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Symmetry groups of mathematical physics

P RUDRA

Department of Physics, University of Kalyani, Kalyani 741 235, India

Abstract. Recent work on Lie's method of extended groups to obtain symmetry groups and invariants of differential equations of mathematical physics is surveyed. As an essentially new contribution one-parameter Lie groups admitted by three-dimensional harmonic oscillator, three-dimensional wave equation, Klein-Gordon equation, two-component Weyl's equation for neutrino and four-component Dirac equation for Fermions are obtained.

Keywords. Mathematical physics; symmetry groups; differential equations; Lie groups; harmonic oscillator; wave equation; Klein-Gordon equation; Weyl equation; Dirac equation.

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1. Introduction

Application of the Lie groups theory to the characterization of atomic spectra by Wigner (1931) followed Weyl's work (1925, 1926) on the representation theory of continuous groups. The symmetry group used in this characterization was the three-dimensional orthogonal group leaving the form $\Sigma_{i=1}^{3} x_i^2$ invariant. The l-degeneracy for the hydrogen atom in a potential $V(r) = \mu/r$ could not be explained on this basis; this gave rise to the concept of accidental degeneracy (McIntosh 1971; Elliott and Dawber 1979), until Fock (1935) showed that the full symmetry group of the non-relativistic hydrogen problem was $O(4)$, the orthogonal group in four-dimension. Full degeneracy of the problem was thus accounted for. Bargmann (1936) showed that besides the angular momentum operator $\mathbf{L}$ an extra operator, the Runge-Lenz vector $\mathbf{A} = \mathbf{L} \times \mathbf{p} - \mu\mathbf{r}/r$, was required for the complete description of the orbit.

The experience with non-relativistic hydrogen problem has led physicists to search for complete symmetry groups of other classical and quantum mechanical systems. Maxwell's electromagnetic equations form one such system of interest of the former class. Bateman (1910) showed that Maxwell's equations are invariant under the 15-parameter conformal group consisting of the 10-parameter inhomogeneous Lorentz group, scale transformation, inversion and 4-parameter acceleration transformations. Use of conformal symmetry in quantum field theory and other branches of physics has been investigated in some detail (Wess 1960; Fulton *et al* 1962). The other physical system that has drawn the attention of physicists is the harmonic oscillator, both isotropic and anisotropic. Jauch and Hill (1940) investigated in detail the isotropic harmonic oscillator and showed that its symmetry group is the unitary unimodular group. Cisneros and McIntosh (1970) made a similar study of the anisotropic harmonic oscillator. It will be appropriate to mention here that a connected problem is to obtain the Casimir invariants (Racah 1951) of the symmetry group. By Noether's theorem (1918) the symmetry groups are characterized by their Casimir invariants, and obtaining the Casimir invariants of the problem was equivalent to obtaining the symmetry group.

Methods of solving both the problems, finding the symmetry group or obtaining the Casimir invariants, remained an art which only the ingeneous could utilize for different physical systems. It somehow escaped most of the physicists that Lie and Scheffers (1891) and Dickson (1924) had long ago given very detailed algorithm of extended group for finding out generators of Lie groups that keep the form of a differential equation invariant. Only recently this method (Olver 1976; Sattinger 1979; Hamermesh 1983) has been used extensively to find symmetry groups of different equations of physics. Wulfman and Wybourne (1976) considered the classical one-dimensional harmonic oscillator and found that its symmetry group is the 8-parameter non-compact Cartan group A_2. Lutzky (1978) showed that a 5-parameter subgroup of this 8-parameter group leaves the action integral invariant, thus having 5 conserved quantities of which only 2 are functionally independent. In a series of papers Leach and coworkers (Leach 1981a; Prince 1983a; Prince and Eliezer 1980, 1981; Prince and Leach 1980) solved the classical problems of N-dimensional time-dependent harmonic oscillator and Kepler motion. Boyer *et al* (1976), and Harnard and Winternitz (1980) studied both linear and non-linear one-dimensional Schrödinger equation and found the corresponding symmetry groups. Vinet (1980) considered the linear hyperbolic equation in two variables. Kalinis and Miller (1974) and Boyer *et al* (1975) applied the method of Lie to time-dependent Schrödinger equation of free particles.

As mentioned before, a connected problem is obtaining the Casimir invariants. The usual method of Noether keeping the action integral invariant has the disadvantage that it does not give the Runge-Lenz vector for Kepler motion. In this method Runge-Lenz vector is obtained if one goes beyond point transformations and involve the velocity as an independent variable (Lévy-Leblond 1971). In the Lie's method of differential equation all the invariants appear as a result of point transformations. The method has been successfully applied to the Kepler problem, time-dependent harmonic oscillator and quadratic hamiltonians (Leach 1978, 1980; Guenther and Leach 1977; Prince 1983b). Makarov *et al* (1967) obtained linear and quadratic invariants of motion for non-relativistic Schrödinger equation. Patera *et al* (1976a, b) used Lie's method to obtain the invariants of continuous subgroups of Poincaré group, for all real algebras of dimension up to 5 and for all real nilpotent algebras of dimension 6. These authors made a distinction between Casimir operators (polynomials in the generators), rational invariants (rational functions of generators) and general invariants (irrational and transcendental functions of the generators). González-Gascon (1977), and González-Gascon and González-López (1983) have applied the theory of partial differential equations to ordinary differential systems of classical mechanics to obtain first integrals and the upper bounds for the number of independent point-like symmetry vectors of differential equations.

In §2 Lie's theory of extended group is described and the symmetry groups for some of the important differential equations of mathematical physics are obtained. Section 3 shows how the invariants are obtained from Lie's theory and in §4 the method is applied to Korteweg-deVries nonlinear equation of soliton physics.

Finally, another aspect of Lie's theory is considered (Eisenhart 1961). This is the concept of one-parameter Lie groups admitted by a complete system of differential equations. If ψ is a solution of the system then all the functions $X_\alpha\psi$ for a set of linear operators $X_\alpha(\alpha = 1, \ldots, r)$ are also solutions when the X_α's satisfy a particular condition. The one-parameter Lie groups generated by the X_α's are said to be admitted by the given system of differential equations. A key theorem states that either the system admits a r-parameter Lie group or solutions of the system are obtained by direct process. In §5 the generators of the one-parameter Lie groups admitted by three-dimensional isotropic harmonic oscillator, three-dimensional wave equation, Klein-Gordon equation, Weyl's equation for neutrino and Dirac equation for spin-1/2 particles are obtained. It should be pointed out that in the last two cases the differential equations are not scalar equations, but are 2- and 4-component equations respectively.

2. Symmetry groups of differential equations

In this section we describe Lie's theory of extended group and obtain the symmetry group that keep the form of the classical Lagrangian equation invariant. We consider a point transformation

$$t' = t + \delta\alpha\xi(q,t), \quad q'_i = q_i + \delta\alpha\eta_i(q,t), \quad i = 1, \ldots, N, \tag{1}$$

in the space $t, q = (q_i)$. The generator of the transformation is

$$X \equiv \xi(q,t)\partial/\partial t + \sum_i \eta_i(q,t)\partial/\partial q_i. \tag{2}$$

The generator for the nth extension group is given by

$$X^{(n)} \equiv \xi\partial/\partial t + \sum_i [\eta_i\partial/\partial q_i + \eta_i^{(1)}\partial/\partial q_i^{(1)} + \ldots + \eta_i^{(n)}\partial/\partial q_i^{(n)}]$$

where $$\eta_i^{(k)}(q, q^{(1)}, \ldots, q^{(k)}, t) = \frac{\mathrm{d}}{\mathrm{d}t}\eta_i^{(k-1)} - q_i^{(k)}\frac{\mathrm{d}}{\mathrm{d}t}\xi, \quad (k = 1, \ldots, n)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \partial/\partial t + \sum_i [q_i^{(1)}\partial/\partial q_i + \ldots + q_i^{(k)}\partial/\partial q_i^{(k-1)}]$$

and $$q_i^{(k)} = (\mathrm{d}/\mathrm{d}t)^k q_i. \tag{3}$$

The finite transformation of the extended group can be expressed as

$$t' = (\exp\alpha X)t, \quad q' = (\exp\alpha X)q, \quad q^{(k)\prime} = (\exp\alpha X^{(k)})q^{(k)}. \tag{4}$$

For an nth order differential equation

$$q_i^{(n)} + g_i(q, q^{(1)}, \ldots, q^{(n-1)}, t) = 0, \tag{5}$$

we have to find the unknown functions ξ and η_i's from the condition

$$X^{(n)}[q_i^{(n)} + g_i(q, q^{(1)}, \ldots, q^{(n-1)}, t)] = 0, \quad i = 1, \ldots, N. \tag{6}$$

By equating the coefficients of powers of $q_i^{(k)}$, $k = 1, \ldots, (n-1)$, to zero, after replacing $q_i^{(n)}$ by $-g_i$ in the left side of (6), we get partial differential equations for ξ and η_i's, whose solutions give the unknown functions.

For the second order Newtonian equation of the form (Leach 1981b)

$$\ddot{q}_i + g_i(q,t) = 0. \quad i = 1, \ldots, N \tag{7}$$

the generator for the extended group is

$$X^{(2)} = \xi \partial/\partial t + \sum_i [\eta_i \partial/\partial q_i + (\dot{\eta}_i - \dot{\xi}\dot{q}_i)\partial/\partial \dot{q}_i + (\ddot{\eta}_i - \ddot{\xi}\dot{q}_i - 2\dot{\xi}\ddot{q}_i)\partial/\partial \ddot{q}_i]. \tag{8}$$

Separating out the terms of the second and the third power of the $\dot{q}_i$'s, we get

$$\partial^2 \xi/\partial q_i \partial q_j = 0, \tag{9}$$

$$\partial^2 \eta_l/\partial q_i \partial q_j - \delta_{jl}\partial^2 \xi/\partial q_i \partial t - \delta_{il}\partial^2 \xi/\partial q_j \partial t = 0. \tag{10}$$

Equations (9) and (10) give

$$\xi = a(t) + \sum_k b_k(t) q_k$$

$$\eta_i = \sum_k [\dot{b}_k q_k q_i + c_{ik}(t) q_k] + d_i(t). \tag{11}$$

Thus the generators of the symmetry group are of the form

$$X = [a + \sum_k b_k q_k]\partial/\partial t + \sum_i [d_i + \sum_k (\dot{b}_k q_k q_i + c_{ik} q_k)]\partial/\partial q_i, \tag{12}$$

where a, b_k, c_{ik}, d_i are determined from (7) and (8). For 2-dimensional Kepler motion (Prince and Eliezer 1981)

$$\ddot{\mathbf{r}} + \mu \mathbf{r}/r^3 = 0, \tag{13}$$

we get the 3 generators for the symmetry group

$$X_1 = \partial/\partial t, \quad X_2 = x\partial/\partial y - y\partial/\partial x, \quad X_3 = t\partial/\partial t + \tfrac{2}{3}x\partial/\partial x + \tfrac{2}{3}y\partial/\partial y \tag{14}$$

with the commutation relations

$$[X_1, X_2] = 0, \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = 0. \tag{15}$$

For 1-dimensional harmonic oscillator (Wulfman and Wybourne 1976)

$$\ddot{x} + x = 0, \tag{16}$$

we get 8 generators for the symmetry group

$$X_1 = (1 + x^2)\sin t\, \partial/\partial x - x\cos t\, \partial/\partial t,$$

$$X_2 = (1 - x^2)\sin t\, \partial/\partial x + x\cos t\, \partial/\partial t,$$

$$X_3 = (1 + x^2)\cos t\, \partial/\partial x + x\sin t\, \partial/\partial t,$$

$$X_4 = (1 - x^2)\cos t\, \partial/\partial x - x\sin t\, \partial/\partial t,$$

$$X_5 = \partial/\partial t, \quad X_6 = x\partial/\partial x, \quad X_7 = x\cos 2t\, \partial/\partial x + \sin 2t\, \partial/\partial t,$$

$$X_8 = -x\sin 2t\, \partial/\partial x + \cos 2t\, \partial/\partial t, \tag{17}$$

forming the non-compact Cartan's algebra A_2.

3. Invariants of symmetry groups

In this section we first explain Lie's method of obtaining the invariant functions of the generators for a Lie group. The problem of finding the invariants reduces to solving a certain set of linear first order partial differential equations. These equations may have polynomial solutions giving rise to Casimir invariants, rational solutions giving rise to rational invariants (*i.e.* ratio of two polynomials), or more general solutions including transcendental functions leading to general invariants (Patera *et al* 1976a, b). If $X_i, i = 1, \ldots, r$ are generators of the symmetry group with structure constants C_{ij}^k

$$[X_i, X_j] = \sum_k C_{ij}^k X_k \tag{18}$$

then the invariants are those functions $F(X)$ for which

$$[X_i, F(X)] = 0, \quad i = 1, \ldots, r. \tag{19}$$

A differential representation of X_i's are

$$X_i \to \chi_i = \sum_{j,k} C_{ij}^k x_k \partial/\partial x_j. \tag{20}$$

The commutation relation (19) is then replaced by the partial differential equation

$$\chi_i F = 0, \quad i = 1, \ldots, r. \tag{21}$$

For a single equation of the form of (21) we are led to the total differential equation

$$\sum_j \mu_{jm} \mathrm{d}x_j / \sum_{jk} x_k C_{ij}^k \mu_{jm} = \text{independent of } m \tag{22}$$

with the subsidiary condition

$$\sum_j C_{ij}^k \mu_{jm} = \lambda_m \mu_{km}, \tag{23}$$

i.e. μ_{km} is an eigenvector of the matrix C_{ij}^k belonging to the eigenvalue λ_m (suppressing the dependence of μ_{jm} and λ_m on i). Then

$$\mathrm{d}x'_m/\lambda_m x'_m = \text{independent of } m, \text{ where } x'_m = \sum_j x_j \mu_{jm}. \tag{24}$$

If the eigenvectors are complete, we get the solution

$$F(x) = \Phi(x_2'^{\lambda_1}/x_1'^{\lambda_2}, x_3'^{\lambda_1}/x_1'^{\lambda_3}, \ldots, x_r'^{\lambda_1}/x_1'^{\lambda_r}). \tag{25}$$

To obtain the first integrals of motion $I(q, \dot{q}, t)$ of (5) we have to solve the partial differential equation

$$X_i^{(1)} I(q, \dot{q}, t) = 0, \quad \mathrm{d}I(q, \dot{q}, t)/\mathrm{d}t = 0, \quad i = 1, \ldots, r. \tag{26}$$

Here $X_i^{(1)}$'s are the first extensions of the generators X_i's. The first member of (26) has the associated Lagrangian system

$$\mathrm{d}t/\xi = \mathrm{d}q_i/\eta_i = \mathrm{d}\dot{q}_i/\eta_i^{(1)}. \tag{27}$$

If u_i's are constants of integrals for the first pair of (27) and v_i's for any other pair, then I is a function of u and v: $I(u, v)$ and $\mathrm{d}v_i/\mathrm{d}u_j = \psi(u, v)$. From these sets of equations the first integral is obtained by quadrature.

In the two-dimensional Kepler motion the generators X_1 and X_2 yield the integrals

$$I_1 = \tfrac{1}{2}\dot{\mathbf{r}}^2 + \mu/r = \text{energy}, \quad I_2 = x\dot{y} - y\dot{x} = z\text{-component of angular momentum.}$$

For the generator X_3 the associated Lagrangian system is

$$dt/t = dx/\tfrac{2}{3}x = dy/\tfrac{2}{3}y = d\dot{x}/(-\tfrac{1}{3}\dot{x}) = d\dot{y}/(-\tfrac{1}{3}\dot{y}) \tag{28}$$

yielding $u_1(x, t) = x^2/t^3, \quad u_2(y, t) = y^2/t^3,$

$$v_1(x, \dot{x}, t) = t\dot{x}^3, \quad v_2(y, \dot{y}, t) = t\dot{y}^3. \tag{29}$$

The corresponding integrals are the x and y components of Runge-Lenz vector

$$x\dot{y}^2 - y\dot{x}\dot{y} - \mu x/r = \text{constant},$$

$$y\dot{x}^2 - x\dot{x}\dot{y} - \mu y/r = \text{constant}. \tag{30}$$

It must be admitted that Lie's method for obtaining the invariants is not as straightforward as Noether's. Nevèrtheless, it has the advantage that it gives us all the invariants of the problem.

4. Symmetry groups of non-linear differential equations

Not all differential equations of mathematical physics are linear. There are a whole lot of nonlinear equations that have come to the attention of physicists (Eilenberger 1981). These are the partial differential equations having soliton-like solutions. A particularly important equation in this set is the Korteweg-deVries equation

$$\partial u/\partial t + u\partial u/\partial x + \partial^3 u/(\partial x)^3 = 0. \tag{31}$$

We now describe the method (Hamermesh 1983) for obtaining the largest local symmetry group of the set of partial differential equations in q dependent variables $u^l, l = 1, \ldots, q$ and n independent variables $x_i, i = 1, \ldots, n$. For this we first construct a space encompassing all the derivatives that appear in the partial differential equations. The number of different kth order partial derivatives is

$$n_k = \binom{n+k-1}{k}$$

and the partial derivatives are denoted by

$$\partial_J = \partial^{|J|}/\partial x_1^{j_1} \ldots \partial x_n^{j_n} \text{ with } J \equiv (j_1, \ldots, j_n), \quad |J| = \sum_i j_i.$$

Here the j_i's are non-negative integers. If X is a generator of the product space (x, u) then the kth extension $X^{(k)}$ of X is given by

$$X = \sum_i \xi^i(x, u)\partial/\partial x_i + \sum_l \varphi_l(x, u)\partial/\partial u^l. \tag{32}$$

$$X^{(k)} = X + \sum_l \sum_{|J| \leq k} \varphi_l^J(x, u^{(k)})\partial/\partial u_J^l. \tag{33}$$

Here

$$\varphi_l^J = D^J(\varphi_l - \sum_i u_i^l \xi^i) + \sum_i u_{J,i}^l \xi^i, \tag{34}$$

where $u_i^l = \partial u^l/\partial x_i, \quad J,i \equiv (j_1, \ldots, j_{i-1}, j_i + 1, j_{i+1}, \ldots, j_n)$ (35)

and $$D^J = D_1{}^{j_1} D_2{}^{j_2} \dots D_n{}^{j_n}, \quad D_i = \partial/\partial x_i + \sum_l \sum_{|J| \leq k} u^l_{J,i} \partial/\partial u^l_J. \tag{36}$$

A system of partial differential equations

$$\Delta^\alpha(x, u^{(k)}) = 0, \quad \alpha = 1, \dots, p \tag{37}$$

has the symmetry group G if for every generator $X \in G$

$$X^{(k)} \Delta^\alpha(x, u^{(k)}) = 0, \quad \alpha = 1, \dots, p. \tag{38}$$

Detailed calculation shows that for the Korteweg-deVries equation (equation (31)) G is a four-parameter Lie group with generators

$$X_1 = \partial/\partial x, \quad X_2 = \partial/\partial t, \quad X_3 = t\partial/\partial x + \partial/\partial u,$$
$$X_4 = x\partial/\partial x + 3t\partial/\partial t + 2u\partial/\partial u \tag{39}$$

with the non-vanishing commutators

$$[X_1, X_4] = X_1, [X_2, X_3] = X_1, [X_2, X_4] = 3X_2,$$
$$[X_3, X_4] = -2X_3. \tag{40}$$

5. One-parameter Lie groups admitted by differential equations

In this section we come to another aspect of Lie's theory (Eisenhart 1961). If we know the solution ψ of a complete system of linear partial differential equations

$$A^\alpha \psi = 0, \quad \alpha = 1, \dots, p, \tag{41}$$

then the functions $X\psi$, where

$$X = \sum_m \xi_m(x) \partial/\partial x_m, \tag{42}$$

will also be solutions of the complete set if

$$[X, A^\alpha] = \sum_\beta \lambda^\alpha_\beta(x) A^\beta. \tag{43}$$

If moreover X_i and X_j are two operators satisfying (43), then their commutator $[X_i, X_j]$ also satisfies (43). These X's form the different one-parameter Lie groups admitted by the complete set of linear partial differential equations. The key theorem is that either the operators X_i, $i = 1, \dots, r$, satisfying (43) form an r-parameter Lie group or the equations are solvable by direct process. The importance of these one-parameter groups lies in the fact that they give the complete set of solutions, if one of the solutions is known. In second order partial differential equations

$$A^\alpha \equiv a^\alpha_0(x) + \sum_i a^\alpha_i(x) \partial/\partial x_i + \sum_{ij} a^\alpha_{ij}(x) \partial^2/\partial x_i \partial x_j, \quad \alpha = 1, \dots, p, \tag{44}$$

we get a set of partial differential equations for the ξ's

$$\sum_m \xi_m \partial a^\alpha_0/\partial x_m = \sum_\beta \lambda^\alpha_\beta a^\beta_0, \quad [\xi_k, a^\alpha_{ij}] + [\xi_i, a^\alpha_{jk}] + [\xi_j, a^\alpha_{ki}] = 0,$$

$$[\xi_i, a_0^\alpha] + \sum_m \xi_m \partial a_i^\alpha/\partial x_m - \sum_m a_m^\alpha \partial\xi_i/\partial x_m - \sum_{mn} a_{mn}^\alpha \partial^2\xi_i/\partial x_m \partial x_n = \sum_\beta \lambda_\beta^\alpha a_i^\beta$$

$$\tfrac{1}{2}([\xi_j, a_i^\alpha] + [\xi_i, a_j^\alpha]) + \sum_m \xi_m \partial a_{ij}^\alpha/\partial x_m$$

$$-\sum_m (a_{im}^\alpha \partial\xi_j/\partial x_m + a_{mj}^\alpha \partial\xi_i/\partial x_m) = \sum_\beta \lambda_\beta^\alpha a_{ij}^\beta$$

$$\alpha, \beta = 1, \ldots, p; \quad i, j, m, k = 1, \ldots, n. \tag{45}$$

We now apply this method to obtain the generators of the one-parameter Lie groups for the three-dimensional isotropic harmonic oscillator, three-dimensional wave equation, Klein-Gordon equation, two-component Weyl equation for neutrino and four-component Dirac equation for Fermions.

5.1 *Three-dimensional isotropic harmonic oscillator*

There is a single equation

$$i\hbar A\psi = 0, \text{ with } A \equiv \partial/\partial t - (i\hbar/2m)\nabla^2 + (im\omega_0^2/2\hbar)(x^2 + y^2 + z^2). \tag{46}$$

Taking $X = \xi_1 \partial/\partial x + \xi_2 \partial/\partial y + \xi_3 \partial/\partial z + \xi_4 \partial/\partial t$, (47)

equation (45) takes the form

$$\xi_1 x + \xi_2 y + \xi_3 z = \lambda(x^2 + y^2 + z^2)/2;$$
$$(i\hbar/2m)\nabla^2\xi_i + \partial\xi_i/\partial t = 0, \quad i = 1, 2, 3; \quad (i\hbar/2m)\nabla^2\xi_4 + \partial\xi_4/\partial t = -\lambda;$$
$$\partial\xi_1/\partial x = \partial\xi_2/\partial y = \partial\xi_3/\partial z = -\lambda/2;$$
$$\partial\xi_4/\partial x = \partial\xi_4/\partial y = \partial\xi_4/\partial z = 0;$$
$$\partial\xi_2/\partial x + \partial\xi_1/\partial y = \partial\xi_3/\partial x + \partial\xi_1/\partial z = \partial\xi_3/\partial y + \partial\xi_2/\partial z = 0. \tag{48}$$

Since the ξ_i's are analytic functions of the real variables $x_1 = x, x_2 = y, x_3 = z, x_4 = t$, we use the necessary and sufficient condition $\partial^2\xi_i/\partial x_m \partial x_n = \partial^2\xi_i/\partial x_n \partial x_m$, and get the 4 generators

$$X_1 = -i(y\partial/\partial z - z\partial/\partial y), \quad X_2 = -i(z\partial/\partial x - x\partial/\partial z),$$
$$X_3 = -i(x\partial/\partial y - y\partial/\partial x), \quad X_4 = -i\partial/\partial t, \tag{49}$$

with the non-vanishing commutators

$$[X_1, X_2] = iX_3, \quad [X_2, X_3] = iX_1, \quad [X_3, X_1] = iX_2. \tag{50}$$

These generators form the unitary group $U(2)$ with X_4 forming an exceptional subgroup.

5.2 *Three-dimensional wave equation*

There is again a single equation

$$A\psi = 0, \text{ with } A \equiv \nabla^2 - \partial^2/(\partial\tau)^2, \text{ where } \tau = ct. \tag{51}$$

With $X = \xi_1 \partial/\partial x + \xi_2 \partial/\partial y + \xi_3 \partial/\partial z + \xi_4 \partial/\partial \tau$, (52) we get the following partial differential equations for the ξ_i's:

$$\partial\xi_1/\partial x = \partial\xi_2/\partial y = \partial\xi_3/\partial z = \partial\xi_4/\partial\tau = \lambda/2;$$
$$\nabla^2\xi_i - \partial^2\xi_i/(\partial\tau)^2 = 0, \quad i = 1, 2, 3, 4;$$
$$\partial\xi_4/\partial z - \partial\xi_3/\partial\tau = \partial\xi_4/\partial y - \partial\xi_2/\partial\tau = \partial\xi_4/\partial x - \partial\xi_1/\partial\tau = 0;$$
$$\partial\xi_1/\partial y + \partial\xi_2/\partial x = \partial\xi_1/\partial z + \partial\xi_3/\partial x = \partial\xi_2/\partial z + \partial\xi_3/\partial y = 0. \tag{53}$$

If we again impose the analyticity condition on the ξ_i's, we get 11 generators

$$X_1 = -i\partial/\partial x, \quad X_2 = -i\partial/\partial y, \quad X_3 = -i\partial/\partial z, \quad X_4 = -i\partial/\partial\tau,$$
$$X_5 = -i(y\partial/\partial z - z\partial/\partial y), \quad X_6 = -i(z\partial/\partial x - x\partial/\partial z),$$
$$X_7 = -i(x\partial/\partial y - y\partial/\partial x), \quad X_8 = x\partial/\partial\tau + \tau\partial/\partial x,$$
$$X_9 = y\partial/\partial\tau + \tau\partial/\partial y, \quad X_{10} = z\partial/\partial\tau + \tau\partial/\partial z,$$
$$X_{11} = x\partial/\partial x + y\partial/\partial y + z\partial/\partial z + \tau\partial/\partial\tau \tag{54}$$

with the non-vanishing commutators

$$[X_1, X_6] = iX_3, [X_1, X_7] = -iX_2, [X_1, X_8] = X_4, [X_2, X_5] = -iX_3,$$
$$[X_2, X_7] = iX_1, [X_2, X_9] = X_4, [X_3, X_5] = iX_2, [X_3, X_6] = -iX_1,$$
$$[X_3, X_{10}] = X_4, [X_4, X_8] = X_1, [X_4, X_9] = X_2, [X_4, X_{10}] = X_3,$$
$$[X_5, X_6] = iX_7, [X_5, X_7] = -iX_6, [X_5, X_9] = iX_{10}, [X_5, X_{10}] = X_9,$$
$$[X_6, X_7] = iX_5, [X_6, X_8] = -iX_{10}, [X_6, X_{10}] = iX_8, [X_7, X_8] = iX_9,$$
$$[X_7, X_9] = -iX_8, \quad [X_8, X_9] = iX_7, \quad [X_8, X_{10}] = -iX_6,$$
$$[X_9, X_{10}] = iX_5, [X_k, X_{11}] = X_k, k = 1, 2, 3, 4. \tag{55}$$

The first 4 generators are those for translations along the x, y, z, τ axes; X_5, X_6 and X_7 are those for rotations in the coordinate space of x, y and z; X_8, X_9 and X_{10} are those for the 3 Lorentz boosts; finally X_{11} is the generator for the scale transformation $x'^i = sx^i$. This 11-parameter Lie group has no exceptional subgroup. The generators $X_1, \ldots, X_{10}$ are those for the inhomogeneous Lorentz group I_4^1 and form a subgroup. The group obtained here does not contain the inversion generator and those for the four-parameter abelian group of acceleration transformations (Wess 1960) of the conformal group. The reason is that the vectors for those generators are not analytic functions of the coordinates and we have obtained here only those that are analytic functions.

5.3 *Klein-Gordon equation*

Here also we have a single equation

$$\hbar^2c^2A\psi = 0, \text{ with } A \equiv \nabla^2 - \partial^2/(\partial\tau)^2 - (mc/\hbar)^2 \tag{56}$$

Taking the coordinates $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = \tau = ct$, and

$$X = \sum_i \xi_i\partial/\partial x_i \tag{57}$$

Equations (45) give

$$\lambda(mc/\hbar)^2 = 0; [\nabla^2 - \partial^2/(\partial\tau)^2]\xi_i = 0,\ i = 1, 2, 3, 4;$$

$$\partial\xi_1/\partial x_1 = \partial\xi_2/\partial x_2 = \partial\xi_3/\partial x_3 = \partial\xi_4/\partial x_4 = 0;$$

$$\partial\xi_4/\partial x_1 - \partial\xi_1/\partial x_4 = \partial\xi_4/\partial x_2 - \partial\xi_2/\partial x_4 = \partial\xi_4/\partial x_3 - \partial\xi_3/\partial x_4 = 0;$$

$$\partial\xi_2/\partial x_3 + \partial\xi_3/\partial x_2 = \partial\xi_3/\partial x_1 + \partial\xi_1/\partial x_3 = \partial\xi_1/\partial x_2 + \partial\xi_2/\partial x_1 = 0. \quad (58)$$

Imposing the analyticity restriction we get the ten generators $X_1, \ldots, X_{10}$ of the inhomogeneous Lorentz group I_4^1 of (54) with the commutation relations of (55). The scaling generator X_{11} is absent because of the presence of the non-zero mass term in A.

5.4 *Two-component Weyl equation for neutrino*

This is again a single equation

$$i\hbar c A\psi = 0, \text{ with } A \equiv \boldsymbol{\sigma}\cdot\nabla + I\partial/\partial\tau. \quad (59)$$

Since the coefficients appearing in A are 2×2 matrices, we write the ξ_i's as 2×2 matrices and expand them in terms of the complete set of bases, the identity matrix I and the 3 components of Pauli matrices σ_i's.

$$\xi_i = \xi_i^0 I + \sum_\alpha \xi_i^\alpha \sigma_\alpha, \quad \lambda = \lambda_0 I + \sum_\alpha \lambda_\alpha\sigma_\alpha, \quad a_0 = 0, \quad a_{ij} = 0,$$

$$a_4 = I, \quad a_i = \sigma_i\ (i = 1, 2, 3). \quad (60)$$

The condition $[\xi_i, a_j] + [\xi_j, a_i] = 0$ (61)

gives $\xi_4 = \xi_4^0 I, \quad \xi_i = \xi_i^0 I + \xi\sigma_i\ (i = 1, 2, 3)$ (62)

The partial differential equations connecting ξ_i^0's, ξ, λ_0, λ_α's are

$$\frac{\partial}{\partial x}(\xi_4^0 - \xi) = \partial\xi_1^0/\partial\tau = -i\partial\xi_2^0/\partial z = i\xi_3^0/\partial y = -\lambda_1 - \partial\xi/\partial x$$

$$\partial(\xi_4^0 - \xi)/\partial y = i\partial\xi_1^0/\partial z = \partial\xi_2^0/\partial\tau = -i\partial\xi_3^0/\partial x = -\lambda_2 - \partial\xi/\partial y$$

$$\partial(\xi_4^0 - \xi)/\partial z = -i\partial\xi_1^0/\partial y = i\partial\xi_2^0/\partial x = \partial\xi_3^0/\partial\tau = -\lambda_3 - \partial\xi/\partial z$$

$$\partial(\xi_4^0 - \xi)/\partial\tau = \partial\xi_1^0/\partial x = \partial\xi_2^0/\partial y = \partial\xi_3^0/\partial z = -\lambda_0 - \partial\xi/\partial\tau. \quad (63)$$

The admissible generators are

$$X_1 = I(-i\partial/\partial x),\ X_2 = I(-i\partial/\partial y),\ X_3 = I(-i\partial/\partial z),\ X_4 = I(-i\partial/\partial\tau),$$

$$X_5 = I[-i(y\partial/\partial z - z\partial/\partial y) + (x\partial/\partial\tau + \tau\partial/\partial x)],$$

$$X_6 = I[-i(z\partial/\partial x - x\partial/\partial z) + (y\partial/\partial\tau + \tau\partial/\partial y)],$$

$$X_7 = I[-i(x\partial/\partial y - y\partial/\partial x) + (z\partial/\partial\tau + \tau\partial/\partial z)],$$

$$X_8 = I(x\partial/\partial x + y\partial/\partial y + z\partial/\partial z + \tau\partial/\partial\tau),$$

$$X_9 = -i\boldsymbol{\sigma}\cdot\nabla \quad (64)$$

with the non-vanishing commutators

$$[X_1,X_5]=X_4,[X_1,X_6]=iX_3,[X_1,X_7]=-iX_2,[X_2,X_5]=-iX_3,$$
$$[X_2,X_6]=X_4,\ [X_2,X_7]=iX_1,[X_3,X_5]=iX_2,\ [X_3,X_6]=-iX_1,$$
$$[X_3,X_7]=X_4,\ [X_4,X_5]=X_1,\ [X_4,X_6]=X_2,\ [X_4,X_7]=X_3,$$
$$[X_5,X_6]=2iX_7,\ [X_5,X_7]=-2iX_6,\ [X_6,X_7]=2iX_5,$$
$$[X_j,X_8]=X_j\ (j=1,2,3,4)$$
$$[X_5,X_9]=i(\sigma_2X_3-\sigma_3X_2)-\sigma_1X_4,\ [X_6,X_9]=i(\sigma_3X_1-\sigma_1X_3)-\sigma_2X_4,$$
$$[X_7,X_9]=i(\sigma_1X_2-\sigma_2X_1)-\sigma_3X_4,\ [X_8,X_9]=-X_4-X_9. \quad (65)$$

Here X_1 to X_4 are the generators for translations along x, y, z, τ axes; X_5, X_6, X_7 are generators for screw transformations along x, y and z directions; X_8 is that for scale transformation and X_9 for helicity transformation. It should be noted that commutators with X_9 have matrices rather than scalars as the structure constants. Thus $X_1,\ldots,X_8$ form the group of the eigenfunctions.

5.5 *Four-component Dirac equation for fermions*

We have here a four-component equation

$$i\hbar cA\psi=0,\ \text{with}\ A\equiv\boldsymbol{\alpha}\cdot\nabla+I\partial/\partial\tau+(imc/\hbar)\beta. \quad (66)$$

The functions a_0, a_i's, a_{ij}'s and ξ_i's are now 4×4 matrices and we express them in the bases formed by the Kronecker direct products $(I\otimes I)$, $(I\otimes\sigma_i)$, $(\sigma_i\otimes I)$, $(\sigma_i\otimes\sigma_j)$. On this basis $\boldsymbol{\alpha}=(\sigma_1\otimes\sigma)$ and $\beta=(\sigma_3\otimes I)$. Solving the set of partial differential equations in ξ_i's embodied in (45) we get the following 12 generators:

$$X_1=-i(I\otimes I)\partial/\partial x,\quad X_2=-i(I\otimes I)\partial/\partial y,\quad X_3=-i(I\otimes I)\partial/\partial z,$$
$$X_4=-i(I\otimes I)\partial/\partial\tau,\quad X_5=-i\boldsymbol{\alpha}\cdot\nabla-(mc/\hbar)(\sigma_2\otimes I)\boldsymbol{\alpha}\cdot(\mathbf{r}\times\nabla)$$
$$X_6=-i(2mc/\hbar)(I\otimes I)(y\partial/\partial z-z\partial/\partial y)+(\sigma_3\otimes I)\left(\alpha_y\frac{\partial}{\partial z}-\alpha_z\frac{\partial}{\partial y}\right)$$
$$X_7=-i(2mc/\hbar)(I\otimes I)(z\partial/\partial x-x\partial/\partial z)+(\sigma_3\otimes I)\left(\alpha_z\frac{\partial}{\partial x}-\alpha_x\frac{\partial}{\partial z}\right)$$
$$X_8=-i(2mc/\hbar)(I\otimes I)(x\partial/\partial y-y\partial/\partial x)+(\sigma_3\otimes I)\left(\alpha_x\frac{\partial}{\partial y}-\alpha_y\frac{\partial}{\partial x}\right)$$
$$X_9=(\sigma_2\otimes I)(\alpha_y\partial/\partial z-\alpha_z\partial/\partial y),\quad X_{10}=(\sigma_2\otimes I)\left(\alpha_z\frac{\partial}{\partial x}-\alpha_x\frac{\partial}{\partial z}\right)$$
$$X_{11}=(\sigma_2\otimes I)\left(\alpha_x\frac{\partial}{\partial y}-\alpha_y\frac{\partial}{\partial x}\right),\quad X_{12}=-i(\sigma_1\otimes I)\boldsymbol{\alpha}\cdot\nabla.$$

6. Discussion

We have surveyed recent activities in determining the symmetry groups for different classical and quantum mechanical systems. The problem reduces to solving a set of coupled partial differential equations. It is true that this is no mean proposition. However, there is a vast literature of classical mathematics on this branch and with a little trial and error symmetry groups for a large number of physical systems have been obtained. On the other hand the classic works of Bateman (1910) and Fock (1935) were only possible as results of individual ingenuity. It has been shown that Lie's method gives an algorithm for the systematic study of the problem.

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Note added in proof:

In equations (33) and (36), x_i, u^l and u^l_j's are to be considered as independent variables. The term $|J| = 0$ is excluded from the summation over $|J|$ in (33), while in (36) it is included.