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A geometric generalization of classical mechanics and quantization

R K VARMA

Physical Research Laboratory, Navrangpura, Ahmedabad 380 009, India

Abstract. A geometrization of classical mechanics is presented which may be considered as a realization of the Hertz picture of mechanics. The trajectories in the f-dimensional configuration space V_f of a classical mechanical system are obtained as the projections on V_f of the geodesics in an $(f+1)$ dimensional Riemannian space V_{f+1}, with an appropriate metric, if the additional $(f+1)th$ coordinate, taken to be an angle, is assumed to be "cyclic".

When the additional (angular) coordinate is not cyclic we obtain what may be regarded as a generalization of classical mechanics in a geometrized form. This defines new motions in the neighbourhood of the classical motions. It has been shown that, when the angular coordinate is "quasi-cyclic", these new motions can be used to describe events in the quantum domain with appropriate periodicity conditions on the geodesics in V_{f+1}.

Keywords. Hertz mechanics; Riemannian space; geometrization; geodesics; classical mechanics; quantization.

1. Introduction

All the laws of classical mechanics were set forth by Newton in the most direct form in his *Principia.* But these laws have continued to be reformulated from different viewpoints leading to further insights into their form and content. Such reformulations which may have appeared to be purely academic at one time have paid rich dividends in terms of further understanding and development of physical principles.

One of the most useful and successful of these is the variational formulation involving the Hamilton principle. This leads to the Lagrangian or the Hamiltonian formulation depending on whether the variational principle is written in the configuration space or in phase space (p, q). Apart from the aesthetic appeal that these variational principles have, the resulting formulations have a down-to-earth practical utility in that they provide a very simple way of writing the equations of motion for any complicated system, a task which becomes highly cumbersome and involved with the methods of vectorial mechanics.

The formal structure of Hamiltonian mechanics, however, bloomed into its own when through the apparatus of canonical transformation the theory was cast into the elegant Hamilton-Jacobi formalism. Making use of the theory of first order partial differential equations, the problem of integration of the equations of motion was replaced by the determination of the complete integral of a partial differential equation—the Hamilton-Jacobi equation—with the equations of motion as its characteristic equations. The determination of the trajectory of a system is then reduced to simple inversions of certain relations. It is well known that Hamilton-Jacobi formalism played a pivotal role in the formulation of quantum mechanics *a la'* Schrödinger.

While the Hamilton principle does carry some geometrical flavour, Gauss (1829) and Hertz (1894) gave formulations of classical mechanics which have a distinct geometrical significance. They introduced a quantity which Hertz called as the "curvature". According to the Gauss-Hertz principle, of all the kinematically possible paths, the actual trajectory was identified to be the one which has the least "curvature". This can be considered as a first attempt at geometrization of classical mechanics.

Hertz (1894), however, went further. Dissatisfied with the concept of potential energy he replaced it with the energy associated with the motion of some "hidden" masses. He then introduced some "constraints" or "connections" to provide a communication between the motions of "hidden" and "visible" masses. His formulation does not seem to have been appreciated and developed further, presumably because he did not get an opportunity, due to his early demise, to clarify his viewpoint. But it must surely be considered as a first systematic attempt at a complete geometrization of classical mechanics. In fact, his point of view proved very much to be prophetic since Einstein's geometric theory of gravitation followed a decade or so later.

In the context of the current search for the hidden variable theories of quantum mechanics, it is pertinent to point out here that Hertz's formulation involving "hidden masses" should in fact be regarded as the first hidden variable theory ever conceived—but the one for classical mechanics.

In this paper, we present what we believe to be a realization of the Hertz picture, where we seek to geometrize classical mechanics in the spirit of the Hertz Fundamental Law, *viz*, "*Systema omne liberum perseverare in statu suo quiescendi vel movendi uniformiter in directissiman*" (Every free system persists in its state of rest or of uniform motion in a *straightest* path). To be sure, the fundamental law as worded above holds only for free systems. But any unfree system is regarded here as a subsystem of a larger free system. So the motion of an unfree system is deducible from the fundamental law applied to a larger free system of which the unfree system is a part.

Implicit in the statement of the fundamental law, with its reference to the "straightest path" for a free system, is the point of view that a free system still follows a straightest path (a geodesic) as in Newton's 1st law, but now in a non-Euclidean space. It is now the geometry of space belonging to a "free" system that will account for the motion of an unfree system which can always be considered as a part of an appropriate former. Clearly, as we shall see, this geometric picture dispenses with the concept of potential energy.

If f be the dimensionality of the configuration space of a classical dynamical unfree system, then we show that it is necessary to introduce a Riemannian space of only one higher dimension, *i.e.* $(f+1)$, to serve as the space for the larger free system. The trajectories of the classical dynamical system are thus realized as the projections on the f-dimensional space, of the geodesics in the $(f+1)$ dimensional Riemannian space with an appropriate metric. The structure of space as defined by its metric then determines the motion.

The geometric point of view has already met with a resounding success in the Einstein theory of gravitation, and should be extended to other areas. A hidden variable geometric formulation of classical mechanics based on this view point also provides a natural framework for a hidden variable crypto-deterministic theory for quantum mechanics. The author has been carrying out these investigations for some time (Varma 1978, 1984).

In §2, we introduce a Riemannian metric in the $(f+1)$ dimensional configuration space (f being the number of degrees of freedom of the classical dynamical system under consideration). We discuss the correspondence between the classical motion and the geodesic in the $(f+1)$ dimensional Riemannian manifold, and obtain an identification for the "potential" in terms of the metric components. A diagonal metric for V_{f+1} is shown here to induce a "potential" for the classical motion in the flat space V_f, while a particular form of nondiagonal metric shown in §3 yields electromagnetic potentials in V_f for a charged particle. We thereby obtain an identification of the electromagnetic potentials in terms of the metric components. In §§4 and 5, we show how quantization can follow from some "periodicity conditions" on the geodesics in the $(f+1)$ dimensional space.

2. Geodesics in $(f+1)$ dimensional Riemannian manifold

In accordance with the discussion in §1 we define an $(f+1)$ dimensional Riemannian space, V_{f+1} where f is the dimensionality of the configuration space of the classical mechanical system. (For an n-particle system in the 3-space, we have $f = 3n$). The additional dimension is taken to be an angular coordinate χ. Let m_i be the "masses" corresponding to the coordinates X_i, then we define a line-element ds in the Riemannian space V_{f+1} by

$$\begin{aligned} \mathrm{d}s^2 &= \frac{1}{2}\sum_i m_i(\mathrm{d}X^i)^2 + \frac{1}{2}g(x^1,\ldots X^f;\chi)\mathrm{d}\chi^2 \\ &= \frac{1}{2}\sum_\alpha g_\alpha(\mathrm{d}X^\alpha)^2 \end{aligned} \tag{1}$$

where $\quad g_\alpha = m_\alpha, \quad \alpha = 1, 2, \ldots f$

$$g_{f+1} = g(X^1, X^2, \ldots X^f, \chi).$$

For a n-particle system in 3-space, the "masses" m cannot all be different. We must have $m_1 = m_2 = m_3 = m^{(1)}$, $m_4 = m_5 = m_6 = m^{(2)}$ etc. where the masses $m^{(j)} = m_{(3j-3+i)}$ correspond to the jth particle as i takes only the values 1, 2, 3. In general, g can be a function of all the variables X^i and χ. But for reasons, given later, it is first taken to be independent of χ, and a function only of the X^i.

A geodesic (or "straightest path") in the space defined by (1) is given by:

$$\delta\int \mathrm{d}s = \delta\int\left[\sum_\alpha g_\alpha(\mathrm{d}X^\alpha)^2\right]^{1/2} = 0 \tag{2}$$

Or if we introduce the Lagrangian

$$\Lambda = \frac{1}{2}\sum_\alpha g_\alpha\left(\frac{\mathrm{d}X^\alpha}{\mathrm{d}t}\right)^2 \tag{3}$$

Equation (2) is equivalent to

$$\begin{aligned} 0 &= \delta\int\left[\sum_\alpha g_\alpha\left(\frac{\mathrm{d}X^\alpha}{\mathrm{d}t}\right)^2\right]^{1/2}\mathrm{d}t \\ &= \delta\int \Lambda^{1/2}\,\mathrm{d}t. \end{aligned} \tag{4}$$

This yields the Euler-Lagrange equations for the geodesic

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \Lambda^{1/2}}{\partial \dot{X}^\alpha}\right) - \frac{\partial \Lambda^{1/2}}{\partial X^\alpha} = 0, \quad \alpha = 1, \ldots, f+1 \tag{5a}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\Lambda^{-1/2}\frac{\partial \Lambda}{\partial \dot{X}^\alpha}\right] - \Lambda^{-1/2}\frac{\partial \Lambda}{\partial X^\alpha} = 0. \tag{5b}$$

The Hamilton principle with the Lagrangian Λ on the otherhand yields the Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \Lambda}{\partial \dot{X}^\alpha}\right) - \frac{\partial \Lambda}{\partial X^\alpha} = 0 \tag{6}$$

If the Lagrangian Λ of the system is time-independent then the energy $\mathscr{E}$ defined as

$$\begin{aligned} \mathscr{E} &= \dot{X}^\alpha \frac{\partial \Lambda}{\partial \dot{X}^\alpha} - \Lambda \\ &= \frac{1}{2}\sum_\alpha g_\alpha \left(\frac{\mathrm{d}X^\alpha}{\mathrm{d}t}\right)^2 \end{aligned} \tag{7}$$

is a constant of the motion and is numerically equal to the Lagrangian Λ. For such a system (5b) reduces to (6); *i.e.*, the geodesics with the Riemannian metric (1) coincide with the trajectories in V_{f+1} corresponding to the Lagrangian Λ.

We shall consider only time-independent systems so that the above identification continues to hold.

2.1 *Projected motion in the f-dimensional configuration space, V_f*

The equations for the geodesic in the $(f+1)$-dimensional Riemannian manifold V_{f+1} in terms of the parameter t are thus:

$$m_i \frac{\mathrm{d}^2 X^i}{\mathrm{d}t^2} = \frac{1}{2}\dot{\chi}^2 \frac{\partial g}{\partial X^i}, \quad i = 1, 2, \ldots, f \tag{8a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(g\dot{\chi}) = 0. \tag{8b}$$

The right side of (8b) vanishes as g is assumed to be independent of χ. This yields

$$p_\chi = g\dot{\chi} = \varepsilon \text{ (a constant)}, \tag{9}$$

where $p_\chi = g\dot{\chi}$ is the momentum conjugate to χ. Using (9) $\dot{\chi}$ can be eliminated from the right side of (8a) in terms of ε and g, giving

$$m_i \frac{\mathrm{d}^2 X^i}{\mathrm{d}t^2} = -\frac{\partial}{\partial X^i}\left(\frac{\varepsilon^2}{2g}\right). \tag{10}$$

It is to be noted that X^i and χ motions are completely decoupled since (10) does not involve χ at all. It may thus be considered to describe the projected geodesics on the f-dimensional space $(X^1, X^2 \ldots X^f)$ and may be identified as the equation of motion of classical mechanics with $\varepsilon^2/2g$ being identified with the potential $V \equiv \varepsilon^2/2g$. Thus one has

$$m_i\frac{\mathrm{d}^2X^i}{\mathrm{d}t^2} = -\frac{\partial V}{\partial X^i}. \tag{11}$$

If this identification is to hold universally then ε must be related to some fundamental constants. As ε is of the dimension of action it is tempting to relate it to $\hbar$. Such an identification may not be quite justifiable since we have so far considered only classical motions. However, such an identification ($\varepsilon = \hbar$ to be precise) does follow when we proceed to construct a crypto-deterministic theory for quantum mechanics based on these concepts (Varma 1978, 1984). With such an identification, it furthermore follows that g should have the form $g = \hbar^2/2V$.

The potential V for the classical motion of a system in the f-dimensional configuration space, being expressible in terms of the metric component g, thus appears as a property of the geometry of the $(f+1)$ dimensional Riemannian manifold V_{f+1}. This then is a realization of geometrization of classical mechanics (what we believe to be) *a là* 'Hertz: The angular coordinate χ, for instance, belongs to Hertz' "hidden mass", which is contained as a factor in the metric component g. Equation (9), which follows from the independence of g on χ, can be considered as a "connection" or a "constraint" which provides a communication between the kinetic energy $\frac{1}{2}g\dot{\chi}^2$ of the hidden mass and the kinetic energy $\sum_i \frac{1}{2}g_i(\mathrm{d}X^i/\mathrm{d}t)^2$ of the visible masses.

3. Riemannian metric in V_{f+1} for the electromagnetic potentials

We next introduce a Riemannian metric in the space V_{f+1} which induces electromagnetic potentials in the flat space V_f. For a system of n particles of masses m_i in the ordinary 3-space the line-element in the space V_{3n+1} is given by

$$\mathrm{d}s^2 = \sum_i \frac{1}{2}m_i(\mathrm{d}\mathbf{X}_i)^2 + \frac{1}{2}g(\mathbf{X}_1, \mathbf{X}_2, \ldots \mathbf{X}_n)\,(\mathrm{d}\chi + \sum_i \beta_i \mathbf{A}'\cdot\mathrm{d}\mathbf{X}_i)^2, \tag{12}$$

where β_i are constants pertaining to the particles m_i, and where g and $\mathbf{A}'$ are, in general, functions of the $\mathbf{X}_i$ and χ. But here they are taken to be functions only of $\mathbf{X}_i$.

If the metric components are independent of the time parameter t, then as shown in §2, the geodesics in the space V_{3n+1} characterized by the line-element (12) coincide with the trajectories in V_{3n+1} induced by the Lagrangian Λ_{EM}

$$\Lambda_{\mathrm{EM}} = \sum \tfrac{1}{2}m_i\dot{\mathbf{X}}_i^2 + \tfrac{1}{2}g(\mathbf{X}_1, \mathbf{X}_2, \ldots \mathbf{X}_n)\,(\dot{\chi} + \sum_i \beta_i \mathbf{A}'\cdot\dot{X}_i)^2 \tag{13}$$

and are given by the Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\Lambda_{\mathrm{EM}}}{\partial\dot{\mathbf{X}}_i}\right) - \frac{\partial\Lambda_{\mathrm{EM}}}{\partial\dot{\mathbf{X}}_i} = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\Lambda_{\mathrm{EM}}}{\partial\dot{\chi}}\right) - \frac{\partial\Lambda_{\mathrm{EM}}}{\partial\chi} = 0. \tag{14}$$

It is useful, in the present case, to transform the equations to the Hamiltonian form.

The canonical momenta corresponding to $\mathbf{X}_i$ and χ are:

$$\mathbf{p}_i = \frac{\partial \Lambda_{EM}}{\partial \dot{\mathbf{X}}_i} = m_i \dot{\mathbf{X}}_i + \beta_i g \mathbf{A}'(\dot{\chi} + \sum_i \beta_i \dot{\mathbf{X}}_i \cdot \mathbf{A}') \tag{15a}$$

$$p_\chi = \frac{\partial \Lambda_{EM}}{\partial \dot{\chi}} = g(\dot{\chi} + \sum_i \beta_i \dot{\mathbf{X}}_i \cdot \mathbf{A}'), \tag{15b}$$

so that the Hamiltonian is given by

$$H = \sum_i \mathbf{p}_i \cdot \dot{\mathbf{X}}_i + p_\chi \dot{\chi} - \Lambda_{EM}$$
$$= \sum_i \frac{1}{2m_i}[\mathbf{p}_i - \beta_i p_\chi \mathbf{A}']^2 + \frac{p_\chi^2}{2g}. \tag{16}$$

The Hamilton equations of motion are:

$$\dot{\mathbf{p}}_i = -\frac{\partial H}{\partial \mathbf{X}_i} = -\frac{\partial}{\partial \mathbf{X}_i}\left[\sum_i \frac{1}{2m_i}(\mathbf{p}_i - \beta_i p_x \mathbf{A}')^2 + p_\chi^2/2g\right], \tag{17a}$$

$$\dot{p}_\chi = -\frac{\partial H}{\partial \chi} = 0, \tag{17b}$$

$$\dot{\mathbf{X}}_i = \frac{\partial H}{\partial \mathbf{p}_i} = \frac{1}{m_i}(\mathbf{p}_i - \beta_i p_\chi \mathbf{A}'), \tag{18a}$$

$$\dot{\chi} = \frac{\partial H}{\partial p_\chi} = \frac{p_\chi}{g} - \mathbf{A}' \cdot \sum_i \frac{\beta_i}{m_i}(p_i - \beta_i p_\chi \mathbf{A}'). \tag{18b}$$

The zero on the right side of (17b) follows from g and $\mathbf{A}'$ and therefore the Hamiltonian being independent of χ. Using (15b) it follows that

$$p_\chi = g(\dot{\chi} + \sum_i \beta_i \dot{\mathbf{X}}_i \cdot \mathbf{A}') = \varepsilon \text{ (a constant).} \tag{19}$$

With this the motion of $\mathbf{X}_i$ and χ are completely decoupled. In fact, with $p_\chi = \varepsilon = \hbar$ the Hamiltonian involves only the variables $\mathbf{X}_i$ (and not χ), and has the form:

$$H = \sum_i \frac{1}{2m_i}[\mathbf{p}_i - \hbar\beta_i \mathbf{A}']^2 + \hbar^2/2g \tag{20}$$

The Hamiltonian (20) now obviously has the form for the classical motion of charged particles in an electromagnetic field provided we identify $\hbar^2/2g$ as the term corresponding to the scalar potential Φ and the quantity $\hbar\beta_i\mathbf{A}'$ as $e\mathbf{A}/c$, $\mathbf{A}$ being the vector potential. Since Φ and $\mathbf{A}$ are produced by charges and currents they must be proportional to the elementary charge e. Thus if we write

$$\mathbf{A} = e\hat{\mathbf{A}} \quad \text{and} \quad \Phi = e\hat{\Phi}, \tag{21}$$

we have the identifications:

$$\hbar\beta\mathbf{A}' = e\mathbf{A}/c = e^2\hat{\mathbf{A}}/c,$$

and

$$\hbar^2/2g = e\Phi = e^2\hat{\Phi}, \tag{22}$$

so that we have

$$\beta = (e^2/\hbar c) = \alpha, \tag{23}$$

where α is the fine structure constant. With these identifications the line element in V_{3n+1} has the form:

$$ds^2 = \sum_i \tfrac{1}{2} m_i (d\mathbf{X}_i)^2 + \tfrac{1}{2} g \, (d\chi + \alpha \hat{\mathbf{A}} \cdot \sum_i d\mathbf{X}_i)^2 \tag{24}$$

and the Hamiltonian has the form:

$$\begin{aligned} H &= \sum_i \frac{1}{2m_i} [\mathbf{p}_i - \alpha p_\chi \hat{\mathbf{A}}]^2 + \frac{p_\chi^2}{2g} \\ &= \sum_i \frac{1}{2m_i} \left(\mathbf{p}_i - \frac{e\mathbf{A}}{c} \right)^2 + e\Phi, \end{aligned} \tag{25}$$

where (19), (21), (22) and (23) have been made use of. The Hamiltonian (25) thus has the required form for particles in an electromagnetic field, with $\mathbf{A}$ and $\mathbf{\Phi}$ being the electromagnetic potentials.

4. Geodesics in V_{f+1} for quantum systems

In the discussion so far, we had assumed the metric component g appearing in the line elements (1) and (12) to be strictly independent of the angular coordinate χ which led to a decoupling of the χ and X-motions. It was shown that the projection of the complete motion in the (X, χ) space onto the X-space was identical with the classical motion provided certain identifications of the "potential" were made in terms of g.

It was shown in Varma (1978, hereafter referred to as paper I) that if g is a slowly varying function of χ, so that p_χ is no more an exact constant of motion, but an adiabatic invariant, the resulting geodesics in the (X, χ) space can be considered to represent the trajectories of quantum systems. Clearly the projections of these trajectories on the X-space will deviate from the corresponding classical trajectories. The deviations were shown to represent quantum effects and were described by a set of Schrödinger-like equations. It may be recalled that these equations describe the behaviour of an ensemble of systems with their trajectories in the neighbourhood of the classical trajectory. The Planck quantum of action $\hbar$ in this formalism was simply the momentum conjugate to the angular coordinate χ.

Recently (Varma 1984, to be referred to as paper II), we have given a more straightforward derivation of the Schrödinger-like equations for the Lagrangians (3) and (13) for the electromagnetic field. The latter also leads to the required form of the Schrödinger equations for a particle in an electromagnetic field.

Having obtained the Schrödinger equations as the ensemble description of the generalised trajectories in V_{f+1}, the quantization and all the other quantum effects follow in the standard way through solutions which are regular, single-valued and satisfy appropriate boundary conditions. However, since these quantum properties by hypothesis, emanate from the generalised trajectories in V_{f+1}, it should be possible to understand quantization and other effects in terms of the properties of individual trajectories in V_{f+1}.

In paper I we have been able to explain the tunnelling of potential barriers in terms of the trajectory in V_{3+1}. We have also given a 'derivation' of the position-momentum 'uncertainty relation' in terms of these trajectories. However, the question as to what is quantization in terms of the properties of trajectories in V_{f+1} which are continuous, deterministic and dense has remained.

4.1 *What is quantization?*

Quantization means the admissibility of certain well-defined discrete set of states of a system. From the Schrödinger equation these are obtained as its eigenstates, which are in turn characterized by solutions which are regular, single-valued and satisfy appropriate boundary conditions. Does there exist a correspondence between the eigenstates of the Schrödinger equation and the properties of the trajectories in V_{f+1}? Does there exist a class of discrete trajectories in V_{f+1} which correspond to the eigenstates of the Schrödinger equation? If so, how are such trajectories to be characterized? Since the Schrödinger wave function in our formalism refers to an ensemble of trajectories, the answer to the above questions is not expected to be simple. Nevertheless, in some of the cases we have studied we find that the 'admissible' discrete states are those which are periodic both in the X-space and in the χ-coordinate simultaneously. We give below an example to explain quantization in the presence of an external magnetic field based on this criterion.

5. Space quantization in an external magnetic field

Consider first a particle in a spherically symmetric potential. The trajectory of such a particle in V_{3+1} is given by (5a) and (5b), with a g which is independent of the angles θ and ϕ, and depends only on $r = |\mathbf{X}|$. The first point to note is that the total angular momentum vector $\mathbf{L} = \mathbf{X} \times \mathbf{p}$, is conserved for such a particle as follows trivially from (5a).

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{L} = \mathbf{X} \times \dot{\mathbf{p}} = 0. \tag{26}$$

This means that all the three components of the angular momentum $\mathbf{L}$ of an individual particle are conserved, whatever they are fixed to be initially.

However, as has been argued in paper II, for an ensemble which represents a quantum state, not all its members, for reasons of the uncertainty principle, can be specified with δ-function distributions for all the components of angular momentum. In fact, it was argued that only one component, say $\mathbf{L}_z$ of the angular momentum can be specified with a δ-function distribution, with $\mathbf{L}$ of the different members of the ensemble distributed uniformly over a cone of angle θ around the Z-axis and having the same magnitude L. Thus the magnitude L^2 and the component L_z are assignable with a δ-function distribution for the ensemble, though it must be emphasized that for every member of the ensemble the vector $\mathbf{L}$ is strictly conserved.

The question of space quantization is then the question of determining the allowed L_z values in relation to the magnitude L; or in other words of determining the different discrete allowed values of θ.

Experimentally, the space quantization of the orientation of the electronic orbit in an atom is manifested through the Zeeman effect. The different discrete orientations described by the magnetic quantum number m are thus referred with respect to the direction of the external magnetic field. Such a quantization of direction should thus be obtainable for the motion in the presence of a magnetic field through appropriate conditions of periodicity.

To determine the quantization of direction consider the trajectory of a charged particle in V_{3+1} both in the absence and presence of an external static magnetic field. Recall that because of our identifications (9) and (19)

$$p_\chi = g\dot{\chi} = \hbar \tag{27a}$$

$$p_{\chi'} = g\left(\dot{\chi} + \frac{e}{2\hbar c}\dot{\mathbf{X}}\cdot\mathbf{A}\right) = \hbar \tag{27b}$$

where $\mathbf{A}$ is the vector potential for the static field and χ' denotes the angular coordinate in the presence of the magnetic field; g is the same in the two cases, since the particle is acted upon by the same electrostatic potential due to the nuclear charge in the atom. From (27a) and (27b), we then obtain on subtracting:

$$\frac{e}{\hbar c}\mathbf{A}\cdot\dot{\mathbf{X}} = (\dot{\chi} - \dot{\chi}'). \tag{28}$$

For a uniform magnetic field $\mathbf{B}$, we have $\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{X}$. Then we get:

$$\frac{e\mathbf{B}}{2mc}\cdot\frac{\mathbf{L}}{\hbar} = (\dot{\chi} - \dot{\chi}'). \tag{29}$$

where we have used

$$\mathbf{A}\cdot\dot{\mathbf{X}} = \tfrac{1}{2}B\times\mathbf{X}\cdot\dot{\mathbf{X}} = \tfrac{1}{2}\mathbf{B}\cdot(\mathbf{X}\times\mathbf{p})/m = \tfrac{1}{2}\mathbf{L}\cdot\mathbf{B}/m,$$

$\mathbf{L}$ being the angular momentum vector, $\mathbf{L} = \mathbf{X}\times\mathbf{P}$ for the particle.

Now if the direction of $\mathbf{L}$ is inclined at an angle θ to the direction of $\mathbf{B}$, the vector $\mathbf{L}$ will precess around the direction of $\mathbf{B}$ with the Larmor frequency $\Omega_L = eB/2mc$. Integrating (29) over the period of a Larmor precession $T(=2\pi/\Omega_L)$ we obtain:

$$2\pi(L\cos\theta/\hbar) = \Delta(\chi - \chi') = \delta\chi, \tag{30}$$

where $\Delta\chi$ is the change in χ (in the absence of magnetic field) and $\Delta\chi'$, the change in χ' (in the presence of magnetic field) over the period of a Larmor precession. $\delta\chi = \Delta(\chi - \chi')$ is thus the change in the angle over a Larmor precession period induced due to the presence of the magnetic field. If we set $\delta\chi = 2\pi m$ with m an integer and measure L in units of $\hbar$, $L = l\hbar$, then we get from (30):

$$\cos\theta = m/l \tag{31}$$

Now $l\hbar$ by definition, is the maximum value of the projection of $\mathbf{L}$ along any direction. Thus given l, the maximum allowed value of $|m|$ from (31) is l. Thus m being an integer takes on values

$$m = -l,\ -(l-1), \ldots .\ 0, \ldots\ (l-1),\ l. \tag{32}$$

Equations (31) and (32), then define the allowed directions of the orientation of the vector $\mathbf{L}$ with respect to the magnetic field $\mathbf{B}$ and are indeed the well-known directions with m being the magnetic quantum number.

The key condition which defines these discrete directions is of course $\delta\chi = 2\pi m$, involving the change in the angular coordinate χ (being integral multiple of 2π). Other quantizations can also be obtained in terms of similar conditions.

Before we close this discussion, we would like to recall how the familiar expression $l(l+1)\hbar^2$ for the eigenvalue of the operator L^2 can be obtained from simply a knowledge of the discrete space orientations of $\mathbf{L}$ given above by (31) and (32). This problem is considered in the problem book on quantum mechanics (Goldman *et al* 1956). It is shown that the equation $\overline{L^2} = l(l+1)\hbar^2$ can be obtained by using elementary equations of probability theory, if one uses the fact that the possible values of the components of the angular momentum along an axis are equal to $m(m = -l, \ldots 0 \ldots l)$ and all these components are equally probable and all axes equivalent. The solution is as follows:

Because of the equivalence of x, y and z axes we have:

$$\overline{L^2} = \overline{L_x^2} + \overline{L_y^2} + \overline{L_z^2} = 3\overline{L_x^2}.$$

From the definition of the average L_x^2 value and the fact of equal probability for all different possible values we have

$$\overline{L_x^2} = \hbar^2 \overline{m^2} = \frac{\hbar^2}{(2l+1)} \sum_{m=-l}^{+l} m^2$$
$$= \tfrac{1}{3} l(l+1)\hbar^2.$$

Therefore,

$$\overline{L_x^2} = 3\overline{L_x^2} = l(l+1)\hbar^2.$$

The reason for pointing out this derivation is to emphasize the fact that $l(l+1)\hbar^2$ is essentially an average value of L^2 with the average taken in the usual sense of the classical probability theory, over the allowed values of the projections. The essential observables are thus the projections $m(m = -l, \ldots 0 \ldots +l)$ and not L^2. In fact, we know of no experiment where L^2 may be directly observed in its eigenstate with the value $l(l+1)\hbar^2$. The average being taken in the sense of the classical probability theory and yielding the values $l(l+1)\hbar^2$ again seems to be consistent with our point of view as developed in papers I and II that the probability in quantum mechanics is already contained within the framework of classical probability theory provided the latter is applied to the trajectories in V_{f+1} rather than in V_f.

6. Summary and epilogue

We have developed here what may be considered as a generalization of classical mechanics in a geometrized form. The classical mechanical trajectories in a space V_f are obtained as projections on V_f of geodesics in a V_{f+1} with a certain metric appropriate for the required classical motion in V_f. The potentials for the latter are expressed in terms of the metric components for V_{f+1}.

The space V_{f+1} differs from V_f in the addition of an angular coordinate χ. When the coordinate χ is "ignorable", the classical mechanical (CM) trajectories in V_f can be regarded as *exact* projections of the geodesics in V_{f+1} with appropriate identification of the potential for the CM motion with the metric components in V_{f+1}. In such a case we simply have a geometrization of classical mechanics through V_{f+1}. This we believe is a realization of the Hertz picture of Mechanics.

When the angular coordinate is not ignorable, we have more than a mere geometrization of classical mechanics. We have a generalization of classical mechanics whereby we introduce new motions in V_{f+1} as also the projections thereof on V_f. But the projections on V_f do not in general form a "closed" system, describable by an equation of motion in V_f.

The main thrust of our line of development is that the quantum effects are essentially the manifestations of the new motions in V_{f+1} as projected onto V_f, such that the projected 'motions' are in the neighbourhood of the classical motion. Furthermore, the discrete quantum states may be identified as a class of discrete geodesics in V_{f+1} satisfying certain periodicity conditions involving specifically the angular coordinate χ.

In particular, we have shown in §5, how the quantization of the orientation of the angular momentum in an atom with respect to an external magnetic field follows from some conditions on $\delta\chi$. Similar conditions can be obtained for other quantizations as, for instance, is implied in the double-slit interference experiment. These points are discussed elsewhere. Suffice it to say at present that the space V_{f+1} as obtained by augmenting the dimensionality of V_f by an angular coordinate seems to be an appropriate space for describing quantum events as its geodesics. We shall also show in a later work how this also leads to 'quantum correlations' and 'quantum non-locality' for a many-particle system.

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