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Quasi invariants and generalized Killing vectors

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Abstract. The connection between quasi-invariants (invariants of a Hamiltonian system defined only on a single constant energy hypersurface) and generalized Killing vector fields associated with the corresponding Jacobi metric is investigated. The results are used to deduce a generalised form of the classical Whittaker problem in two degrees of freedom.

Keywords. Invariants; Killing vectors; Jacobi metric; Whittaker problem.

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1. Introduction

The search for invariants of classical mechanics (CM) systems has a long history. The question of whether a given CM system is integrable or not has been posed at various times in the past and a variety of methods have been developed to attempt to solve this problem (Stackel 1893; Eisenhart 1948; Whittaker 1944; Ankiewicz and Pask 1983; Ablowitz *et al* 1978, 1980; Ramani *et al* 1982; Chang *et al* 1981, 1982). After the publication of the celebrated KAM theorem (Abraham and Marsden 1978) and with the results of many numerical investigations becoming known, a slightly more general form of the integrability problem has come to the fore. The KAM theorem assures us that certain invariant tori persist under sufficiently small perturbations. In effect, this implies that for certain classes of initial conditions, the Hamiltonian possesses additional invariants. It is also well known that these additional constants of motion are not necessarily present for other choices of initial conditions. For example, numerical investigations with the Henon-Heiles Hamiltonian, given by

$$H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{x^2}{2} + \frac{y^2}{2} + x^2 y - \frac{y^3}{3}, \tag{1}$$

show that the system possesses a second constant of motion for sufficiently small energies, leading to invariant tori, while at higher energies, the tori break up, implying the non-existence of additional conserved quantities (Henon and Heiles 1964; Gustavson 1966).

Such a situation obviously poses the following problem: Given a CM system and a set of initial conditions, does the CM system possess additional conserved quantities for motions belonging to the given set of initial conditions? The problem as stated appears to be too general to be tackled; however, there may exist weaker forms of the problem which may be amenable to analysis. In this paper, one particular form of the problem, where the set of initial conditions, is assumed to define a constant energy surface is considered. Hall (1982) has already given most of the results for such a case, however, his method differs from ours in several aspects. In particular, our approach has the advantage that it is easily generalisable.

An invariant, if one exists, in such a case, will be called a quasi-invariant. More precisely, we have

Definition: Let $L = L(x^i, \dot{x}^i)$ be the Lagrangian of a conservative CM system with n degrees of freedom. A function $I(x_i, \dot{x}_i)$ is said to be a quasi-invariant at energy E if $dI/dt = 0$ on the hypersurface $H = E$.

Obviously, a quasi-invariant I is an invariant iff I is quasi-invariant for all E. Quasi-invariants at a given E as usual form a Lie algebra under the operation of Poisson's brackets.

For studying such quasi-invariants, the most useful formulation of classical mechanics is in terms of the associated Jacobi metric (Whittaker 1944; Pin 1975; Abraham and Marsden 1978). This formulation allows us to naturally isolate a given energy E and to study the quasi-invariants through generalised Killing vectors.

2. The equations governing a quasi-invariant

Let L be given by

$$L = a_{ij}\frac{\dot{x}^i \dot{x}^j}{2} - V(x). \tag{2}$$

We know that the Euler-Lagrange equations of motion for L at a given energy E is completely equivalent to the geodesic equations on a Riemannian manifold of dimension n (= number of degrees of freedom) described by the metric

$$ds^2 = g_{ij}\, dx^i dx^j, \tag{3}$$

where $\quad g_{ij} = (E - V)a_{ij}$.

Now, if I is a quasi-invariant at $H = E$, then I defines a one-parameter group of transformations which is a subgroup of the invariance group of the geodesic equations. In turn, this implies that the quasi-invariant generates a Killing vector field on the Riemannian manifold.

For seeing the precise connection between the two, consider the Killing equations for a vector field $\xi_i(x)$: (Eisenhart 1966)

$$\xi_{i;j} + \xi_{j;i} = 0, \tag{4}$$

where the semicolon refers to the covariant derivative with respect to g_{ij}. These equations are equivalent to

$$\xi_{i;j}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0, \tag{5}$$

and hence to

$$\left(\xi_i \frac{dx^i}{ds}\right)_{;j} \frac{dx^j}{ds} = 0, \tag{6}$$

where we have used the geodesic equations in the form:

$$\left(\frac{dx^i}{ds}\right)_{;j} \frac{dx^j}{ds} = 0. \tag{7}$$

If we now define $I = \xi_i \mathrm{d}x^i/\mathrm{d}s$, then $\mathrm{d}I/\mathrm{d}s = 0$. Note also, from the definition of $\mathrm{d}s$ that $\mathrm{d}s/\mathrm{d}t \neq 0$, and hence, $\mathrm{d}I/\mathrm{d}t = 0$, *i.e.*, I is a quasi-invariant at $H = E$. We can turn this argument around and associate with every quasi-invariant a vector field ξ_i which satisfies (5). Note however, that such a vector field is not in general a classical Killing vector field—the ξ_i's are functions of not only x^i's but also $\mathrm{d}x^i/\mathrm{d}s$. For this reason, a generalised Killing vector field will no longer necessarily obey (4).

In the rest of the paper, we specialise to the case when $a_{ij} = \delta_{ij}$, so that the metric is given by

$$\mathrm{d}s^2 = (E - V)(\delta_{ij}\mathrm{d}x^i\,\mathrm{d}x^j) \equiv g(\delta_{ij}\mathrm{d}x^i\,\mathrm{d}x^j). \tag{8}$$

In such a case, we have the following constraint on the $\mathrm{d}x^i/\mathrm{d}s$:

$$\delta_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}s}\frac{\mathrm{d}x^j}{\mathrm{d}s} = \frac{1}{g}. \tag{9}$$

Obviously, the easiest way to treat the constraint is to work in polar co-ordinates.

3. First order quasi-invariant for $n = 2$

We now consider the Lagrangian

$$L = \frac{\dot{x}^2}{2} + \frac{\dot{y}^2}{2} - V(x, y), \quad x \equiv x^1, \quad y \equiv x^2, \tag{10}$$

and assume that

$$I = \xi_1\frac{\mathrm{d}x}{\mathrm{d}s} + \xi_2\frac{\mathrm{d}y}{\mathrm{d}s}, \tag{11}$$

where the ξ_i's are functions of the x^i's alone, *i.e.* they define a classical Killing vector field and the quasi-invariant is of first order in $\mathrm{d}x^i/\mathrm{d}s$ (or $\mathrm{d}x^i/\mathrm{d}t$). Define $g = (E - V)$ so that $g_{ij} = g\delta_{ij}$. It turns out that it is easier to work with the covariant components ξ^i, rather than the contravariant components ξ_i. In terms of these, the equations become (Eisenhart 1966)

$$\xi^1\frac{\partial g}{\partial x} + \xi^2\frac{\partial g}{\partial y} + g\frac{\partial \xi^1}{\partial x} = 0,$$

$$\xi^1\frac{\partial g}{\partial x} + \xi^2\frac{\partial g}{\partial y} + g\frac{\partial \xi^2}{\partial y} = 0,$$

$$\frac{\partial \xi^1}{\partial y} + \frac{\partial \xi^2}{\partial x} = 0. \tag{12}$$

From these equations, it immediately follows that (ξ^1, ξ^2) form a conjugate pair of solutions of Laplace's equation in 2 dimensions, *i.e.* $\xi^1 + i\xi^2$ is an analytic function of $x + iy$. To simplify the equations, we transform to a new set of coordinates (α, β), given by

$$\frac{\partial \alpha}{\partial x} = \frac{\xi^1}{(\xi^1)^2 + (\xi^2)^2}, \quad \frac{\partial \alpha}{\partial y} = \frac{\xi^2}{(\xi^1)^2 + (\xi^2)^2},$$

$$\frac{\partial \beta}{\partial x} = -\frac{\partial \alpha}{\partial y}, \quad \frac{\partial \beta}{\partial y} = \frac{\partial \alpha}{\partial x}. \tag{13}$$

It is easy to see that the above system is integrable; note also that α, β are also conjugate solutions of Laplace's equation.

If we denote the components of the various quantities in the (α, β) system by a $\sim$, then,

$$\tilde{\xi}^1 = 1, \tilde{\xi}^2 = 0,$$

$$ds^2 = \tilde{g}(d\alpha^2 + d\beta^2),$$

$$\tilde{g} = \frac{g}{|\nabla\alpha|^2} = \frac{g}{|\nabla\beta|^2}. \tag{14}$$

Writing the Killing equations in this system of coordinates leads to

$$\frac{\partial\tilde{g}}{\partial\alpha} = 0 \Rightarrow \tilde{g} = \phi(\beta) \Rightarrow g = \phi(\beta)\cdot|\nabla\alpha|^2. \tag{15}$$

In terms of the contravariant components, we have,

$$\tilde{\xi}_1 = \frac{g}{|\nabla\alpha|^2}, \quad \tilde{\xi}_2 = 0,$$

and hence,

$$I = \frac{g}{|\nabla\alpha|^2}\frac{d\alpha}{ds} = \frac{\sqrt{2}}{|\nabla\alpha|^2}\frac{d\alpha}{dt} = \sqrt{2}\,p_\alpha, \tag{16}$$

where p_α denotes the momentum conjugate to α. Thus, if there exists a pair of conjugate solutions to Laplace's equation such that $g = (E - V) = \phi(\beta)\cdot|\nabla\alpha|^2$, then p_α is a quasi-invariant for the CM system.

In terms of the Hamilton-Jacobi equation, it is easy to see why a quasi-invariant exists in such a situation. In fact, the *H-J* equation reads

$$\left(\frac{1}{|\nabla\alpha|^2}\left(\frac{\partial S}{\partial\beta}\right)^2 - \phi(\beta)|\nabla\alpha|^2\right) + \frac{1}{|\nabla\alpha|^2}\left(\frac{\partial S}{\partial\alpha}\right)^2 = 0, \tag{17}$$

which is clearly of separable type.

It is also instructive to note that the Hamiltonian written in (α, β) coordinates is given by

$$\tilde{H} = |\nabla\alpha|^2\left(\frac{p_\alpha^2}{2} + \frac{p_\beta^2}{2} - \phi(\beta)\right) + E, \tag{18}$$

which is equivalent to

$$H^1 = |\nabla\alpha|^2\left(\frac{p_\alpha^2}{2} + \frac{p_\beta^2}{2} - \phi(\beta)\right), \tag{19}$$

so that α seems to appear as an "ignorable" coordinate.

Finally, consider (15)

$$E - V = \phi(\beta)\cdot|\nabla\alpha|^2 = \phi(\beta)\cdot|\nabla\beta|^2. \tag{20}$$

Clearly, the quasi-invariant will exist for all E iff this equation is satisfied for all E. This in turn is possible iff $|\nabla\alpha|^2$ (and hence $|\nabla\beta|^2$) is a function of β alone. We therefore write,

$$\frac{\partial \alpha}{\partial x} = K(\beta)\cos\theta(\alpha, \beta); \quad \frac{\partial \alpha}{\partial y} = K(\beta)\sin\theta(\alpha, \beta),$$

$$\frac{\partial \beta}{\partial x} = -\frac{\partial \alpha}{\partial y}; \quad \frac{\partial \beta}{\partial y} = \frac{\partial \alpha}{\partial x}. \tag{21}$$

The integrability condition for these equations implies that θ is independent of β; *i.e.*, the tangent vector to α = constant curve is always in the same direction. This immediately implies that the α = constant curve is a straight line, which yields a special case of Whittaker's theorem: "A Hamiltonian system $H = (p_x^2/2) + (p_y^2/2) + V(x, y)$ possesses an invariant linear in the velocities iff $V = f(x)$ or $V = f(r)$ or $V = f(\theta)$, where (x, y) denote Cartesian coordinates and (r, θ) denote polar coordinates".

4. Second order quasi-invariant for $n = 2$

We take

$$L = \frac{\dot{x}^2}{2} + \frac{\dot{y}^2}{2} - V(x, y),$$

$$I = \phi(x, y) + \xi_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}s}\frac{\mathrm{d}x^j}{\mathrm{d}s}. \tag{22}$$

The corresponding ξ_i, defined by $I = \xi_i \, \mathrm{d}x^i/\mathrm{d}s$ is then given by

$$\xi_i = \phi(x, y)/(\mathrm{d}x^i/\mathrm{d}s) + \xi_{ij}(\mathrm{d}x^j/\mathrm{d}s). \tag{23}$$

Since ξ_{ij} is a symmetric tensor, we know that its eigenvectors are orthogonal to one another. We therefore work in coordinates (α, β) which are aligned parallel to the principal directions of ξ_{ij}. In terms of these coordinates, we have

$$L = \frac{\dot{\alpha}^2}{2|\nabla\alpha|^2} + \frac{\dot{\beta}^2}{2|\nabla\beta|^2} - V(\alpha, \beta),$$

$$I = \phi(\alpha, \beta) + \xi_{11}(\mathrm{d}\alpha/\mathrm{d}s)^2 + \xi_{22}(\mathrm{d}\beta/\mathrm{d}s)^2, \tag{24}$$

with the Jacobi metric given by

$$\mathrm{d}s^2 = g\left(\frac{\mathrm{d}\alpha^2}{|\nabla\alpha|^2} + \frac{\mathrm{d}\beta^2}{|\nabla\beta|^2}\right). \tag{25}$$

As a consequence we have the relation

$$\left[\frac{1}{|\nabla\alpha|^2}(\mathrm{d}\alpha/\mathrm{d}s)^2 + \frac{1}{|\nabla\beta|^2}(\mathrm{d}\beta/\mathrm{d}s)^2\right] = \frac{1}{g}. \tag{26}$$

Hence, the $(\mathrm{d}\beta/\mathrm{d}s)^2$ term in the expression for I can be eliminated in terms of $(\mathrm{d}\alpha/\mathrm{d}s)^2$. Therefore, without loss of generality we assume that

$$I = \phi(\alpha, \beta) + \xi_{11}(\mathrm{d}\alpha/\mathrm{d}s)^2. \tag{27}$$

The generalised Killing equations (equation 6), then yield

$$g\frac{\partial\phi}{\partial\alpha}+2|\nabla\alpha|^2\xi_{11;1}=0,$$

$$g\frac{\partial\phi}{\partial\beta}=0,$$

$$|\nabla\alpha|^2\xi_{11;1}=2|\nabla\beta|^2\xi_{12;2}$$

$$\xi_{11;2}+2\xi_{12;1}=0, \tag{28}$$

where the semicolon refers to the covariant derivative w.r.t. the Jacobi metric. Putting $\xi_{11}=\sigma$, and using the relations

$$\xi_{11;1}=\sigma\frac{\partial}{\partial\alpha}\ln\left(\frac{\sigma g}{2|\nabla\alpha|^2}\right),\quad \xi_{11;2}=\sigma\frac{\partial}{\partial\beta}\left(\frac{g}{2|\nabla\beta|^2}\right),$$

$$\xi_{12;1}=-\frac{\sigma}{2}\frac{\partial}{\partial\beta}\ln\left(\frac{g}{2|\nabla\alpha|^2}\right),\quad \xi_{12;2}=\sigma\frac{|\nabla\alpha|^2}{2}\frac{\partial}{\partial\alpha}\left(\frac{g}{2|\nabla\alpha|^2}\right), \tag{29}$$

we get

$$g=(|\nabla\alpha|^2K(\alpha)+|\nabla\beta|^2H(\beta))\,[F(\beta)-\phi(\alpha)]$$

$F(\beta)$ arbitrary function of β,

$$\sigma=\frac{g^2}{4|\nabla\alpha|^4K(\alpha)}, \tag{30}$$

where $K(\alpha)$ and $H(\beta)$ are arbitrary functions of their arguments satisfying

$$K(\alpha)\cdot|\nabla\alpha|^2=H(\beta)\cdot|\nabla\beta|^2. \tag{31}$$

We can always choose a new coordinate system $(\tilde{\alpha},\tilde{\beta})$ where $K(\alpha)=1/2$, $H(\beta)=1/2$, *i.e.* $|\nabla\tilde{\alpha}|^2=|\nabla\tilde{\beta}|^2$. Together with the relation $\nabla\tilde{\alpha}\cdot\nabla\tilde{\beta}=0$, these relations imply that $\tilde{\alpha}$ and $\tilde{\beta}$ are conjugate solutions of Laplace's equation. Thus we have, "The Lagrangian system

$$L=\frac{\dot{x}^2}{2}+\frac{\dot{y}^2}{2}-V(x,y), \tag{32}$$

possesses a second order quasi-invariant iff

$$g=|\nabla\alpha|^2\phi(\alpha)+|\nabla\beta|^2\psi(\beta), \tag{33}$$

where α, β are arbitrary conjugate solutions of Laplace's equation and where ϕ and ψ are arbitrary functions of their arguments; in such a case, the quasi-invariants are given by

$$\frac{p_\alpha^2}{2}-\phi(\alpha)\quad\text{and}\quad\frac{p_\beta^2}{2}-\psi(\beta)." \tag{34}$$

The Hamiltonian in these coordinates becomes

$$\tilde{H}=|\nabla\alpha|^2\left(\frac{p_\alpha^2}{2}+\frac{p_\beta^2}{2}-\phi(\alpha)-\psi(\beta)\right)+E, \tag{35}$$

which is of separable type. Once again, it is easy to see that the quasi-invariant becomes an invariant if either

(i) $|\nabla\alpha|^2$ (and hence $|\nabla\beta|^2$) is independent of α;
(ii) $|\nabla\alpha|^2 = |\nabla\beta|^2 = 1/(F(\alpha)+G(\beta))$

The first case yields as before the potentials separable in Cartesian or in polar coordinates, while the second case yields potentials separable in elliptic coordinates ($|\nabla\alpha|^2 = 1/(\alpha^2-\beta^2)$) or in parabolic coordinates ($|\nabla\alpha|^2 = 1/(\alpha^2+\beta^2)$) (Ankiewicz and Pask 1983).

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