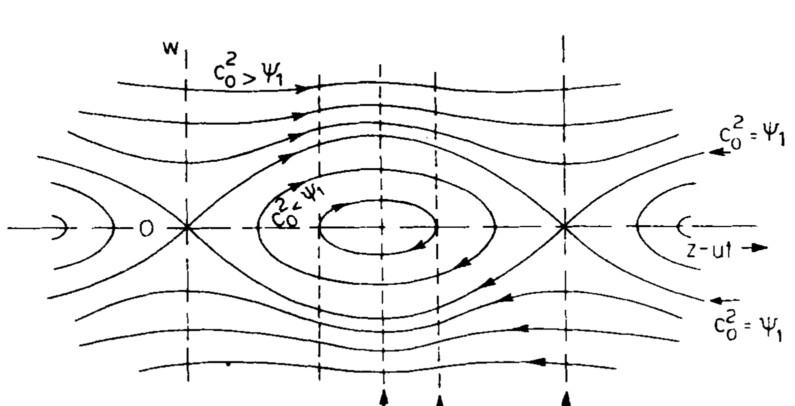

Figure 1. Phase-orbits of the electrons for $\mathbf{A}_\perp$ and $\mathbf{v}_\perp = 0$.

velocity moments of the distribution function f,

Density: $n_- = \int f \, d\mathbf{v}$,

Flow: $\mathbf{\Gamma} = \mathbf{\Gamma}_z + \mathbf{\Gamma}_\perp$

$= \int f \mathbf{v}_z d\mathbf{v} + \int f \mathbf{v}_\perp \, d\mathbf{v}$.

Using (5) for f, these moments can be evaluated by following Flynn and Allis (1971). The results obtained are given below:

$$n_- = n_0 \sum_{k=0}^{\infty} \frac{(\beta u^2)^k}{k!} (\eta + \mu^2)^k \cdot \sum_{l=0}^{\infty} \frac{(-2\beta u^2)^l}{l!} (\mu^2)^l \times \sum_{p=0}^{l} \frac{(-)^p}{2^p} {}^lC_p \, M_{k+l+p}, \tag{7}$$

$$\mathbf{\Gamma}_\perp = n_0 u \boldsymbol{\mu} \sum_{k=0}^{\infty} \frac{(\beta u^2)^k}{k!} (\eta + \mu^2)^k \cdot \sum_{l=0}^{\infty} \frac{(-2\beta u^2)^l}{l!} (\mu^2)^l \times \sum_{p=0}^{l} \frac{(-)^p}{2^p} {}^lC_p (M_{k+l+p} - M_{k+l+p+1}), \tag{8}$$

and

$$\mathbf{\Gamma}_z = \mathbf{e}_z u(n_- - n_0). \tag{9}$$

The derivations of these results are indicated in appendix B.

In (7) to (9), the quantities η and μ stand for the longitudinal and the transverse part of the potentials,

$$\eta = \frac{2e^2}{mu^2} (\phi - uA_z),$$

$$\boldsymbol{\mu} = \frac{e\mathbf{A}_\perp}{mu}, \tag{10}$$

and M_a represents the confluent hypergeometric functions,

$$M_a = {}_1F_1(a; \tfrac{1}{2}; -\beta u^2),$$

which has been referred by Allis (1969) as the plasma dispersion function of the *a*th order. In particular,

$$M_0 = 1, \text{ and } M_1 = -\tfrac{1}{2} \mathrm{Re}[Z'(\beta^{1/2}u)],$$

where $Z(x)$ is the plasma dispersion function defined by Fried and Conte (1961). We note that the function M_a has a zeros, and M_1 is < 0 for $\beta u^2 > 0{\cdot}857$.

Equations (7) to (9) are the most general expressions for the charge and current densities having nonlinearities of all order in the potentials. In fact, they are the small potential expansions of density and flow. We shall, however, retain nonlinearities up to the cubic order in the series (7) to (9), thereby obtaining,

$$n_- = n_0(1 - A_0\eta + A_1\eta^2 + A_2\mu^2 + A_3\eta^3 + A_4\eta\mu^2), \tag{11}$$

$$\mathbf{\Gamma}_\perp = n_0 u \boldsymbol{\mu}(B_0 - B_1\eta - B_2\mu^2 - B_3\eta^2), \tag{12}$$

$$\mathbf{\Gamma}_z = \mathbf{e}_z u(n_- - n_0). \tag{13}$$

In these expressions, the different coefficients are given by the following expressions:

$$
\begin{aligned}
A_0 &= -\beta u^2 M_1; \quad A_1 = (\beta u^2)^2 \frac{M_2}{2} \\
A_2 &= -\beta u^2 (M_1 - M_2); \quad A_3 = (\beta u^2)^3 \frac{M_3}{6} \\
A_4 &= -(\beta u^2)^2 (M_2 - M_3) \\
B_0 &= M_0 - M_1; \quad B_1 = -\beta u^2 (M_1 - M_2) \\
B_2 &= \beta u^2 (M_1 - 2M_2 + M_3); \ B_3 = -(\beta u^2)^2 \frac{M_2 - M_3}{2}.
\end{aligned} \tag{14}
$$

We notice that

$$
\begin{aligned}
A_4 &= 2B_3, \\
A_2 &= B_1.
\end{aligned} \tag{15}
$$

3. The coupled wave equations

It has been pointed out by Allis (1969) that the average electron density $\langle n_- \rangle$ in the plasma need not be equal to the equilibrium density n_0, when there exists a d.c. flow carried by the trapped as well as the resonant electrons. In that case, we write

$$\langle n_- \rangle = n_+ = n_0(1 + C_2), \tag{16}$$

where C_2 is a constant which represents the d.c. flow of the resonant and the trapped electrons. The justification for the validity of (16) can be seen in the following way: When we have a wave form

$$\Psi = A \sin(z - ut) - B,$$

A being the amplitude, and B, a constant, then the zero of the function Ψ does not coincide with the point of inflexion of the curve Ψ. In this case, the average value of Ψ, *i.e.* $\langle \Psi \rangle$ does not vanish, and consequently the average electron density $\langle n_- \rangle$ does not become equal to n_0. Now, referring to figure 1, we notice that the distribution function f becomes exactly Maxwellian, every time the potential goes to zero, *i.e.*, along lines, similar to that marked 0 in figure 1. If this marked line, which represents the zero of potential, is situated symmetrically between the top and the bottom of the wave, then there is no average flux of particles in the trapped region. But if it is placed asymmetrically, which essentially means that the zero of the potential is at a point different from the point of inflexion, then due to this unbalance a net flux of particle arises. This is the origin of the resonant and trapped particle current. As already explained, $\langle n_- \rangle$ becomes different from n_0 when this happens, and we can use (16) for $\langle n_- \rangle$. Because of space charge neutrality, $\langle n_- \rangle$ is always equal to n_+, and it can be shown that the constant C_2 is algebraically related to the constant B in the wave form, measuring the deviation of the zero position of the potential from its point of inflexion. Taking average of (13), and using (16), we obtain

$$\langle \Gamma_z \rangle = \Gamma_0 = (\langle n_- \rangle - n_0)u = n_0 u C_2.$$

Thus C_2 is a measure of the average d.c. flow, due to resonant and trapped particle current. The possibility of generating wave form in the plasma, which will give rise to non-zero value of Γ_0 is an entirely different question, and will not be discussed here.

Using (11) to (13) and (16) the charge and the current densities can now be written as

$$\rho = e(n_+ - n_-) = -en_0 G(\eta, \mu^2),$$

$$\mathbf{j}_z = -en_0 u\, G(\eta, \mu^2)\mathbf{e}_z,$$

$$\mathbf{j}_\perp = -en_0 u\, H(\eta, \mu^2)\boldsymbol{\mu},$$

where,

$$G(\eta, \mu^2) = -C_2 - A_0\eta + A_1\eta^2 + A_2\mu^2 + A_3\eta^2 + A_4\eta\mu^2,$$

and

$$H(\eta, \mu^2) = B_0 - B_1\eta - B_2\mu^2 - B_3\eta^2.$$

Using these expressions for ρ and $\mathbf{j}$ in the Maxwell's equations (3) and (4), the following wave equations for the potentials η and $\boldsymbol{\mu}$ are obtained:

$$\frac{\partial^2\eta}{\partial t^2} - c^2\frac{\partial^2\eta}{\partial z^2} = 2\omega_0^2\alpha^2 G(\eta, \mu^2), \tag{17}$$

$$\frac{\partial^2\boldsymbol{\mu}}{\partial t^2} - c^2\frac{\partial^2\boldsymbol{\mu}}{\partial z^2} = -\omega_0^2 H(\eta, \mu^2)\boldsymbol{\mu}, \tag{18}$$

where

$$\omega_0^2 = \frac{n_0 e^2}{\varepsilon_0 m},$$

and

$$\alpha^2 = 1 - \frac{c^2}{u^2}.$$

We notice that when $C_2 = 0$, ω_0 reduces to the plasma frequency $\omega_p = (n_+ e^2/\varepsilon_0 m)^{1/2}$. Also, the phase velocity u is always greater than the light velocity for a transverse wave. So α is always positive and less than unity.

If we put $\eta = 0$ in (17), it is not satisfied. Hence we conclude that purely transverse nonlinear waves cannot exist in a plasma, a fact already noted by Wang and Lojko (1963).

4. Equations for wave modulation

We solve (17) and (18), by the method of multiple scales, following Nayfeh (1973) and Nayfeh and Mook (1979). Neglecting all nonlinear terms in G and H, the resulting linear equations admit periodic solutions of the form,

$$\eta = \eta(\theta_1),\ \boldsymbol{\mu} = \boldsymbol{\mu}(\theta_2),$$

where the phases θ_1, θ_2 are given by,

$$\theta_{1,2} = k_{1,2}z - \omega_{1,2}t,$$

and

$$\omega_1/k_1 = \omega_2/k_2 = u.$$

The frequencies and the wavenumbers appearing in these solutions satisfy the following linear dispersion relations:

$$\omega_1^2 = 2\omega_0^2 A_0,$$

$$\omega_2^2 = \frac{\omega_0^2}{\alpha^2} B_0. \tag{19}$$

When the nonlinear terms in G and H are retained, the wave train will be modulated, making the amplitude, frequency and wavenumber slowly varying functions of space and time. If z and t are measured on scales which are typical of the wavelength and period of the wave, then we assume that the amplitude, frequency and the wavenumber vary on a scale denoted by

$$Z = \varepsilon z, \quad T = \varepsilon t,$$

where ε is a small parameter. Following then the principle of the multiple scales method, we consider η and μ to be functions of both the fast and the slow variables and the smallness parameter ε, in addition to their phases,

$$\eta = \eta(\theta_1, Z, T; \varepsilon), \; \mu = \mu(\theta_2, Z, T; \varepsilon). \tag{20}$$

The frequency and the wavenumber are now considered as functions of slow scale variables,

$$\omega_{1,2} = -\frac{\partial \Theta_{1,2}}{\partial T}, \; k_{1,2} = \partial \Theta_{1,2}/\partial Z, \tag{21}$$

where $\quad \theta_{1,2} = \varepsilon^{-1}\, \Theta_{1,2}(Z, T).$ (22)

We now make the following expansions in powers of the small parameter ε:

$$\eta = \varepsilon\eta_1(\theta_1, Z, T; \varepsilon) + \varepsilon^2\eta_2 + \varepsilon^3\eta_3 + \dots,$$

$$\mu = \varepsilon\mu_1(\theta_2, Z, T; \varepsilon) + \varepsilon^2\mu_2 + \varepsilon^3\mu_3 + \dots, \tag{23}$$

$$\frac{\partial}{\partial t} = -\omega_{1,2}\frac{\partial}{\partial \theta_{1,2}} + \varepsilon\frac{\partial}{\partial T} + \dots,$$

$$\frac{\partial}{\partial z} = k_{1,2}\frac{\partial}{\partial \theta_{1,2}} + \varepsilon\frac{\partial}{\partial z} + \dots, \tag{24}$$

and assume the constant C_2 to be a small quantity of the order of ε. Using (23) and (24) in the wave equations (17) and (18) and keeping terms up to the cubic order, we equate terms having the same order in ε. In this way, we obtain three different equations, of which the equation of order ε is the following:

$$\frac{\partial^2 \eta_1}{\partial \theta_1^2} + \eta_1 + \frac{\varepsilon^{-1} C_2}{A_0} = 0,$$

$$\frac{\partial^2 \mu_1}{\partial \theta_2^2} + \mu_1 = 0. \tag{25}$$

The solutions of (25) are

$$\eta_1 + \frac{\varepsilon^{-1} C_2}{A_0} = P(Z, T)\exp(i\theta_1) + \text{c.c.}$$

$$\mu_1 = Q(Z, T)\exp(i\theta_2) + \text{c.c.} \tag{26}$$

where P and $\mathbf{Q}$ are the complex amplitudes of the longitudinal and transverse modes, the c.c. denotes the complex conjugates.

Using solutions (26) of the ε-order equation in the two other derived equations of ε^2-order and ε^3-order (which we have not written here), we can solve the latter equations successively. In this procedure of solution, certain secular terms are generated which must be eliminated. The conditions of eliminating these secular terms yields the following equations for the amplitude, frequency and wavenumber variation with slow scales:

$$\frac{\partial}{\partial T}(\omega_1 P^2)+c^2\frac{\partial}{\partial Z}(k_1 P^2) = -4i\omega_0^2\alpha^2\frac{A_1\varepsilon^{-1}C_2}{A_0}P^2,$$

$$\frac{\partial}{\partial T}(\omega_2 Q^2)+c^2\frac{\partial}{\partial Z}(k_2 Q^2) = -i\omega_0^2\frac{B_1\varepsilon^{-1}C_2}{A_0}Q^2, \tag{27}$$

$$\frac{\partial^2 P}{\partial T^2}-c^2\frac{\partial^2 P}{\partial Z^2} = \omega_0^2\alpha^2 P(\sigma_0+\sigma_1 PP^*+\sigma_2\mathbf{Q}\cdot\mathbf{Q}^*),$$

$$\frac{\partial^2 \mathbf{Q}}{\partial T^2}-c^2\frac{\partial^2 \mathbf{Q}}{\partial Z^2} = \omega_0^2\mathbf{Q}(\delta_0+\delta_1\mathbf{Q}\cdot\mathbf{Q}^*+\delta_2 PP^*), \tag{28}$$

where

$$\sigma_0 = \left(\frac{4A_1^2}{A_0^3}+\frac{6A_3}{A_0^2}\right)\varepsilon^{-2}C_2^2,$$

$$\sigma_1 = \frac{20A_1^2}{3A_0}+6A_3,$$

$$\sigma_2 = \frac{8A_1A_2}{A_0}-\frac{8A_2B_1}{B_0}\cdot\frac{k_2^2}{k_1^2-4k_2^2}+4A_4, \tag{29}$$

and,

$$\delta_0 = \left(\frac{A_1B_1}{A_0^3}+\frac{B_3}{A_0^2}\right)\varepsilon^{-2}C_2^2,$$

$$\delta_1 = \frac{2A_2B_1}{A_0}+\frac{A_2B_1}{A_0}\frac{k_1^2}{k_1^2-4k_2^2}+3B_2$$

$$\delta_2 = \frac{2A_1B_1}{A_0}-\frac{2B_1^2}{B_0}\frac{k_2^2}{k_1^2-4k_2^2}+2B_3, \tag{30}$$

and * denotes the complex conjugate quantities. Because of the relations given in (15), we note that

$$\sigma_2 = 4\delta_2. \tag{31}$$

The modulation equations are obtained from (27) and (28) by expressing the complex amplitude P and $\mathbf{Q}$ in terms of a real amplitude and a phase,

$$P(Z,T) = \tfrac{1}{2}\eta_0(Z,T)\exp[i\beta_1(Z,T)]$$

$$\mathbf{Q}(Z,T) = \tfrac{1}{2}\boldsymbol{\mu}_0(Z,T)\exp[i\beta_2(Z,T)]. \tag{32}$$

In terms of the real amplitudes η_0 and $\boldsymbol{\mu}_0$, we can write the wave form given in (26) as,

$$\eta = -\frac{C_2}{A_0}+a\cos(\theta_1+\beta_1)+\ldots,$$

$$\boldsymbol{\mu} = \mathbf{b}\cos(\theta_2+\beta_2)+\ldots, \tag{33}$$

where,

$$a = \varepsilon\eta_0 \quad \text{and} \quad \mathbf{b} = \varepsilon\mu_0.$$

We now introduce the nonlinear frequency $\tilde{\omega}_{1,2}$ and the nonlinear wavenumber $\tilde{k}_{1,2}$, and we want to describe the wave form in (33) in terms of these quantities, instead of the linear values $\omega_{1,2}$, $k_{1,2}$ and the phases $\beta_{1,2}$. Writing

$$\theta_{1,2}+\beta_{1,2} = \varepsilon^{-1}\Phi_{1,2}(Z,T)$$

$$\tilde{\omega}_{1,2} = -\frac{\partial\Phi_{1,2}}{\partial T},\ \tilde{k}_{1,2} = \frac{\partial\Phi_{1,2}}{\partial Z}, \tag{34}$$

we immediately obtain, after using (21) and (22), the following relations:

$$\tilde{\omega}_{1,2} = \omega_{1,2}-\varepsilon\frac{\partial\beta_{1,2}}{\partial T},$$

$$\tilde{k}_{1,2} = k_{1,2}+\varepsilon\frac{\partial\beta_{1,2}}{\partial Z}. \tag{35}$$

Finally, we use (32) and (35) together with the linear dispersion relations (19) in (27) and (28), and eliminate the phase $\beta_{1,2}$. In this way, we obtain the following equations, after some algebra:

$$\varepsilon^2\left(\frac{\partial^2 a}{\partial T^2}-c^2\frac{\partial^2 a}{\partial Z^2}\right) = a\left[\tilde{\omega}_1^2-c^2\tilde{k}_1^2-\alpha^2\omega_0^2\left(2A_0+g_1-\frac{\sigma_1 a^2}{4}-\frac{\sigma_2 b^2}{4}\right)\right], \tag{36}$$

$$\varepsilon^2\left(\frac{\partial^2 \mathbf{b}}{\partial T^2}-c^2\frac{\partial^2 \mathbf{b}}{\partial Z^2}\right) = \mathbf{b}\left[\tilde{\omega}_2^2-c^2\tilde{k}_2^2-\omega_0^2\left(B_0+g_2-\frac{\delta_1 b^2}{4}-\frac{\delta_2 a^2}{4}\right)\right],$$

$$\frac{\partial}{\partial T}(\tilde{\omega}_1 a^2)+c^2\frac{\partial}{\partial Z}(\tilde{k}_1 a^2) = 0,$$

$$\frac{\partial}{\partial T}(\tilde{\omega}_2 b^2)+c^2\frac{\partial}{\partial Z}(\tilde{k}_2 b^2) = 0. \tag{37}$$

The quantities $\tilde{\omega}_{1,2}$ and $\tilde{k}_{1,2}$ should obey the consistency relations,

$$\frac{\partial\tilde{k}_{1,2}}{\partial T}+\frac{\partial\tilde{\omega}_{1,2}}{\partial Z} = 0, \tag{38}$$

which can be derived from the relations in (34). In (36), the quantities g_1, g_2 are given by,

$$g_1 = \frac{4A_1}{A_0}C_2-\left(\frac{4A_1^2}{A_0^3}+\frac{6A_3}{A_0^2}\right)C_2^2,$$

$$g_2 = \frac{B_1}{A_0}C_2-\left(\frac{A_1B_1}{A_0^3}+\frac{B_3}{A_0^2}\right)C_2^2. \tag{39}$$

Equations (36) to (38) are the final modulation equations, which we shall analyse in the remaining sections for the stability of wave form.

5. Monochromatic waves and their stability

For a uniform wave train, the frequency, amplitude and the wave number remain constant. Putting these constant values

$$a = a^{(0)},\ \mathbf{b} = \mathbf{b}^{(0)},\ \tilde{\omega}_{1,2} = \omega^{(0)}_{1,2},\ \tilde{k}_{1,2} = k^{(0)}_{1,2},$$

in the modulation equations (36) to (38), we obtain the following nonlinear dispersion relations:

$$\omega_1^{(0)2} = \omega_0^2\left(2A_0 + g_1 - \sigma_1\frac{a^{(0)2}}{4} - \sigma_2\frac{b^{(0)2}}{4}\right)$$

$$\alpha^2\omega_2^{(0)2} = \omega_0^2\left(B_0 + g_2 - \delta_1\frac{b^{(0)2}}{4} - \delta_2\frac{a^{(0)2}}{4}\right). \tag{40}$$

The structure of these dispersion relations is easily understood. The first terms with A_0 and B_0 give the linear frequencies, the second terms with g_1 and g_2 denote the nonlinear frequency shift due to the d.c. flow of resonant and trapped electrons, and the last pair of terms with σ_1, σ_2 and δ_1, δ_2 indicate the amplitude dependent nonlinear frequency shift for the longitudinal and the transverse modes. Equations (40) are coupled through the amplitude dependent terms containing the quantities $\sigma_{1,2}$, and $\delta_{1,2}$. Thus the sign of these frequency shifts can be estimated by using (14), (15), (29), (30), and using the asymptotic expansions for M_a for different values of a (Magnus and Oberhettinger 1949), for the case when $\beta u^2 \gg 1$. Since βu^2, as given by (6), is the ratio of the phase velocity to the electron thermal velocity, $\beta u^2 \gg 1$ occurs either for large u, or for small T, or both. In this way, we have estimated the sign of the quantities $g_{1,2}$, $\sigma_{1,2}$, $\delta_{1,2}$ in the following cases:

Case (i): Only quadratic nonlinearity. $\beta u^2 \gg 1$.

g_1 and g_2 have the sign of C_2,

$$\sigma_1 > 0,\ \sigma_2 > 0,\ \delta_1 > 0,\ \delta_2 > 0. \tag{41}$$

Case (ii): Quadratic and cubic non-linearity both appearing together. $\beta u^2 \gg 1$.

g_1 and g_2 have the sign of C_2,

$$\sigma_1 < 0,\ \sigma_2 > 0,\ \delta_1 < 0,\ \delta_2 > 0. \tag{42}$$

Using (41) and (42) as well as the magnitudes of the ratio of the two amplitudes $a^{(0)}$ and $b^{(0)}$, the sign and magnitude of the nonlinear frequency shifts can be estimated.

We now investigate whether this uniform wave train is stable under small perturbations. For this we set,

$$a = a^{(0)} + a^{(1)},\quad \mathbf{b} = \mathbf{b}^{(0)} + \mathbf{b}^{(1)},$$

$$\tilde{\omega}_{1,2} = \omega^{(0)}_{1,2} + \omega^{(1)}_{1,2},\quad \tilde{k}_{1,2} = k^{(0)}_{1,2} + k^{(1)}_{1,2},$$

in (36) to (38) and linearise them. The quantities with superscripts 1 denote the perturbations. The linearized equations so obtained may be solved by assuming solutions of the form

$$\omega_1^{(1)},\, k_1^{(1)},\, a^{(1)} \sim \exp[i\lambda_1(z - U_1^{(0)}T)],$$

$$\omega_2^{(1)},\, k_2^{(1)},\, b^{(1)} \sim \exp[i\lambda_2(z - U_2^{(0)}T)],$$

giving rise to the following dispersion relation connecting the modulation wavenumbers $\varepsilon\lambda_{1,2}$ with the modulation speeds $U_{1,2}^{(0)}$:

$$\left[\varepsilon^2\lambda_1^2(c^2 - U_1^{(0)2}) - 4\omega_1^{(0)2}\left(\frac{c^2}{u} - U_1^{(0)}\right)^2 - \alpha^2\omega_0^2\sigma_1\frac{a^{(0)2}}{2}(c^2 - U_1^{(0)2})\right]$$
$$\times\left[\varepsilon^2\lambda_2^2(c^2 - U_2^{(0)2})^2 - 4\omega_2^{(0)2}\left(\frac{c^2}{u} - U_2^{(0)}\right)^2 - \omega_0^2\delta_1\frac{b^{(0)2}}{2}(c^2 - U_2^{(0)2})\right]$$
$$= (c^2 - U_1^{(0)2})(c^2 - U_2^{(0)2})\alpha^2\omega_0^4\frac{\sigma_2\delta_2}{4}a^{(0)2}b^{(0)2}. \tag{43}$$

This equation can be solved approximately by first neglecting the small terms in (43) containing $\varepsilon^2\lambda_{1,2}^2$ and $a^{(0)2}$, $b^{(0)2}$. This gives

$$U_1^{(0)} = U_2^{(0)} \approx \frac{c^2}{u}.$$

We use this approximate values of $U_{1,2}^{(0)}$ again in the small terms of (43). After some rearrangement, we obtain,

$$(X - G_1)(Y - G_2) = D, \tag{44}$$

where,

$$X = \left(\frac{c^2}{u} - U_1^{(0)}\right)^2,$$

$$Y = \left(\frac{c^2}{u} - U_2^{(0)}\right)^2,$$

$$G_1 = \frac{c^2\alpha^4}{4\omega_1^{(0)2}}\left(\varepsilon^2\lambda_1^2c^2 - \frac{\omega_0^2\sigma_1 a^{(0)2}}{2}\right),$$

$$G_2 = \frac{c^2\alpha^4}{4\omega_2^{(0)2}}\left(\varepsilon^2\lambda_2^2c^2 - \frac{\omega_0^2\delta_1 b^{(0)2}}{2\alpha^2}\right),$$

$$D = \frac{c^2\alpha^6\omega_0^4}{64\omega_1^{(0)2}\omega_2^{(0)2}}\cdot\sigma_2\delta_2 a^{(0)2}b^{(0)2}.$$

Let us note here that $X > 0$, $Y > 0$ and $D > 0$.

A plot of X *vs* Y of (44) is a rectangular hyperbola. If no part of this plot falls with in the positive quadrant, then stable solutions of (44) cannot be obtained. The condition under which this can occur is given by,

$$G_1 < 0,\ G_2 < 0 \text{ and } G_1G_2 > D.$$

That is,

$$\varepsilon^2\lambda_1^2c^2 - \frac{\omega_0^2}{2}\sigma_1 a^{(0)2} < 0$$

$$\varepsilon^2\lambda_2^2c^2 - \frac{\omega_0^2}{2\alpha^2}\delta_1 b^{(0)2} < 0$$

and $$\left(\varepsilon^2\lambda_1^2c^2-\frac{\omega_0^2}{2}\sigma_1 a^{(0)2}\right)\left(\varepsilon^2\lambda_2^2c^2-\frac{\omega_0^2}{2\alpha^2}\delta_1 b^{(0)2}\right) > \frac{\omega_0^4}{4\alpha^2}\sigma_2\delta_2 a^{(0)2}b^{(0)2}. \tag{45}$$

Inequalities (45) lead at once to the following criterion for modulational instability:

(a) $\sigma_1 > 0$ and $\delta_1 > 0$.

(b) For zero modulation wavenumber (*i.e.* $\lambda_1 = \lambda_2 = 0$),

$$\sigma_1\delta_1 > \sigma_2\delta_2.$$

(c) Modulation helps to stabilize the solutions.

(d) When $\beta u^2 \gg 1$, use of (41) and (42) shows that quadratic nonlinearity alone produces instability, whereas quadratic and cubic nonlinearity together give a stable solution.

6. Steady profile solution, Solitary waves

We shall follow Whitham (1974) to find out the conditions under which solitary wave solution of the set (36) to (38) may exist, which propagates without changing their shape. To do this, we consider the modulation equations (36) to (38) in a moving frame, and express all the quantities as functions of the moving coordinates

$$\zeta_{1,2} = Z - U_{1,2}T:$$
$$a = a(Z-U_1T),\quad \mathbf{b} = \mathbf{b}(Z-U_2T),$$
$$\tilde{\omega}_{1,2} = \tilde{\omega}_{1,2}(Z-U_{1,2}T),\quad \tilde{k}_{1,2} = \tilde{k}_{1,2}(Z-U_{1,2}T), \tag{46}$$

where $U_{1,2}$ are the nonlinear group velocities of the two modes. Using (46), (37) and (38) can be integrated to give

$$(c^2\tilde{k}_1-\tilde{\omega}_1U_1)a^2 = R_1,$$
$$(c^2\tilde{k}_2-\tilde{\omega}_2U_2)b^2 = R_2, \tag{47}$$

and $$\tilde{\omega}_{1,2}-\tilde{k}_{1,2}U_{1,2} = S_{1,2}, \tag{48}$$

where $R_{1,2}$ and $S_{1,2}$ are integration constants.

Let us consider a solitary wave form for which the amplitudes $a \to 0, b \to 0$ as $\zeta_{1,2} \to \pm\infty$. This gives from (47), $R_{1,2} = 0$, so that

$$U_{1,2} = c^2\tilde{k}_{1,2}/\tilde{\omega}_{1,2}. \tag{49}$$

Since $S_{1,2}$ are constants, (48) together with (49) show that $\tilde{\omega}_{1,2}$ and $\tilde{k}_{1,2}$ remain constant for this kind of wave form. Under this condition, (36) can be integrated and combined together to yield the following result:

$$\varepsilon^2\left[(U_1^2-c^2)\left(\frac{da}{d\zeta_1}\right)^2+4\alpha^2(U_2^2-c^2)\left(\frac{d\mathbf{b}}{d\zeta_2}\right)^2\right]$$
$$= a^2[\tilde{\omega}_1^2-c^2\tilde{k}_1^2-\alpha^2\omega_0^2(2A_0+g_1)]$$

$$+4\alpha^2 b^2[\tilde{\omega}_2^2 - c^2\tilde{k}_2^2 - \omega_0^2(B_0 + g_2)]$$

$$+\frac{\alpha^2\omega_0^2}{8}(\sigma_1 a^4 + 2\sigma_2 a^2 b^2 + 4\delta_1 b^4). \tag{50}$$

Now, the linear group velocities of the two modes are both less than c, so that the nonlinear group velocities $U_{1,2}$ can be taken as less than c. Since $\alpha^2 > 0$, this means that the left side of (50) is always negative. On the right side, terms with second power of the amplitude should dominate the last term with fourth power of the amplitude as $a \to 0$, $b \to 0$ at $\zeta_{1,2} \to \pm\infty$. This gives the following condition:

$$a^2[\tilde{\omega}_1^2 - c^2\tilde{k}_1^2 - \alpha^2\omega_0^2(2A_0 + g_1)]$$
$$+4\alpha^2 b^2[\tilde{\omega}_2^2 - c^2\tilde{k}_2^2 - \omega_0^2(B_0 + g_2)] < 0. \tag{51}$$

At the maximum value of the amplitudes $a = a_m$, $b = b_m$, the derivatives on the left side of (50) vanish. Therefore, the maximum values a_m and b_m of the soliton amplitude is governed by the following inequality, found by using (51) in (50):

$$\sigma_1 a_m^4 + 2\sigma_2 a_m^2 b_m^2 + 4\delta_1 b_m^2 > 0. \tag{52}$$

When (52) is satisfied, solitary waves having maximum amplitudes a_m and b_m will propagate. Condition (52) is satisfied for any values of a_m and b_m, if $\sigma_1 > 0$, $\sigma_2 = 4\delta_2 > 0$, and $\delta_1 > 0$, which is the case when the nonlinearity is quadratic. For other combinations of σ_1, σ_2 and δ_1, relative magnitudes of a_m and b_m will determine the solitary wave formation.

If we use the nonlinear dispersion relations (40), and the relation (31), we can rewrite (52) in the following alternative form:

$$a_m^2(\omega_1^{(m)} - \omega_1)\omega_1 + 4\alpha^2 b_m^2(\omega_2^{(m)} - \omega_2)\omega_2 < \frac{\omega_0^2}{2}(a_m^2 g_1 + 4b_m^2 g_2), \tag{53}$$

where, we have written $\omega_{1,2}^{(0)} = \omega_{1,2}^{(m)}$ at $a^{(0)} = a_m$, $b^{(0)} = b_m$ and $\omega_{1,2}^m + \omega_{1,2} \approx 2\omega_{1,2}$, approximately. Since the functions $\omega_{1,2}''(k_{1,2})$ have the signs of the linear frequencies $\omega_{1,2}$, prime representing differentiation with reference to the argument, and since the quantities $(\omega_{1,2}^{(m)} - \omega_{1,2})$ denote the nonlinear frequency shift at the maximum amplitude, the two terms on the left side of (53), taken separately with $g_1 = g_2 = 0$, represent the Lighthill (1965) criterion for instability for the longitudinal and transverse modes. Hence we see that condition (53) is nothing but the modified Lighthill criterion for instability for the coupled nonlinear mode in the presence of the resonant and the trapped particle current in the plasma. For small C_2, (39) show that $g_{1,2}$ have the sign of C_2. It therefore follows that the presence of d.c. flow generated by the average motion of the resonant and trapped electrons favours modulational instability and solitary wave propagation in the plasma.

In conclusion, we may note the all-important role of the dispersion functions M_a of various orders of nonlinearity in determining the nature of the wave form and their stability. These functions depend only on the quantity βu^2, which is the ratio of the phase velocity to the electron thermal velocity.

Appendix A

Verification of (5).

We seek a plane wave solution of (1) and (2) of the form

$$f = f(\mathbf{v}, z-ut),\ \phi = \phi(z-ut),\ \mathbf{A} = \mathbf{A}(z-ut). \tag{A1}$$

Introducing the wave-frame coordinate

$$Z = z - ut,$$

we obtain from (1) and (2), the following:

$$w\frac{\partial f}{\partial Z} = -\frac{e}{m}\left[\frac{\mathrm{d}}{\mathrm{d}Z}(\phi - uA_z - \mathbf{v}_\perp \cdot \mathbf{A}_\perp)\frac{\partial f}{\partial w} + w\frac{\mathrm{d}\mathbf{A}_\perp}{\mathrm{d}Z}\cdot\frac{\partial f}{\partial \mathbf{v}_\perp}\right],$$

i.e.,

$$\frac{\partial f}{\partial Z} = -e\frac{\mathrm{d}}{\mathrm{d}Z}(\phi - uA_z)\frac{\partial f}{\partial \mathscr{E}}, \tag{A2}$$

where

$$\mathscr{E} = \tfrac{1}{2}m(v_\perp^2 + w^2) \tag{A3}$$

is the kinetic energy in the wave-frame.

Now, linearizing (A2), by writing

$$f(Z, \mathbf{v}) = n_0 f_0(\mathbf{v}) + n_1(Z) f_1(\mathbf{v}) + \dots,$$

and then integrating w.r.t. Z, we obtain

$$n_1 f_1 = -en_0(\phi - uA_z)\frac{\partial f_0}{\partial \mathscr{E}}.$$

Hence, the total distribution function is given by

$$f = n_0\left[f_0 - e(\phi - uA_z)\frac{\partial f_0}{\partial \mathscr{E}} + \dots\right]. \tag{A4}$$

The right side of (A4) is nothing but the first two terms of the Taylor expansion of the function

$$f(E) = f[\mathscr{E} - e(\phi - uA_z)]. \tag{A5}$$

Hence the nonlinear distribution function in the wave frame can be taken as

$$f = n_0\left(\frac{\beta}{\pi}\right)^{3/2}\exp\left(-\frac{E}{kT}\right), \tag{A6}$$

where

$$E = \mathscr{E} - e(\phi - uA_z) \tag{A7}$$

is the total energy in the wave-frame. Transforming the wave-frame energy E to the laboratory frame energy E', we get

$$E' = E + uP_z + \tfrac{1}{2}mu^2, \tag{A8}$$

where the relations

$$E = \frac{P_z^2}{2m} + \frac{P_\perp^2}{2m},$$

and

$$\mathbf{P}_\perp = m\mathbf{v}_\perp - e\mathbf{A}_\perp, \tag{A9}$$

have been used. Using (A8) and (A9) in (A6), we obtain (5).

Appendix B

Derivation of (7), (8) and (9).

The electron density is given by the moment of the distribution function (5):

$$n_- = \int f\,d\mathbf{v} = \int d\mathbf{v}_\perp \int_{-\infty}^{+\infty} dw f$$
$$= n_0 \left(\frac{\beta}{\pi}\right)^{3/2} \int d\mathbf{v}_\perp \exp\left[-\beta\left(\mathbf{v}_\perp - \frac{e}{m}\mathbf{A}_\perp\right)^2\right]$$
$$\times \int_{-\infty}^{+\infty} dw \frac{1}{2} \exp(-\beta c_0^2)[\exp(-\beta u^2 - 2\beta u c_0)$$
$$+ \exp(-\beta u^2 + 2\beta u c_0)]. \tag{B1}$$

Now, using the standard series expansions of Hermite and Laguerre polynomials, one can write (Magnus and Oberhettinger 1949),

$$\exp(-\beta u^2 - 2\beta u c_0) + \exp(-\beta u^2 + 2\beta u c_0)$$
$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{2^{2n} n!}{(2n)!} (-\beta u^2)^n L_n^{-1/2}(\beta c_0^2), \tag{B2}$$

where $L_n^{-1/2}$ is the Laguerrer polynomial of argument βc_0^2. The quantity βc_0^2 is given by (6). By using (10), this can be expressed as

$$\beta c_0^2 = \beta w^2 - \beta u^2(\eta + \mu^2) + 2\beta u \boldsymbol{\mu} \cdot \mathbf{v}_\perp. \tag{B3}$$

We next use the sum rule for the Laguerre polynomial

$$L_n^\alpha(x+y) = \exp(y) \sum_{k=0}^{\infty} \frac{(-y)^k}{k!} L_n^{\alpha+k}(x), \tag{B4}$$

and the relation (B3) to express $L_n^{-1/2}(\beta c_0^2)$ in terms of $L_n^\alpha(\beta w^2)$. In this way, the right side of (B2), becomes

$$2\exp(\beta c_0^2 - \beta w^2) \sum_{n,k,l} \frac{(-\beta u^2)^n}{\left(\frac{1}{2}\right)_n} \frac{(\beta u^2)^k (\eta + \mu^2)^k}{k!}$$
$$\times \frac{(-2\beta u \boldsymbol{\mu} \cdot \mathbf{v}_\perp)^l}{l!} L_n^{-1/2+k+l}(\beta w^2), \tag{B5}$$

where $\quad \left(\frac{1}{2}\right)_n = \Gamma\left(\frac{1}{2} + n\right) \Big/ \Gamma\left(\frac{1}{2}\right).$

Using (B5) for the right side of (B2), and the standard integral

$$\int_0^x dx \exp(-x) x^{\gamma-1} L_n^\alpha(x) = \frac{\Gamma(\gamma)\Gamma(1+\alpha+n-\gamma)}{n!\,\Gamma(1+\alpha-\gamma)}$$

in the right side of (B1), one can carry out the integrals in w and $\mathbf{v}_\perp$ in successive order. Then carrying out the summation over the integer n, using the result

$$\sum_{n=0}^{\infty} \frac{(k+l)_n}{\left(\frac{1}{2}\right)_n} \cdot \frac{(-\beta u^2)^n}{n!} = M_{k+l},$$

where $\quad M_{k+l} \equiv {}_1F_1(k+l; \frac{1}{2}; -\beta u^2)$

is the confluent hypergeometric function, we obtain the result given in (7).

Equation (8) can be similarly obtained with only a slight, but otherwise straightforward, modification of the $v_\perp$ -integral.

Equation (9) needs a comment which we explain here. Writing the distribution function (5) as

$$f = n_0\left(\frac{\beta}{\pi}\right)\exp[-\beta(\mathbf{v}_\perp - \frac{e}{m}\mathbf{A}_\perp)^2]F,$$

where

$$F = \left(\frac{\beta}{\pi}\right)^{1/2} \exp[-\beta(u+c_0)^2], \tag{B6}$$

we get for the z component of flow Γ_z,

$$\begin{aligned}\Gamma_z &= \int f v_z \, d\mathbf{v} \\ &= \int d\mathbf{v}_\perp \, n_0\left(\frac{\beta}{\pi}\right)\exp\left[-\beta\left(\mathbf{v}_\perp - \frac{e}{m}\mathbf{A}_\perp\right)^2\right] \int_{-\infty}^{+\infty} F v_z \, dv_z.\end{aligned} \tag{B7}$$

Next we write

$$\begin{aligned}\int_{-\infty}^{+\infty} F\, v_z \, dv_z &= \int_{-\infty}^{+\infty} F(u+w)\,dw \\ &= u\int_{-\infty}^{+\infty} F\,dw + \int_{-\infty}^{+\infty} Fw\,dw.\end{aligned} \tag{B8}$$

The first integral on the right side of (B8) can be evaluated as outlined above. When used in (B7), this gives the value $n_- u$. To evaluate the second integral of (B8), we use (6a) to change the integration variable from w to c_0. In doing so, we note that in the trapped region, F has a discontinuity at $w = 0$, which gives rise to a branch point singularity at

$$c_0 = \pm(\psi_1)^{1/2},$$

by phase mixing of the trapped particle orbits, ψ_1 being the top of the potential well. This is already explained in §2. Using this fact, we can write

$$\int_{-\infty}^{+\infty} Fw\,dw = \int_{-\infty}^{+\infty} F c_0\,dc_0 - \int_{-\psi_1^{1/2}}^{+\psi_1^{1/2}} F c_0\,dc_0.$$

The second integral on the right side above represents the phase-mixed current, which we shall neglect here, as this produces only initial damping of the wave. The first integral $\int_{-\infty}^{+\infty} Fc_0\,dc_0$ can be easily evaluated, which together with (B7) and (B8) yield the result given in (9).

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