Pramāṇa, Vol. 23, No. 6, December 1984, pp. 695–701. © Printed in India.

On higher order computations in strong coupling

CARL M BENDER, ASHOK DAS* and HWA-AUN LIM*

Department of Physics, Washington University, St. Louis, MO 63130, USA

* Department of Physics and Astronomy, University of Rochester, Rochester, NY 14627, USA

MS received 16 April 1984; revised 10 August 1984

Abstract. An algorithm which simplifies manipulation of higher order graphs in strong coupling is presented.

Keywords. Lattice field theories; strong coupling.

PACS No. 11·15 Ha, 11·30 Pb

Lattice field theories have received much attention in recent years (Wilson 1974, 1975; Creutz 1984). One of the advantages, among many others, of such theories is that one can perform strong coupling calculations (Banks *et al* 1976; Balian *et al* 1974, 1975; Bender *et al* 1979). Many theories have been studied in the strong coupling regime and recently we have carried out the strong coupling calculation in a particular theory up to the fourteenth order (Bender *et al* 1984).

The rules for computing graphs were given earlier. We follow here the notations of Bender *et al* (1981). As one goes to higher orders of calculation, more and more topologically nontrivial graphs begin to contribute. One must, therefore, develop a method of handling such graphs in an efficient way. We present, in this paper, an algorithm which we believe to be efficient. We discuss below our notation and work out a few nontrivial graphs.

For simplicity and no loss of generality we work with a supersymmetric Wess-Zumino model in two space-time dimensions (Bender *et al* 1983). Let us consider a hypercubical lattice of spacing a in N dimensions. Since our theory involves both fermions and bosons, we define the inverse propagators using a symmetric difference for the fermion line (Bender *et al* 1984). Thus

$$---\circ\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\circ--- = S = \frac{1}{2a^{1+N}} \sum_{k=1}^{N} \{\gamma_k(1_k) - \gamma_k(-1_k)\},$$

where 1_k is a N component row matrix with only the kth element nonzero and equal to unity. Thus in two space-time dimensions the fermion line would be

$$S = \frac{1}{2a^3}[\gamma_1(1,0) + \gamma_2(0,1) - \gamma_1(-1,0) - \gamma_2(0,1)].$$

We, then, define the boson line from the observation that two fermion lines in series is equivalent to a boson line. That is

$$\int \mathrm{d}^N z \, \not{\partial}_x \delta^N(x-z) \, \not{\partial}_z \delta^N(z-y) = \partial_x^2 \delta^N(x-y)\mathbf{1}.$$

This leads to the definition of the boson line as

$$= G = \frac{1}{4a^{2+N}} \sum_{k=1}^{N} \{(2_k) + (-2_k) - 2(0_k)\}.$$

Thus for example, in one dimension the boson line would be

$$G = \frac{1}{4a^3}[(2) + (-2) - 2(0)].$$

Similarly, in two space-time dimensions the boson line would be

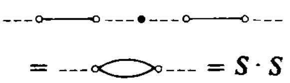
$$G = \frac{1}{4a^4}[(2, 0) + (0, 2) + (-2, 0) + (0, -2) - 4(0, 0)].$$

Any strong coupling graph consists of vertices, boson lines and fermion lines. However, to calculate them one further needs the concepts of (i) dot product (ii) cross product and (iii) outer product. These were earlier explained by Bender *et al* (1981) and we simply present those in our notation.

(i) *Dot product*: Given two elements on the lattice denoted by the row matrices $(i_1, i_2 \ldots i_N)$ and $(j_1, j_2 \ldots j_N)$, their dot product is defined as

$$(i_1, i_2 \ldots i_N) \cdot (j_1, j_2, \ldots j_N) = \delta_{i_1 j_1} \delta_{i_2 j_2} \ldots \delta_{i_N j_N} (i_1, i_2 \ldots i_N).$$

Thus for example, the dot product of two fermion propagators is

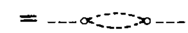


$$= S \cdot S = \frac{1}{4a^{2+2N}} \sum_{k=1}^{N} \{(1_k) + (-1_k)\} \mathbf{1}.$$

Here we have used the fact that the square of any Dirac matrix is unity. Explicitly in two dimensions

$$S \cdot S = \frac{1}{4a^6}[(1, 0) + (0, 1) + (-1, 0) + (0, 1)] \mathbf{1}.$$

Similarly one can calculate the dot product of two boson lines



$$= G \cdot G = \frac{1}{16a^{4+2N}} \sum_{k=1}^{N} \{(2_k) + (-2_k) + 4N(0_k)\}.$$

In one dimension the explicit formula is

$$G \cdot G = \frac{1}{16a^6}[(2) + (-2) + 4(0)].$$

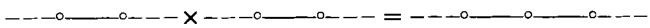
Similarly, in two dimensions it is

$$G\cdot G=\frac{1}{16a^8}[(2,0)+(0,2)+(-2,0)+(0,-2)+16(0,0).$$

(ii) *Cross product*: For two lattice elements defined by $(i_1, i_2, \ldots i_N)$ and $(j_1, j_2, \ldots j_N)$ the cross product is defined as

$$(i_1, i_2, \ldots i_N)\times(j_1, j_2 \ldots j_N)=a^N(i_1+j_1, i_2+j_2 \ldots i_N+j_N)$$

Let us work out the cross product of two fermion lines



$$S\times S=\frac{1}{2a^{1+N}}\left[\sum_{k=1}^{N}\{\gamma_k(1_k)-\gamma_k(-1_k)\}\right]$$

$$\times\frac{1}{2a^{1+N}}\left[\sum_{k'=1}^{N}\{\gamma_{k'}(1_{k'})-\gamma_{k'}(-1_{k'})\}\right],$$

$$=\frac{a^N}{4a^{2+2N}}\sum_{k=1}^{N}[(2_k)+(-2_k)-2(0_k)]\mathbf{1},$$

$$=\frac{1}{4a^{2+N}}\sum_{k=1}^{N}[(2_k)+(-2_k)-2(0_k)]\mathbf{1}.$$

We note that this is nothing but a boson line except for the unit matrix **1**. Thus in two space-time dimensions

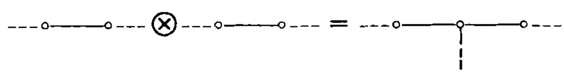
$$S\times S=\frac{1}{4a^4}[(2,0)+(0,2)+(-2,0)+(0,-2)-4(0,0)]\,\mathbf{1}.$$

It is worth pointing out here that both dot product and cross product are by definition commutative. We next define the outer product which is not commutative.
(iii) *Outer product*: For the two lattice elements $(i_1, i_2 \ldots i_N)$ and $(j_1, j_2, \ldots j_N)$ the outer product is defined as

$$(i_1,i_2 \ldots i_N)\otimes(j_1,j_2 \ldots j_N)=(i_1+j_1, i_2+j_2 \ldots i_N+j_N)_{(i_1,i_2\ldots i_N)}$$

Thus we immediately see that it is a matrix of matrices. Furthermore, the outer product or the convolution clearly involves three points—the end points are represented in the main bracket and the middle point in the lower bracket. In other words, the matrix elements of the bigger matrix represent the middle point whereas the submatrices at each element of the bigger matrix correspond to the end points. Note here that the outer product is not commutative.

As an example, let us write down the outer product of two fermion lines in two space-time dimensions.



$$S\otimes S=\frac{1}{2a^3}[\gamma_1(1,0)+\gamma_2(0,1)-\gamma_1(-1,0)-\gamma_2(0,1)]$$

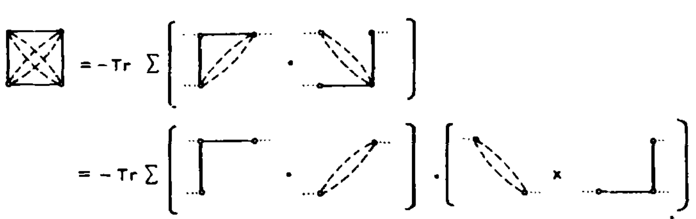
$$\otimes\frac{1}{2a^3}[\gamma_1(1,0)+\gamma_2(0,1)-\gamma_1(-1,0)-\gamma_2(0,-1)],$$

$$=\frac{1}{4a^6}\{[\gamma_1^2(2,0)+\gamma_1\gamma_2(1,1)-\gamma_1^2(0,0)-\gamma_1\gamma_2(1,-1)]_{(1,0)}$$
$$+[-\gamma_1\gamma_2(1,1)+\gamma_2^2(0,2)+\gamma_1\gamma_2(-1,1)-\gamma_2^2(0,0)]_{(0,1)}$$
$$+[\gamma_1^2(-2,0)+\gamma_1\gamma_2(-1,-1)-\gamma_1^2(0,0)-\gamma_1\gamma_2(-1,1)]_{(-1,0)}$$
$$+[\gamma_2^2(0,-2)+\gamma_1\gamma_2(1,-1)-\gamma_2^2(0,0)-\gamma_1\gamma_2(-1,-1)]_{(0,-1)}\}.$$

Written out explicitly this has the form

$$S\otimes S=\frac{1}{4a^6}\begin{bmatrix} [\,0\,] & \begin{bmatrix} 0 & 0 & \gamma_2^2 & 0 & 0 \\ 0 & \gamma_1\gamma_2 & 0 & -\gamma_1\gamma_2 & 0 \\ 0 & 0 & -\gamma_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & [\,0\,] \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -\gamma_1\gamma_2 & 0 & 0 & 0 \\ \gamma_1^2 & 0 & -\gamma_1^2 & 0 & 0 \\ 0 & \gamma_1\gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & [\,0\,] & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_1\gamma_2 & 0 \\ 0 & 0 & -\gamma_1^2 & 0 & \gamma_1^2 \\ 0 & 0 & 0 & -\gamma_1\gamma_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ [\,0\,] & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\gamma_2^2 & 0 & 0 \\ 0 & -\gamma_1\gamma_2 & 0 & \gamma_1\gamma_2 & 0 \\ 0 & 0 & \gamma_2^2 & 0 & 0 \end{bmatrix} & [\,0\,] \end{bmatrix}$$

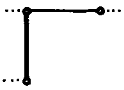
As an application let us now calculate the value of a nontrivial graph that occurs in the 12th order of the strong coupling expansion. Graphically, one can breakdown the nontrivial graph to simpler component graphs in the following way.



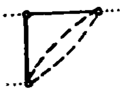
In the above, the summation stands for sum over all matrix elements. In the first bracket the dot product is for the end points whereas in the second bracket the cross product is at the middle point. The factor of a^4 arises because each integration leads to a factor of a^2. Although there are three end point integrations, translation invariance reduces the number of integrations to two.

To evaluate the above vacuum graph, therefore, let us tabulate the individual elements.

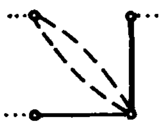
$$= G\cdot G = \frac{1}{16a^8}[(2,0)+(0,2)+(-2,0)+(0,-2)+16(0,0)],$$



$$\begin{aligned}= S\otimes S = \frac{1}{4a^6}\{&[\gamma_1^2(2,0)+\gamma_1\gamma_2(1,1)-\gamma_1^2(0,0)-\gamma_1\gamma_2(1,-1)]_{(1,0)}\\ &+[-\gamma_1\gamma_2(1,1)+\gamma_2^2(0,2)+\gamma_1\gamma_2(-1,1)-\gamma_2^2(0,0)]_{(0,1)}\\ &+[\gamma_1^2(-2,0)+\gamma_1\gamma_2(-1,-1)\\ &-\gamma_1^2(0,0)-\gamma_1\gamma_2(-1,1)]_{(-1,0)}\\ &+[\gamma_2^2(0,-2)+\gamma_1\gamma_2(1,-1)\\ &-\gamma_2^2(0,0)-\gamma_1\gamma_2(-1,-1)]_{(0,-1)}\}.\end{aligned}$$



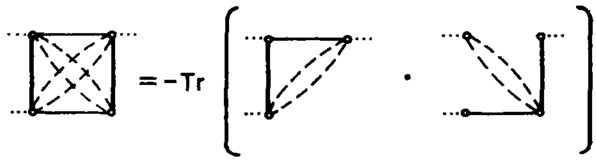
$$\begin{aligned}&= [(S\otimes S)\cdot(G\cdot G]_{\text{end points}}\\ &= \frac{1}{64a^{14}}1\{[(2,0)-16(0,0)]_{(1,0)}\\ &\quad+[(0,2)-16(0,0)]_{(0,1)}\\ &\quad+[(-2,0)-16(0,0)]_{(-1,0)}\\ &\quad+[(0,-2)-16(0,0)]_{(0,-1)}\}\end{aligned}$$



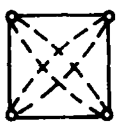
$$\begin{aligned}&= [(G\cdot G)\times(S\otimes S)]_{\text{middle point}}\\ &= \frac{1}{64a^{12}}\{[\gamma_1^2(2,0)+\gamma_1\gamma_2(1,1)-\gamma_1^2(0,0)-\gamma_1\gamma_2(1,-1)]_{(3,0)}\\ &\quad+[\gamma_2^2(0,2)+\gamma_1\gamma_2(-1,1)-\gamma_2^2(0,0)-\gamma_1\gamma_2(1,1)]_{(0,3)}\\ &\quad+[\gamma_1^2(-2,0)+\gamma_1\gamma_2(-1,-1)-\gamma_1^2(0,0)-\gamma_1\gamma_2(-1,1)]_{(-3,0)}\\ &\quad+[\gamma_2^2(0,-2)+\gamma_1\gamma_2(1,-1)-\gamma_2^2(0,0)-\gamma_1\gamma_2(-1,-1)]_{(0,-3)}\\ &\quad+[\gamma_2^2(0,2)+\gamma_1\gamma_2(-1,1)-\gamma_2^2(0,0)-\gamma_1\gamma_2(1,1)]_{(2,1)}\\ &\quad+[\gamma_2^2(0,-2)+\gamma_1\gamma_2(1,-1)-\gamma_2^2(0,0)-\gamma_1\gamma_2(-1,-1)]_{(2,-1)}\\ &\quad+[\gamma_2^2(0,2)+\gamma_1\gamma_2(-1,1)-\gamma_2^2(0,0)-\gamma_1\gamma_2(1,1)]_{(-2,1)}\\ &\quad+[\gamma_2^2(0,-2)+\gamma_1\gamma_2(1,-1)-\gamma_2^2(0,0)-\gamma_1\gamma_2(-1,-1)]_{(-2,-1)}\\ &\quad+[\gamma_1^2(2,0)+\gamma_1\gamma_2(1,1)-\gamma_1^2(0,0)-\gamma_1\gamma_2(1,-1)]_{(1,2)}\\ &\quad+[\gamma_1^2(-2,0)+\gamma_1\gamma_2(-1,-1)-\gamma_1^2(0,0)-\gamma_1\gamma_2(-1,1)]_{(-1,2)}\\ &\quad+[\gamma_1^2(2,0)+\gamma_1\gamma_2(1,1)-\gamma_1^2(0,0)-\gamma_1\gamma_2(1,-1)]_{(1,-2)}\\ &\quad+[\gamma_1^2(-2,0)+\gamma_1\gamma_2(-1,-1)-\gamma_1^2(0,0)-\gamma_1\gamma_2(-1,1)]_{(-1,-2)}\end{aligned}$$

$$+[16\gamma_1^2(2,0)+16\gamma_1\gamma_2(1,1)-17\gamma_1^2(0,0)-16\gamma_1\gamma_2(1,-1)$$
$$-\gamma_1\gamma_2(-1,1)+\gamma_1^2(-2,0)+\gamma_1\gamma_2(-1,-1)]_{(1,0)}$$
$$+[16\gamma_2^2(0,2)+16\gamma_1\gamma_2(-1,1)-17\gamma_2^2(0,0)-16\gamma_1\gamma_2(1,1)$$
$$-\gamma_1\gamma_2(-1,-1)+\gamma_2^2(0,-2)+\gamma_1\gamma_2(1,-1)]_{(0,1)}$$
$$+[16\gamma_1^2(-2,0)+16\gamma_1\gamma_2(-1,-1)-17\gamma_1^2(0,0)-16\gamma_1\gamma_2(-1,1)$$
$$-\gamma_1\gamma_2(1,-1)+\gamma_1^2(2,0)+\gamma_1\gamma_2(1,1)]_{(-1,0)}$$
$$+[16\gamma_2^2(0,-2)+16\gamma_1\gamma_2(1,-1)-17\gamma_2^2(0,0)-16\gamma_1\gamma_2(-1,-1)$$
$$-\gamma_1\gamma_2(1,1)+\gamma_2^2(0,2)+\gamma_1\gamma_2(-1,1)]_{(0,-1)}\}$$

Thus

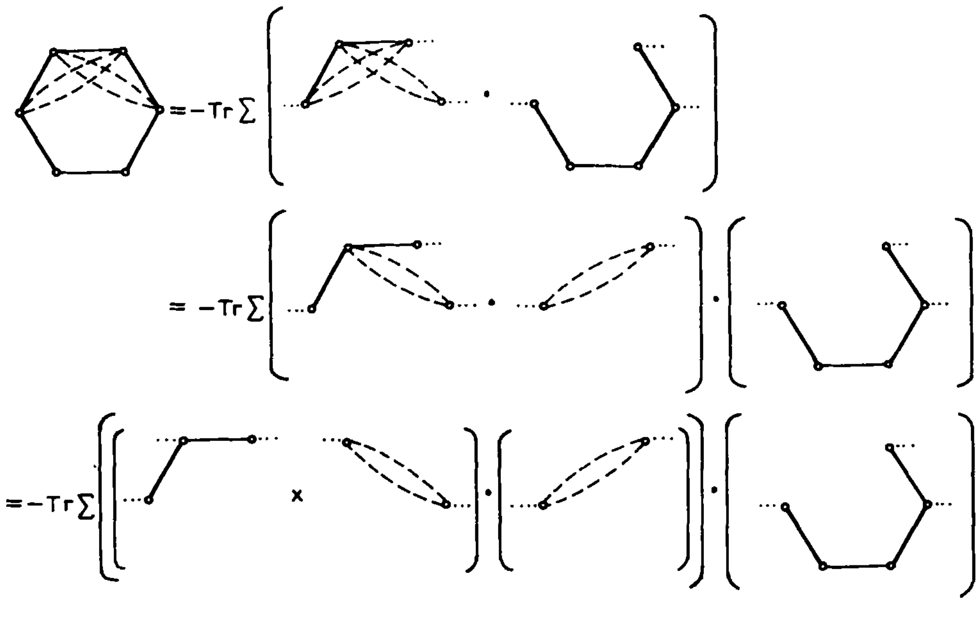


$$= -\frac{1}{64^2a^{26}}\mathrm{Tr}\,\mathbf{1}\,\{[16(2,0)+272(0,0)]_{(1,0)}$$
$$+[16(0,2)+272(0,0)]_{(0,1)}$$
$$+[16(-2,0)+272(0,0)]_{(-1,0)}$$
$$+[16(0,-2)+272(0,0)]_{(0,-1)}\}$$

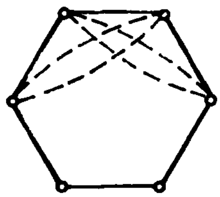


$$= -a^4\frac{\mathrm{Tr}\,\mathbf{1}}{64^2a^{26}}\cdot 4(16+272) = -\frac{9}{16a^{22}}$$

As a final example, we indicate how the following nontrivial graph of order 14 is evaluated.



Once decomposed this way, it is a matter of simple algebra to evaluate the value of the graph which turns out to be



$$= \frac{1441}{2048\, a^{24}}$$

Acknowledgements

This work was supported by the US Department of Energy, AD and HL are supported by a Junior Investigator Award. The authors thank Judith Mack for typing the manuscript.

References

Balian R, Drouffe J M and Itzykson C 1974 *Phys. Rev.* **D10** 3376
Balian R, Drouffe J M and Itzykson C 1975 *Phys. Rev.* **D11** 2104
Balian R, Drouffe J M and Itzykson C 1979 *Phys. Rev.* **D19** 2514
Banks T, Susskind L and Kogut J 1976 *Phys. Rev.* **D13** 1043
Bender C M, Cooper F, Guralnik G S and Sharp D H 1979 *Phys. Rev.* **D19** 1865
Bender C M, Cooper F, Guralnik G S, Roskies R and Sharp D H 1981 *Phys. Rev.* **D23** 2976
Bender C M, Das A and Cooper F 1983 *Phys. Rev. Lett.* **50** 397
Bender C M, Das A, Lim H and Simmons L Jr 1984 *Phys. Lett.* **B134** 225
Creutz M 1984 *Quarks and gluons on a lattice* (Cambridge: University Press)
Wilson K G 1974 *Phys. Rev.* **D10** 2445
Wilson K G 1975 *New Phenomena in subnuclear physics* (ed.) A Zichichi (New York: Plenum)