Pramāṇa, Vol. 23, No. 5, November 1984, pp. 541–546. © Printed in India.

Superposition of special non-abelian potentials*

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MS received 13 September 1983; revised 30 May 1984

Abstract. The superposition of the non-abelian potentials of the form $A'^{\mu} = A\alpha^{\mu} + a\beta^{\mu}$ and $B'^{\mu} = B\gamma^{\mu} + b\eta^{\mu}$ are considered and the necessary as well as the sufficient conditions are obtained. The significance of the conditions is discussed and the constrained isotopic spins of the perturbation potentials $(a\beta^{\mu}, b\eta^{\mu})$ are shown to be necessary for the superposition of these potentials.

Keywords. Superposition; non-abelian; Yang-Mills; perturbation; isotopic spin.

PACS No. 11·15q

1. Introduction

In the classical domain of sizes and attainable field strengths there is plenty of evidence for the validity of linear superposition. However, in the atomic and subatomic domain there are small quantum mechanical nonlinear effects which modify the interaction between charged particles (Jackson 1975). For Yang-Mills (YM) fields the majority of classical solutions are either sourceless or generated by single source (quark) potentials. The superposition for the sourceless YM equations has been discussed by Melia and Lo (1978) who showed that despite nonlinearity, the sourceless YM gauge particle may have some of the properties of a conventional vector particle like the photon. On the other hand for YM theory with sources, superposition has been noticed (Mandula 1976) for the Coulomb-type potentials with parallel sources in isospin space. Recently Gorski (1982) extended it to consider the superposition of abelian YM potentials.

In this paper we study the superposition of non-abelian potentials and determine the conditions, under which the system of non-abelian sources generates a field similar to the superposition of non-abelian potentials of separate sources. The necessary and sufficient conditions for the superposition of non-abelian potentials obtained are more general than those for the abelian case (Gorski 1982), and the constraints on the perturbation potentials are obtained explicitly. In §2 the YM field equations associated with two potentials are considered and the non-abelian character of the potentials is discussed. In §3 the superposition of these potentials is explained and the necessary as well as sufficient conditions obtained. The significance of the conditions obtained is also discussed.

2. YM field equations and non-abelian potentials

The field strength tensor $f_a^{\mu\nu}$ in terms of a gauge potential $\mathbf{A}_a^{\mu}$ may be defined as

$$f^a_{\mu\nu} = \partial_\mu \mathbf{A}^a_\nu - \partial_\nu \mathbf{A}^a_\mu + \mathbf{A}^a_\mu \times \mathbf{A}^a_\nu, \qquad (1)$$

* Work under the project HCS/DST/1081/81

where the gauge coupling parameter is taken as unity and the cross-product is in the gauge group space such that $(\mathbf{D}\times\mathbf{E})^a = f^{abc}\mathbf{D}^b\mathbf{E}^c$, where f^{abc} are the structure constants of the group. In the present case the gauge group being SU(2), $f^{abc}\to\varepsilon^{abc}$, where ε^{abc} is the purely antisymmetric tensor. Thus the YM field equations in the presence of an external source density $\mathbf{j}^a_\nu$ may be written as

$$\partial^\mu f^a_{\mu\nu} + \mathbf{A}^{\mu a}\times f^a_{\mu\nu} = \mathbf{j}^a_\nu. \tag{2}$$

Similarily for another source density k^a_ν, we may write

$$\partial^\mu h^a_{\mu\nu} + \mathbf{B}^{\mu a}\times h^a_{\mu\nu} = \mathbf{k}^a_\nu, \tag{3}$$

where

$$h^a_{\mu\nu} = \partial_\mu \mathbf{B}^a_\nu - \partial_\nu \mathbf{B}^a_\mu + \mathbf{B}^a_\mu\times\mathbf{B}^a_\nu. \tag{4}$$

Therefore, we begin with the two YM field equations (2) and (3) which are expressed in terms of the gauge potentials $\mathbf{A}^a_\mu$ and $\mathbf{B}^a_\mu$ respectively (henceforth the superscript a will be suppressed). The gauge potentials $\mathbf{A}_\mu$ and $\mathbf{B}_\mu$ are essentially abelian in the sense that there exists a gauge for each potential (Brandt and Neri 1978) in which the cross-products

$$\mathbf{A}_\mu\times\mathbf{A}_\nu = \mathbf{B}_\mu\times\mathbf{B}_\nu = 0. \tag{5}$$

However, a small perturbation in the abelian potentials causes instability in the fields (Weiss 1979) and the fields fluctuate. For instance, consider $\mathbf{a}_\mu(\mathbf{x},t) = \mathbf{a}_\mu(\mathbf{x})\exp(-i\omega t)$ as the perturbation in the abelian potential $\mathbf{A}_\mu$, where ω the frequency of the eigenmodes consists of real and imaginary parts as $\omega = \omega_R + \omega_I$. When ω is real for all modes then $\mathbf{a}_\mu$ remains small for all times (Chang and Weiss 1979) and the field remains unchanged to the order $\mathbf{a}_\mu(x,t)$. However, if one or more modes have complex ω, then the perturbation grows exponentially with time and changes the character of solution $\mathbf{A}_\mu$ drastically such that the potential

$$\mathbf{A}'_\mu = \mathbf{A}_\mu + \mathbf{a}_\mu, \tag{6}$$

will have non-abelian character. Similarly, the abelian potential $\mathbf{B}_\mu$ will be made non-abelian by introducing a perturbation $\mathbf{b}_\mu(\mathbf{x},t)$.

$$\mathbf{B}'_\mu = \mathbf{B}_\mu + \mathbf{b}_\mu. \tag{7}$$

Thus in terms of the non-abelian gauge potentials $\mathbf{A}'_\mu$ and $\mathbf{B}'_\mu$ (2) and (3) may be written as

$$\partial^\mu F_{\mu\nu} + \mathbf{A}'^\mu\times F_{\mu\nu} = \mathbf{j}'_\nu, \tag{8}$$

and

$$\partial^\mu H_{\mu\nu} + \mathbf{B}'^\mu\times H_{\mu\nu} = \mathbf{k}'_\nu, \tag{9}$$

where

$$F_{\mu\nu} = \partial_\mu\mathbf{A}'_\nu - \partial_\nu\mathbf{A}'_\mu + \mathbf{A}'_\mu\times\mathbf{A}'_\nu, \tag{10}$$

and

$$H_{\mu\nu} = \partial_\mu\mathbf{B}'_\nu - \partial_\nu\mathbf{B}'_\mu + \mathbf{B}'_\mu\times\mathbf{B}'_\nu, \tag{11}$$

and $\mathbf{j}'_\nu$ and $\mathbf{k}'_\nu$ are now the non-abelian sources.

3. Theory

3.1 *Superposition*

We now consider the superposition of non-abelian potentials $\mathbf{A}'^\mu$ and $\mathbf{B}'^\mu$ in terms of which (8) and (9) are defined and determine under what conditions the system of non-abelian sources generates the field which is similar to that of the superposition of separate sources. For this the sum of the two non-abelian potentials $\mathbf{A}'^{\mu}$ and $\mathbf{B}'^{\mu}$ should satisfy (8) or (9) with sources $\mathbf{j}'_{\nu}+\mathbf{k}'_{\nu}$ and then obtain

$$\partial^{\mu}(\partial_{\mu}\mathbf{C}_{\nu}-\partial_{\nu}\mathbf{C}_{\mu}+\mathbf{C}_{\mu}\times\mathbf{C}_{\nu})+\mathbf{C}^{\mu}\times(\partial_{\mu}\mathbf{C}_{\nu}-\partial_{\nu}\mathbf{C}_{\mu}+\mathbf{C}_{\mu}\times\mathbf{C}_{\nu})=\mathbf{j}'_{\nu}+\mathbf{k}'_{\nu}, \tag{12}$$

where
$$\mathbf{C}^{\mu}=\mathbf{A}'^{\mu}+\mathbf{B}'^{\mu}. \tag{13}$$

Since (12) is nonlinear and nonlinearity destroys the superposition principle, the nonlinear terms in (12) should vanish. Then $\mathbf{C}^{\mu}$ defined by (13) is the non-abelian potential obtained due to superposition.

3.2 *Linearisation*

In order to know the specific form of $\mathbf{C}^{\mu}$, to describe a non-abelian potential obtained due to superposition, $\mathbf{C}^{\mu}$ should satisfy a linear equation. This linear equation is obtained from (12), provided the nonlinear terms in it vanish. Thus

$$\partial^{\mu}\partial_{\mu}\mathbf{C}_{\nu}-\partial^{\mu}\partial_{\nu}\mathbf{C}_{\mu}=\mathbf{j}'_{\nu}+\mathbf{k}'_{\nu}, \tag{14}$$

is the linear equation which the non-abelian potential $\mathbf{C}^{\mu}$ should satisfy and

$$\partial^{\mu}(\mathbf{C}_{\mu}\times\mathbf{C}_{\nu})+\mathbf{C}^{\mu}\times(\partial_{\mu}\mathbf{C}_{\nu}-\partial_{\nu}\mathbf{C}_{\mu}+\mathbf{C}_{\mu}\times\mathbf{C}_{\nu})=0, \tag{15}$$

i.e., the vanishing of nonlinear terms, is the condition which must hold so that (14) is satisfied and the superposition is made possible. Therefore, the linearisation condition given by (15) is the sufficient condition which allows the superposition of two non-abelian potentials $\mathbf{A}'^{\mu}$ and $\mathbf{B}'^{\mu}$. Using (13) the linearisation condition (15) may be separated into:

$$2\mathbf{A}'_{\mu}\times\partial^{\mu}\mathbf{A}'_{\nu}+\partial^{\mu}\mathbf{A}'_{\mu}\times\mathbf{A}'_{\nu}-\mathbf{A}'^{\mu}\times\partial_{\nu}\mathbf{A}'_{\mu}+\mathbf{A}'^{\mu}\times(\mathbf{A}'_{\mu}\times\mathbf{A}'_{\nu})=0, \tag{16}$$

$$2\mathbf{B}'_{\mu}\times\partial^{\mu}\mathbf{B}'_{\nu}+\partial^{\mu}\mathbf{B}'_{\mu}\times\mathbf{B}'_{\nu}-\mathbf{B}'^{\mu}\times\partial_{\nu}\mathbf{B}'_{\mu}+\mathbf{B}'^{\mu}\times(\mathbf{B}'_{\mu}\times\mathbf{B}'_{\nu})=0, \tag{17}$$

and

$$\begin{aligned}&2\mathbf{A}'^{\mu}\times\partial_{\mu}\mathbf{B}'_{\nu}+\partial^{\mu}\mathbf{A}'_{\mu}\times\mathbf{B}'_{\nu}-\mathbf{A}'^{\mu}\times\partial_{\nu}\mathbf{B}'_{\mu}+\mathbf{A}'^{\mu}\times(\mathbf{A}'_{\mu}\times\mathbf{B}'_{\nu})\\&\quad+2\mathbf{B}'^{\mu}\times\partial_{\mu}\mathbf{A}'_{\nu}+\partial^{\mu}\mathbf{B}'_{\mu}\times\mathbf{A}'_{\nu}-\mathbf{B}'^{\mu}\times\partial_{\nu}\mathbf{A}'_{\mu}+\mathbf{B}'^{\mu}\times(\mathbf{B}'_{\mu}\times\mathbf{A}'_{\nu})\\&\quad+\mathbf{A}'^{\mu}\times(\mathbf{B}'_{\mu}\times\mathbf{B}'_{\nu})+\mathbf{A}'^{\mu}\times(\mathbf{B}'_{\mu}\times\mathbf{A}'_{\nu})+\mathbf{B}'^{\mu}\times(\mathbf{A}'_{\mu}\times\mathbf{A}'_{\nu})\\&\quad+\mathbf{B}'^{\mu}\times(\mathbf{A}'_{\mu}\times\mathbf{B}'_{\nu})=0.\end{aligned} \tag{18}$$

Equations (16) and (17) are the linearity conditions for the individual potentials $\mathbf{A}'_{\mu}$ and $\mathbf{B}'_{\mu}$ which satisfy (8) and (9) respectively. In addition, we observe that a more interesting condition for $\mathbf{A}'_{\mu}$ and $\mathbf{B}'_{\mu}$ is depicted by (18), the analysis of which fixes the constraint on the perturbation potentials. It may be noted that (16), (17) and (18) follow from the nonlinear part *i.e.*, (15), of (12).

3.3 *Analysis*

In order to analyse (18) we review equation (8) and consider that

$$\mathbf{A}^{\mu}=\mathbf{A}\alpha^{\mu},\quad \mathbf{a}^{\mu}=\mathbf{a}\beta^{\mu}, \tag{19}$$

$$\mathbf{B}^{\mu}=\mathbf{B}\gamma^{\mu},\quad \mathbf{b}^{\mu}=\mathbf{b}\eta^{\mu}, \tag{20}$$

where **A** and **B** are constant vectors in isospin space and **a** and **b** the corresponding constant perturbations, α^μ, β^μ, γ^μ and η^μ are the four vectors in Minkowaski space. Thus on substituting (8), (19) and (20) the condition (18) may be rewritten as

$$\begin{aligned}\{ (\mathbf{A}\times\mathbf{B})&[2\alpha^\mu\partial_\mu\gamma_\nu - 2\gamma^\mu\partial_\mu\alpha_\nu + \gamma_\nu\partial^\mu\alpha_\mu - \alpha_\nu\partial^\mu\gamma_\mu - \alpha^\mu\partial_\nu\gamma_\mu + \gamma^\mu\partial_\nu\alpha_\mu] \\ &+ (\mathbf{a}\times\mathbf{b})[2\beta^\mu\partial_\mu\eta_\nu - 2\eta^\mu\partial_\mu\beta_\nu + \eta_\nu\partial^\mu\beta_\mu - \beta_\nu\partial^\mu\eta_\mu - \beta^\mu\partial_\nu\eta_\mu + \eta^\mu\partial_\nu\beta_\mu] \\ &+ (\mathbf{A}\times\mathbf{b})[2\alpha^\mu\partial_\mu\eta_\nu - 2\eta^\mu\partial_\mu\alpha_\nu + \eta_\nu\partial^\mu\alpha_\mu - \alpha_\nu\partial^\mu\eta_\mu - \alpha^\mu\partial_\nu\eta_\mu + \eta^\mu\partial_\nu\alpha_\mu] \\ &+ (\mathbf{B}\times\mathbf{a})[2\gamma^\mu\partial_\mu\beta_\nu - 2\beta^\mu\partial_\mu\gamma_\nu + \beta_\nu\partial^\mu\gamma_\mu - \gamma_\nu\partial^\mu\beta_\mu - \gamma^\mu\partial_\nu\beta_\mu + \beta^\mu\partial_\nu\gamma_\mu] \\ &+ R_T\} = 0 \qquad (21)\end{aligned}$$

where other terms denoted by R_T are

$$\begin{aligned}R_T = {} & [\mathbf{A}\times(\mathbf{A}\times\mathbf{B})(\alpha^\mu\alpha_\mu\gamma_\nu - \alpha^\mu\gamma_\mu\alpha_\nu)] + \{\mathbf{A}\leftrightarrow\mathbf{B}(\alpha\leftrightarrow\gamma)\} \\ & + [\mathbf{A}\times(\mathbf{A}\times\mathbf{b})(\alpha^\mu\alpha_\mu\eta_\nu - \alpha^\mu\eta_\mu\alpha_\nu)] + \{\mathbf{A}\leftrightarrow\mathbf{b}(\alpha\leftrightarrow\eta)\} \\ & + [\mathbf{A}\times(\mathbf{B}\times\mathbf{b})(\alpha^\mu\gamma_\mu\eta_\nu - \alpha^\mu\eta_\mu\gamma_\nu)] + \{\mathbf{A}\rightarrow\mathbf{a}(\alpha\rightarrow\beta)\} \\ & + [\mathbf{A}\times(\mathbf{a}\times\mathbf{B})(\alpha^\mu\beta_\mu\gamma_\nu - \alpha^\mu\gamma_\mu\beta_\nu)] + \{(\mathbf{A}\leftrightarrow\mathbf{B},\ \mathbf{a}\rightarrow\mathbf{b})(\alpha\leftrightarrow\gamma, \beta\rightarrow\eta)\} \\ & + [\mathbf{A}\times(\mathbf{a}\times\mathbf{b})(\alpha^\mu\beta_\mu\eta_\nu - \alpha^\mu\eta_\mu\beta_\nu)] + \{\mathbf{A}\rightarrow\mathbf{B}(\alpha\rightarrow\gamma) + \mathbf{A}\rightarrow\mathbf{a}(\alpha\rightarrow\beta) \\ & + \mathbf{A}\rightarrow\mathbf{b}(\alpha\rightarrow\eta)\} + [\mathbf{B}\times(\mathbf{A}\times\mathbf{a})(\gamma^\mu\alpha_\mu\beta_\nu - \gamma^\mu\beta_\mu\alpha_\nu)] + \{\mathbf{B}\rightarrow\mathbf{b} \\ & \times(\gamma\rightarrow\eta)\} + [\mathbf{a}\times(\mathbf{A}\times\mathbf{B})(\beta^\mu\alpha_\mu\gamma_\nu - \beta^\mu\gamma_\mu\alpha_\nu)] + [\mathbf{a}\times(\mathbf{A}\times\mathbf{b}) \\ & \times(\beta^\mu\alpha_\mu\eta_\nu - \beta^\mu\eta_\mu\alpha_\nu)] + [\mathbf{a}\times(\mathbf{a}\times\mathbf{B})(\beta^\mu\beta_\mu\gamma_\nu - \beta^\mu\gamma_\mu\beta_\nu)] \\ & + \{(\mathbf{a}\rightarrow\mathbf{b})(\beta\rightarrow\eta)\}. \qquad (22)\end{aligned}$$

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In (22) the terms in { } may be obtained by interchange ($\leftrightarrow$) and change ($\rightarrow$) of letters indicated in the preceeding [ ] brackets. The condition (18), which now has the form (21) will be necessary as well as sufficient if each of the terms vanishes identically. The first term vanishes if

$$\mathbf{A}\times\mathbf{B} = 0, \qquad (23a)$$

which means the abelian potentials should have parallel isotopic spin directions. Mandula (1976) has considered such type of abelian potentials, or,

$$\mathbf{A}\times\mathbf{B} \neq 0, \qquad (23b)$$

but the coefficient to $(\mathbf{A}\times\mathbf{B})$ vanishes. This leads to the condition that α^μ must be parallel to γ^μ in the Minkowaski space, *i.e.*

$$\alpha^\mu = \xi(x)\gamma^\mu, \qquad (24)$$

where $\xi(x)$ is some scalar function. Similarly, the vanishing of the other three terms in (21) leads to

$$\mathbf{a} \| \mathbf{b}, \tag{25a}$$

i.e., the perturbation potentials have parallel isospins or

$$\beta_\mu \| \eta_\mu, \quad i.e., \quad \beta_\mu = \chi(x)\eta_\mu, \tag{25b}$$

where $\chi(x)$ is again a scalar function.

$$\mathbf{A} \| \mathbf{b}, \tag{26a}$$

$$\alpha^\mu = \xi'(x)\eta^\mu. \tag{26b}$$

$$\mathbf{B} \| \mathbf{a}, \tag{27a}$$

or

$$\gamma^\mu = \chi'(x)\beta^\mu. \tag{27b}$$

It is observed from (22) that R_T will vanish when the coefficient or the vector triple products in the isotopic spin space vanishes. The former leads to

$$\alpha^\mu \alpha_\mu = \alpha_\mu \beta^\mu = 0 \tag{28}$$

while the latter (*i.e.*, the vanishing of vector triple products in isospin space) shows that the isotopic spin of abelian potential and the perturbation potential must be parallel which combined with (25a)–(27a) shows that, one of the conditions for superposition of non-abelian potentials is that all isotopic spins should have parallel directions. However, (25b)–(27b) lead to the following non-abelian potentials,

$$\mathbf{A}'^\mu = (\mathbf{A} + \xi(x)\mathbf{a})\alpha^\mu, \tag{29}$$

and

$$\mathbf{B}'^\mu = (\chi(x)\mathbf{B} + \xi'(x)\chi(x)\mathbf{b})\alpha^\mu, \tag{30}$$

which can be superposed. In forming (29) and (30) no restrictions to the directions of isotopic spin are imposed and as such the system has non-abelian properties (Ardoz 1978). Using (13), (29) and (30) the form of the non-abelian potential obtained as a result of superposition is

$$\mathbf{C}^\mu = (\mathbf{A} + \chi(x)\mathbf{B})\alpha^\mu + (\xi(x)\mathbf{a} + \xi'(x)\chi(x)\mathbf{b})\alpha^\mu, \tag{31}$$

from which we observe that in the abelian limit, *i.e.* in the limit of vanishing perturbations, the form $\mathbf{C}^\mu$ reduces to

$$\mathbf{C}'^\mu = (\mathbf{A} + \chi(x)\mathbf{B})\alpha^\mu, \tag{32}$$

which agrees with the abelian form obtained by Gorski (1982) and (31) determines the perturbation on the abelian $\mathbf{C}'^\mu$.

4. Conclusion

We have studied the superposition of non-abelian potentials formed from the corresponding abelian ones by introducing small perturbations. The potentials are described as special since they have the forms determined by (19) and (20). It may be noted that one of the conditions for the superposition of non-abelian potentials is that the isotopic spin corresponding to the abelian potentials as well as perturbations should be parallel, then the potential obtained as a result of superposition would be

$$\mathbf{C}^{\mu} = \mathbf{A}^{\mu}\left(1+\frac{\phi_2\phi_3}{\phi_4}\right)+\mathbf{B}^{\mu}\left(1+\frac{\phi_1\phi_4}{\phi_3}\right), \tag{33}$$

where ϕ's are the scalar functions, which incidently shows that if $\phi_2/\phi_1 = (\phi_4/\phi_3)^2$, non-abelian potentials cannot be constructed. The other condition, however, determines that non-abelian potentials of the type (29) and (30) may be superposed without requiring the constraint of the parallel isospin of all the potentials. The form of abelian potentials as well as those of perturbation potentials given by (19) and (20) do fix the direction of isospin but the restriction of the parallelism is not imposed. The form of the non-abelian potential obtained due to superposition is given by (31), which in the abelian limit reduces to Gorski (1982) type for the abelian potentials.

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