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Bethe-Salpeter equation with the sine-Gordon interaction

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Abstract. An attempt is made to study the interaction Hamiltonian, $H_{\text{int}} = G\psi^2(x)U(\phi(x))$ in the Bethe-Salpeter framework for the confined states of the ψ particles interacting *via* the exchange of the U field, where $U(\phi) = \cos(g\phi)$. An approximate solution of the eigenvalue problem is obtained in the instantaneous approximation by projecting the Wick-rotated Bethe-Salpeter equation onto the surface of a four-dimensional sphere and employing Hecke's theorem in the weak-binding limit. We find that the spectrum of energies for the confined states, $E = 2m + B$ (B is the binding energy), is characterized by $E \sim n^6$, where n is the principal quantum number.

Keywords. Bethe-Salpeter equation; sine-Gordon interaction; instantaneous approximation; Fock's transformation variables; Hecke's theorem; weak-binding limit; solitonic kernel; confinement.

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1. Introduction

The problem of determining the dynamics which, in quantum chromodynamics (QCD), yields the picture of a hadron being a bound state of quarks has drawn considerable attention in the recent past (Quigg and Rosner 1976; Eichten *et al* 1978, 1980; Buchmuller *et al* 1980; Grosse and Martin 1980). This is, in general, an important problem in grand unified theories, where, to give one example, preons may be regarded as composites of pre-preons (Pati *et al* 1981). It is in these and similar contexts that general criteria have sought to be developed which might govern the formation of composites (Weinberg and Witten 1980).

While earlier investigations of the characteristics of bound two-body systems made use of confining potentials which increase with spatial separation within the framework of non-relativistic Schrödinger equation, it has also been recognized that the intrinsically relativistic nature of QCD compels that similar investigations be taken up in a relativistic framework. In the latter context, the use of the Bethe-Salpeter (BS) equation suggests itself naturally and a number of authors have utilised this framework for their studies (Alabiso and Schierholz 1977; Henley 1979).

The use of the BS formalism, as has been noted by Henley, presents certain peculiar problems, chief among which are the choice of a kernel and the tractability of the resulting equation. In this connection, the need to use a kernel which might be said to describe a relativistic harmonic oscillator has been stressed. Indeed, Alabiso and Schierholz (1977) and Henley (1979) succeeded in discovering soluble kernels which can be so described.

This paper is motivated by the question: in the BS framework, could a confined state of two quarks be brought about by a nonlinear "field" to which the gluons condense because of their self-interaction? Thus, we study the interaction Hamiltonian, $H_{\text{int}} = (G\psi^2(x)U(\phi(x))$ in the BS framework for the confined states of the ψ particles interacting *via* the exchange of the U field. We must now choose the interaction, $U(\phi)$.

Let us recall that, in $1+1$ dimensions at least, the sine-Gordon (SG) interaction is one of the most extensively studied interactions and that it has a rather special status: the SG equation and its 1-soliton solution can be transformed, respectively, to an equation and its soliton-like solution belonging to the general class of ϕ^{2n} theories (Malik *et al* 1983). Thus, an attempt to explore the properties of the SG interaction in $3+1$ dimensions seems worthwhile.

Guided by the above considerations, we assume, in § 2, that the interaction, $U(\phi)$, is of the SG type. It is pertinent now to note that the infinity suppression mechanism in localizable nonpolynomial theories is so strong that in many cases it makes these theories finite as opposed to making them merely renormalizable (Salam 1971). Indeed, in the present case, we find that the SG interaction does lead to a tractable BS equation without any renormalization problems. We proceed as follows. Employing a technique due to Okubo (1954), we construct the function in momentum space which describes the propagation of the SG interaction. This function, the superpropagator, is the kernel to be used in the BS equation. The BS equation is set up in the ladder approximation, in §3, and analysed in the instantaneous approximation. The latter approximation implies that, as has been noted earlier (Biswas *et al* 1982) the relativistic effects in our investigation are embodied only in the kinematic sense. The analysis of our equation is carried out by using Fock's transformation variables, which project it onto the surface of a four-dimensional sphere. We are thus able to obtain an approximate solution for the eigenvalue problem in the weak binding limit, using Hecke's theorem. We find that the SG interaction used as a kernel in the BS equation does lead to confined states, the energy spectra of which is characterized by $E \sim n^6$, where $E = 2m + B$ (B is the binding energy) and n is the principal quantum number. Section 4 gives a brief discussion of our results.

2. The superpropagator corresponding to the SG interaction

We assume that $\phi(x)$ is a massless neutral scalar field which interacts with itself through the SG interaction

$$L_{\text{int}} = U(\phi(x)) = \cos(g\phi), \tag{1}$$

where g is the minor coupling constant. The x-space propagator for the field $U(x)$ may then be written as

$$\begin{aligned} F(x) &= \langle 0|T[U(\phi(x))U(\phi(0))]|0\rangle \\ &= \sum_K \frac{g^{4K} D^{2K}(x)}{(2K)!} \\ &= \cosh[g^2 D(x)], \end{aligned} \tag{2}$$

where $D(x)$ is the propagator for the field $\phi(x)$. We have used

$$\langle 0|T[\phi^m(x)\phi^n(0)]|0\rangle = n!\, D^n(x)\,\delta_{mn},$$

where the $n!$ factor arises from the possibility of pairing factors in the bracket in $n!$ distinct ways. We note that there is no Borel ambiguity involved in obtaining (2) (see, *e.g.* Biswas *et al* 1972).

By analytic continuation we get, following Okubo (1954),

$$g^2 = -i\lambda \quad (\lambda > 0) \tag{3}$$

whence

$$F(x) = \cos[\lambda D(x)],$$
$$= \frac{1}{\pi}\int \frac{\sin(\lambda u)}{u + D(x)}\,\mathrm{d}u. \tag{4}$$

Then $F(p)$, the Fourier transform of $F(x)$, is given by

$$F(p) = \frac{1}{\pi}\int f(p)\sin(\lambda u)\,\mathrm{d}u, \tag{5}$$

where

$$f(p) = \int \frac{\exp(ipx)}{u + D(x)}\,\mathrm{d}^4x, \tag{6}$$

and, for small values of the minor coupling constant,

$$D(x) = -\frac{i}{(2\pi)^4}\int \frac{\exp(ipx)}{p^2 - i\varepsilon}\,\mathrm{d}^4p. \tag{6a}$$

Equation (6) can be solved in different ways: it can be shown that $f(p)$ satisfies the integral equation

$$f(p) = \frac{(2\pi)^4}{u}\delta^4(p) + \frac{i}{(2\pi)^4 u}\int \frac{\mathrm{d}^4p'}{(p-p')^2} f(p'),$$

whence on setting

$$f(p) = \frac{(2\pi)^4}{u}\delta^4(p) + h(p),$$

one may obtain

$$h(p) = \frac{i}{u^2p^2} + \frac{i}{(2\pi)^4 u}\int \frac{h(p')}{(p-p')^2}\,\mathrm{d}^4p'. \tag{7}$$

Then, make the ansatz

$$h(p) = \int_0^\infty \frac{q(t)}{(p^2 + 16\pi^2 ut)^3}\,\mathrm{d}t,$$

and proceed to determine $q(t)$ (see Okubo 1954). Alternatively, (7) may be wick-rotated converted into a differential equation and solved directly (Biswas *et al* 1973). We refer the reader to these papers for details, and simply quote the result here:

$$f(p) = \frac{(2\pi)^4\delta^4(p)}{u} + \frac{iu^{-5/2}}{2\pi(p^2)^{1/2}} K_1\left(\frac{(p^2)^{1/2}}{2\pi u^{1/2}}\right), \tag{8}$$

where $K_1(x)$ is the usual modified Bessel function. Thus (5) becomes

$$F(p) = (2\pi)^4\delta^4(p) + \frac{i}{2\pi^2(p^2)^{1/2}} \int du\, K_1\left(\frac{(p^2)^{1/2}}{2\pi u^{1/2}}\right) u^{-5/2} \sin(\lambda u). \tag{9}$$

Equation (9) is our expression for the superpropagator which generally represents the propagation of an arbitrary number of ϕ quanta (for techniques of calculation of superpropagators and general features of nonlinear Lagrangians, see Salam 1971).

3. The BS equation with the SG kernel in the instantaneous approximation

We consider the interaction Hamiltonian

$$H_{\text{int}} = G\psi^2(x)U(x), \tag{10}$$

where $\psi(x)$ is a neutral scalar field of mass m, G is the renormalized major coupling constant and $U(x)$ is the SG interaction

$$U(x) = \cos(g\phi), \tag{11}$$

where ϕ is a massless neutral scalar field and g is the minor coupling constant. Assuming that the two-body confined states result owing to the exchange of the $U(x)$ "field" alone (ladder approximation), the BS equation for the confined state of the ψ-particles in momentum space has the form (Biswas *et al* 1973)

$$\tau(p) = \frac{G^2}{(2\pi)^4 i} \frac{1}{(\frac{1}{2}E+p)^2+m^2} \frac{1}{(\frac{1}{2}E-p)^2+m^2} \int d^4p' F(p-p')\tau(p'), \tag{12}$$

where E is the sum of the momenta of the particles forming the bound state, p their relative momentum and $F(p)$ is given by (9).

Since the factor that corresponds to the superpropagator in momentum space is the Fourier transform of $F(x)-1$, we can drop the $\delta^4(p)$ term from (9). Thus, (12) yields:

$$\begin{aligned}
&[(\tfrac{1}{2}E+p)^2+m^2][(\tfrac{1}{2}E-p)^2+m^2]\tau(p) \\
&= \frac{G^2}{2^6\pi^7} \int_{-\infty}^{+\infty} du\, u^{-3} \sin(\lambda u) \int_0^\infty \frac{dt}{\left(t^2+\dfrac{1}{4\pi^2 u}\right)^{3/2}} \\
&\times \int \frac{d^4p'}{(p-p')^2} \cos[\{(p-p')^2\}^{1/2}t]\tau(p'),
\end{aligned} \tag{13}$$

where we have used a suitable representation for the K_1 function (Gradshteyn and Ryzhik 1965). Specialising to the frame,

$$E = (0, 0, 0, iE).$$

Equation (13) reduces, in the instantaneous approximation, to

$$\begin{aligned}
&[\mathbf{p}^2-(p_0+\tfrac{1}{2}E)^2+m^2][\mathbf{p}^2-(p_0-\tfrac{1}{2}E)^2+m^2]\tau(p) \\
&= \frac{G^2}{2^6\pi^7} \int du \int dt\, f(u,t) \int \frac{d^4p'}{(\mathbf{p}-\mathbf{p}')^2} \cos[\{(\mathbf{p}-\mathbf{p}')^2\}^{1/2}t]\tau(p'),
\end{aligned} \tag{14}$$

where

$$f(u, t) = \frac{u^{-3} \sin(\lambda \mu)}{\left(t^2 + \frac{1}{4\pi^2 u}\right)^{3/2}}. \tag{14a}$$

Note also that we may now set

$$E = 2m + B > 2m, \tag{14b}$$

where B is the binding energy.

We note that the choice of the K_1 function exercised above is crucial: the form of (14) immediately suggests that, in the weak binding approximation, one may now proceed to reduce this equation following, for example, Biswas *et al* (1973). Essentially, we proceed as follows (to make this note reasonably self-contained, the details are given in Appendix A). Performing the p_0' integration, we obtain an $O(3)$-symmetric equation. Using spherical harmonics and separating the angular variables, we then get an equation for the radical part of the BS wavefunction. This equation is projected on to the surface of a four-dimensional sphere by employing Fock's transformation variables. The equation thus obtained is solved by functions related to Gegenbauer polynomials. The application of Hecke's theorem then yields

$$1 = \frac{-G^2}{2^7 \pi^5 m^2 cn} \int du \int dt\, f(u, t) \times \int_{-1}^{+1} \frac{dx(1-x^2)^{1/2} C_{n-1}^1(x)}{(1-x)} \cos\left[\frac{mct}{\sqrt{2}} (1-x)^{1/2}\right], \tag{15}$$

where

$$c = \left(\frac{E^2}{4m^2} - 1\right)^{1/2}. \tag{16}$$

Substituting (14a) into (15) and carrying out the t-integration (Gradshteyn and Ryzhik 1963), we obtain

$$1 = \frac{-G^2}{2^6 2^{1/2} \pi^4 mn} \int_{-\infty}^{+\infty} du\, u^{-5/2} \sin(\lambda u) \times \int_{-1}^{+1} \frac{dx(1-x^2)^{1/2} C_{n-1}^1(x)(1-x)^{1/2}}{(1-x)} K_1\left(\frac{mc(1-x)^{1/2}}{2\, 2^{1/2} \pi u^{1/2}}\right). \tag{17}$$

Now, in the weak binding limit we approximate $K_1(z)$ as

$$K_1(z) \cong \frac{1}{z} + \frac{z}{2}[\ln(\tfrac{1}{2}z) + A], \tag{18}$$

where

$$A = -\tfrac{1}{2}[\psi(1) + \psi(2)], \tag{18a}$$

$\psi(x)$ being Euler's ψ function; the reason for retaining the second term in the approximation for K_1 is apparent below. Substituting (18) into (17) we get

$$1 = \frac{-G^2}{2^6 2^{1/2} \pi^4 mn} [I_1 + I_2 + I_3 + I_4], \tag{19}$$

where

$$I_1 = \frac{2\cdot 2^{1/2}\pi}{mc}\int_{-\infty}^{+\infty} du\, u^{-2}\sin(\lambda u)\int_{-1}^{+1} dx(1-x^2)^{1/2}(1-x)^{-1}C_{n-1}^{1}(x)$$
$$= 0;$$
$$I_2 = \frac{mc}{4\cdot 2^{1/2}\pi}\int_{-\infty}^{+\infty} du\, u^{-3}\sin(\lambda u)\int_{-1}^{+1} dx(1-x^2)^{1/2}C_{n-1}^{1}(x)\ln(1-x)^{1/2};$$
$$I_3 = -\frac{mc}{4\cdot 2^{1/2}\pi}\int_{-\infty}^{+\infty} du\, u^{-3}\sin(\lambda u)\ln u^{1/2}\int_{-1}^{+1} dx\,(1-x^2)^{1/2}C_{n-1}^{1}(x)$$
$$= -\frac{mc}{4\cdot 2^{1/2}\pi}\cdot\frac{\pi}{4}\int_{-\infty}^{+\infty} du\, u^{-3}\sin(\lambda u)\ln u, \quad (n=1)$$
$$= 0; \quad (n>1)$$
$$I_4 = \frac{mc}{4\cdot 2^{1/2}\pi}\left[A+\ln\left(\frac{mc}{4\cdot 2^{1/2}\pi}\right)\right]\int_{-\infty}^{+\infty} du\, u^{-3}\sin(\lambda u)$$
$$\times\int_{-1}^{+1} dx\,(1-x^2)^{1/2}C_{n-1}^{1}(x)$$
$$= \frac{mc}{4\cdot 2^{1/2}\pi}\left[\ln c + A + \ln\left(\frac{m}{4\cdot 2^{1/2}\pi}\right)\right]\left(\frac{-\lambda^2\pi^2}{4}\right) \quad (n=1)$$
$$= 0. \quad (n>1)$$

We now proceed to carry out the above integrations for arbitrary n. First, the $n = 1$ case. The x-integral in I_2 may be performed through the substitution $x = \cos\theta$:

$$\int_{-1}^{+1} dx(1-x^2)^{1/2}\ln(1-x)^{1/2} = \frac{\ln 2}{2}\int_0^{\pi} d\theta\,\sin^2\theta + \int_0^{\pi} d\theta\,\sin^2\theta\,\ln\left(\sin^2\frac{\theta}{2}\right) = -\frac{\pi}{8}(2\ln 2 - 1);$$

thus,

$$I_2^{(n=1)} = -\frac{\pi mcg^4}{2^6\cdot 2^{1/2}}(2\ln 2 - 1). \tag{20}$$

The u-integral in $I_3^{(n=1)}$ may be performed by parts, to reduce the integrand to the form $(u^{-1}\sin(\lambda u)\ln u)$, whence,

$$I_3^{(n=1)} = \frac{\pi mcg^4}{2^5\cdot 2^{1/2}}(C + \ln\lambda - \tfrac{3}{2}), \tag{21}$$

where C is Euler's constant. $I_4^{(n=1)}$ is trivial to calculate:

$$I_4^{(n=1)} = \frac{\pi mcg^4}{2^4\cdot 2^{1/2}}\left[\ln c + A + \ln\left(\frac{m}{4\cdot 2^{1/2}\pi}\right)\right]. \tag{22}$$

Substituting (20), (21) and (22) into (19) and recalling that $I_1 = 0$ for all n, we have, for $n = 1$,

$$\frac{1}{c} = \frac{\gamma\pi}{2}\left[\frac{1}{4}(2\ln 2 - 1) - \frac{1}{2}\left(C + \ln\lambda - \frac{3}{2}\right) - \left\{\ln c + A + \ln\left(\frac{m}{4{\cdot}2^{1/2}\pi}\right)\right\}\right], \quad (23)$$

where

$$\gamma = G^2 g^4 / 2^{10}\pi^4. \quad (24)$$

For $n > 1$, I_1, I_3 and I_4 in (19) do not contribute; we are left with

$$1 = \frac{-G^2 c}{2^9\pi^5 n}\left(-\lambda^2\frac{\pi}{2}\right)\int_{-1}^{+1}(\mathrm{d}x(1-x^2)^{1/2}C^1_{n-1}(x)\ln(1-x)^{1/2}. \quad (25)$$

If we use the following representation for $C^\lambda_n(x)$,

$$C^\lambda_n(x) = \frac{(-1)^n}{2^n}\frac{\Gamma(2\lambda+n)\Gamma\left(\frac{2\lambda+1}{2}\right)(1-x^2)^{1/2-\lambda}}{\Gamma(2\lambda)\Gamma\left(\frac{2\lambda+1}{2}+n\right)n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}[(1-x^2)^{\lambda+n-1/2}], \quad (26)$$

the integral in (25) can be evaluated for arbitrary n (Appendix B). Thus, we have the results:

Binding energy,

$$B \simeq mc^2 = m\left[\frac{2(n-1)n(n+1)}{\gamma\pi}\right]^2 \quad (n \geqslant 2). \quad (27)$$

The state $n = 1$ may be estimated by assuming $\lambda = m = 1$ in (23); further let $\pi\gamma = 1$. It is then easy to check that $c = 0{\cdot}5$ satisfies the reduced (23), and we have

$$B \simeq 0{\cdot}25, \quad (27a)$$

which conforms to the pattern of the rest of the spectrum.

4. Discussion

In this note we have attempted to obtain approximate solutions to the problem of two-scalar particles confined *via* the exchange of the SG field in the BS formalism. To obtain analytic solutions, we had to make the following simplifying assumptions: velocity of propagation of the interaction is infinite (instantaneous approximation), binding energy is small, and $p/m \ll$ binding energy (p is the relative momentum of the particles forming the confined state). Intuitively, it seems that the "instantaneous" exchange of interaction scarcely allows the participating particles to move with respect to each other.

We find that the spectrum of energies for the confined states is characterized by $E \sim n^6$, where n is the principal quantum number. This result may be naively understood as follows. Consider first the bound state of two particles being brought about by the exchange of a single particle. The kernel used in the BS equation in this case may be pictured as a sort of spring that keeps the constituents bound. Extending this analogy, it seems that the SG kernel, representing as it does the exchange of an arbitrary number of quanta, may be pictured as a sort of "composite spring" of great strength. We conclude this note by drawing attention to Henley's (1979) remarks on the relativistic harmonic oscillator in this connection.

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Appendix A

Reduction of (14) in the weak binding approximation by employing Fock's transformation variables and Hecke's theorem.

Since the right side of (14) is a function of $\mathbf{p}$ alone, we can introduce a function $S(\mathbf{p})$:

$$S(\mathbf{p}) = [\mathbf{p}^2 - (p_0 + \tfrac{1}{2}E)^2 + m^2][\mathbf{p}^2 - (p_0 - \tfrac{1}{2}E)^2 + m^2]\tau(p). \tag{A1}$$

Substituting (A1), into (14), performing the p_0'-integration and defining

$$\chi(\mathbf{p}) = \frac{S(\mathbf{p})}{(\mathbf{p}^2 + m^2)^{1/2}\left(\mathbf{p}^2 + m^2 - \dfrac{E^2}{4}\right)}, \tag{A2}$$

we obtain

$$(\mathbf{p}^2 + m^2)^{1/2}\left(\mathbf{p}^2 + m^2 - \frac{E^2}{4}\right)\chi(\mathbf{p})$$

$$= + \frac{G^2}{2^7\pi^6}\int \mathrm{d}u \int \mathrm{d}t\, f(u, t) \int \frac{\mathrm{d}^3p' \cos[\{(\mathbf{p} - \mathbf{p}')^2\}^{1/2} t]}{(\mathbf{p} - \mathbf{p}')^2}\chi(\mathbf{p}'). \tag{A3}$$

The $O(3)$-symmetry of the above equation enables one to write

$$\chi(\mathbf{p}) = q_l(p)\, Y_l^{\mu}(\theta, \phi) \quad (|\mathbf{p}| = p), \tag{A4}$$

whence, upon using the properties of spherical harmonics, one can separate out the angular variables to obtain

$$z(\rho)(1 + c^2\rho^2)^{1/2} q_l(\rho)$$

$$= + \frac{G^2}{m^2 c 2^7 \pi^6}\int \mathrm{d}u \int \mathrm{d}t\, f(u, t)$$

$$\times \int \frac{\mathrm{d}\rho'\, \rho'^2 \sin\theta'\, \mathrm{d}\theta'\, \mathrm{d}\phi'\, q_l(\rho') P_l(\cos\theta') \cos[mct(\rho^2 + \rho'^2 - 2\rho\rho'\cos\theta')^{1/2}]}{[\rho^2 + \rho'^2 - 2\rho\rho'\cos\theta']}, \tag{A5}$$

where

$$\rho = \frac{p}{mc},$$

$$z(\rho) = (\rho^2 - 1), \quad \text{and}$$

$$c = \left(\frac{E^2}{4m^2} - 1\right)^{1/2}. \tag{A6}$$

Note that, for small binding, (14) now implies

$$B \simeq mc^2. \tag{A7}$$

Equation (A5) may be solved in the weak-binding limit by following the procedure given by Fock: by means of the transformations

$$\rho = \tan(\tfrac{1}{2}\alpha), \quad \rho' = \tan(\tfrac{1}{2}\alpha'), \tag{A8}$$

it is projected onto the surface of a four-dimensional sphere, leading to,

$$\begin{aligned} &z(\alpha)\sec^4(\tfrac{1}{2}\alpha)\left(\frac{a + b\cos\alpha}{1 + \cos\alpha}\right) q_l(\tan\tfrac{1}{2}\alpha) \\ &= + \frac{G^2}{2^9\pi^6 m^2 c}\int \mathrm{d}u \int \mathrm{d}t\, f(u, t) \\ &\times \int \frac{\mathrm{d}\Omega_4' \sec^4(\tfrac{1}{2}\alpha') q_l(\tan\tfrac{1}{2}\alpha') P_l(\cos\theta')}{[1 - \cos\Theta]} \\ &\quad \times \cos\left[\frac{mct}{\sqrt{2}}(1 - \cos\Theta)^{1/2}(1 + \tan^2(\tfrac{1}{2}\alpha) + \tan^2(\tfrac{1}{2}\alpha')\right. \\ &\quad \left. + \tan^2(\tfrac{1}{2}\alpha)\tan^2(\tfrac{1}{2}\alpha'))^{1/2}\right] \end{aligned} \tag{A9}$$

where Θ is the angle between two unit four-dimensional vectors of polar angles $(\alpha, 0, 0)$ and $(\alpha', \theta', \phi')$:

$$\cos\Theta = \cos\alpha\cos\alpha' + \sin\alpha\sin\alpha'\cos\theta', \quad \text{and}$$

$$z(\alpha) = \left(1 - \frac{2}{\sec^2(\tfrac{1}{2}\alpha)}\right);$$

$\mathrm{d}\Omega_4'$ is the four-dimensional solid angle:

$$\mathrm{d}\Omega_4' = \sin^2\alpha' \sin\theta' \,\mathrm{d}\alpha' \,\mathrm{d}\theta' \,\mathrm{d}\phi',$$

$$a = (1 + \tfrac{1}{2}c^2), \quad \text{and}$$

$$b = (1 - \tfrac{1}{2}c^2).$$

We now adopt the small-binding approximation. Thus, $a = b \simeq 1$;

$$z(\alpha) = \left(1 - \frac{2}{\sec^2(\tfrac{1}{2}\alpha)}\right) = 1 - \frac{2}{1 + \left(\dfrac{p}{mc}\right)^2} \simeq -1, \tag{A9}$$

provided $p/m \ll c.$ (A10)

The approximation in (A10) is not inconsistent with the instantaneous approximation within which we are working: the infinite velocity of propagation of the interaction scarcely allows the participating particles to move w.r.t. each other. Thus, the function

$$\sigma(\alpha, \alpha') = \tan^2(\tfrac{1}{2}\alpha) + \tan^2(\tfrac{1}{2}\alpha') + \tan^2(\tfrac{1}{2}\alpha)\tan^2(\tfrac{1}{2}\alpha'), \tag{A11}$$

may also be neglected.

If we now introduce

$$H(\alpha) = \sec^4(\tfrac{1}{2}\alpha)\, q_l(\tan\tfrac{1}{2}\alpha), \tag{A12}$$

equation (A9) reduces to

$$H(\alpha) = \frac{-G^2}{2^9\pi^6 m^2 c}\int du \int dt\, f(u,t) \times \int \frac{d\Omega'_4 H(\alpha')p_l(\cos\theta')}{[1-\cos\Theta]}\cos\left[\frac{mct}{\sqrt{2}}(1-\cos\Theta)^{1/2}\right]. \tag{A13}$$

The solution of (A13) can be taken as

$$H(\alpha) = P^{(2)}_{n-1,l}(\cos\alpha), \quad n = 1, 2, 3, \ldots \tag{A14}$$

where the functions $P^{(2)}_{n-1,1}(\cos\alpha)$ are related to the Gegenbauer polynomials in a simple manner:

$$P^{(2)}_{n,l}(\cos\alpha) = \frac{1}{n+1}\sin^l\alpha\, C^{l+1}_{n-l}(\cos\alpha). \tag{A15}$$

Substituting (A14) into (A13), and applying Hecke's theorem (Erdelyi 1953), we obtain

$$1 = \frac{-G^2}{2^7\pi^5 m^2 cn}\int du \int dt\, f(u,t) \times \int_{-1}^{+1}\frac{dx\,(1-x^2)^{1/2}C^1_{n-1}(x)}{(1-x)}\cos\left[\frac{mct}{\sqrt{2}}(1-x)^{1/2}\right], \tag{A16}$$

which is equation (15) of the text.

Appendix B

Evaluation of the integral in (25) for $n \geq 2$.

$$I = \int_{-1}^{+1} dx\,(1-x^2)^{1/2}C^1_{n-1}(x)\ln(1-x)^{1/2}. \tag{B1}$$

Substituting the representation for $C_{n-1}(x)$ given in (26), we have

$$I = \frac{(-1)^{n-1}}{2^n}\frac{\Gamma(n+1)\Gamma(\frac{3}{2})}{\Gamma(n+\frac{1}{2})(n-1)!}\int_{-1}^{+1} dx\ln(1-x)\frac{d^{n-1}}{dx^{n-1}}(1-x^2)^{n-1/2}. \tag{B2}$$

Since (Gradshteyn and Ryzhik 1965)

$$\frac{d^{n-1}}{dx^{n-1}}(1-x^2)^{1/2} = (-1)^{n-1}\frac{(2n-1)!!}{n}\sin(n\cos^{-1}x). \tag{B3}$$

Equation (B2) may be written as

$$I = \frac{\Gamma(\frac{3}{2})(2n-1)!!}{2^n\,\Gamma(n+\frac{1}{2})}\int_{-1}^{+1} dx\,\ln(1-x)\sin(n\cos^{-1}x). \tag{B4}$$

The substitution $\cos^{-1}x = 2\theta$ yields

$$I = \frac{\Gamma(\frac{3}{2})(2n-1)!!}{2^{n-1}\Gamma(n+\frac{1}{2})}\int_0^{\pi/2} d\theta\,\sin(2\theta)\sin(2n\,\theta)\ln(2\sin^2\theta), \tag{B5}$$

which is trivially reduced to

$$I = \frac{\Gamma(\frac{3}{2})(2n-1)!!}{2^{n-1}\Gamma(n+\frac{1}{2})}\int_0^{\pi/2} d\theta\,[\cos\{2(n-1)\theta\}-\cos\{2(n+1)\theta\}] \times \ln(\sin\theta). \tag{B6}$$

The integral in (B6) can now be performed, to yield

$$I = -\frac{\pi}{2(n-1)(n+1)}, \tag{B7}$$

where we have used

$$2^n\Gamma(n+\tfrac{1}{2}) = \pi^{1/2}(2n-1)!!. \tag{B8}$$

Substitution of (B1), (B7) and (25) into (A7) then yields the general result given in (27).

References

Alabiso C and Schierholz G 1977 *Nucl. Phys.* **B126** 461

Basu D and Biswas S N 1969 *J. Math. Phys.* **10** 2104

Biswas S N, Malik G P and Sudarshan E C G 1972 *Phys. Rev.* **D7** 2884

Biswas S N, Chaudhuri R N and Malik G P 1973 *Phys. Rev.* **D8** 1808

Biswas S N, Datta K and Goyal A 1982 *Phys. Rev.* **D25** 2199

Buchmuller W, Grunberg G and Tye S H H 1980 *Phys. Rev. Lett.* **45** 103

Coleman S 1975 *Lectures given at the Erice Summer School*

Eichten E, Gottfried K, Kinoshita T, Lane K D and Yan T M 1978 *Phys. Rev.* **D17** 3090

Eichten E, Gottfried K, Kinoshita T, Lane K D and Yan T M 1980 *Phys. Rev.* **D21** 203

Erdelyi A (ed) 1953 *Higher transcendental functions* (Bateman Manuscript Project) (McGraw Hill) vol. II

Gradshteyn I S and Ryzhik I M 1965 *Table of integrals, series and products* (New York: Academic)

Grosse R and Martin A 1980 *Phys. Rep.* **60** 341

Henley J R 1979 *Phys. Rev.* **D20** 2532

Malik G P, Subba Rao J and Johri G 1983 *Pramana* **20** 429

Okubo S 1954 *Prog. Theor. Phys.* (*Kyoto*) **11** 80

Pati J C, Salam A and Strathdee J 1981 *Nucl. Phys.* **B185** 416

Quigg C and Rosener J L 1976 *Phys. Rep.* **56** 167

Salam A (Reviewer) 1971 *Nonpolynomial Lagrangians renormalization and gravity* (New York: Gordon and Breach)

Weinberg S and Witten E 1980 *Phys. Lett.* **B96** 59